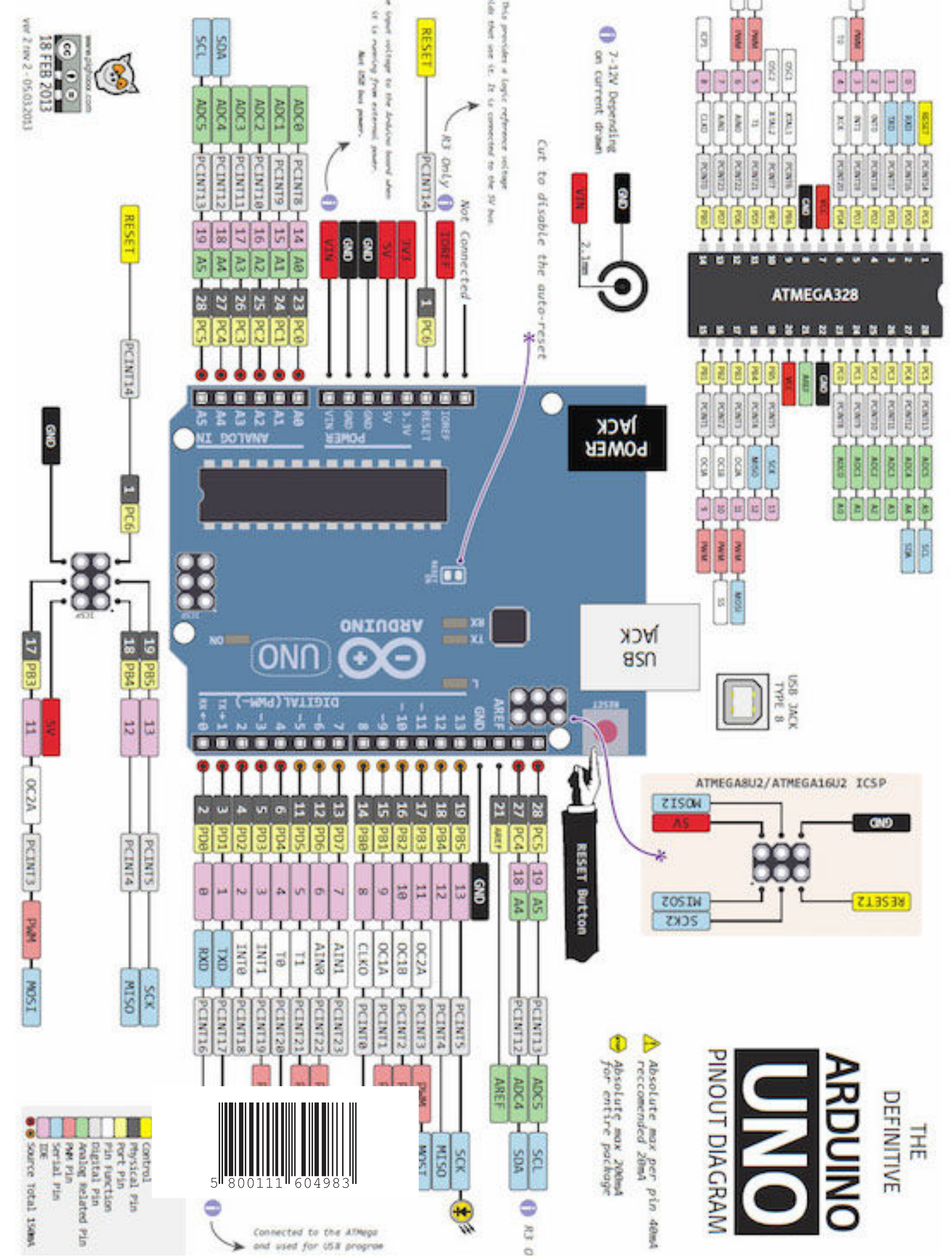

Dynamic System Modeling and Control 2015 Edition

Hugh Jack

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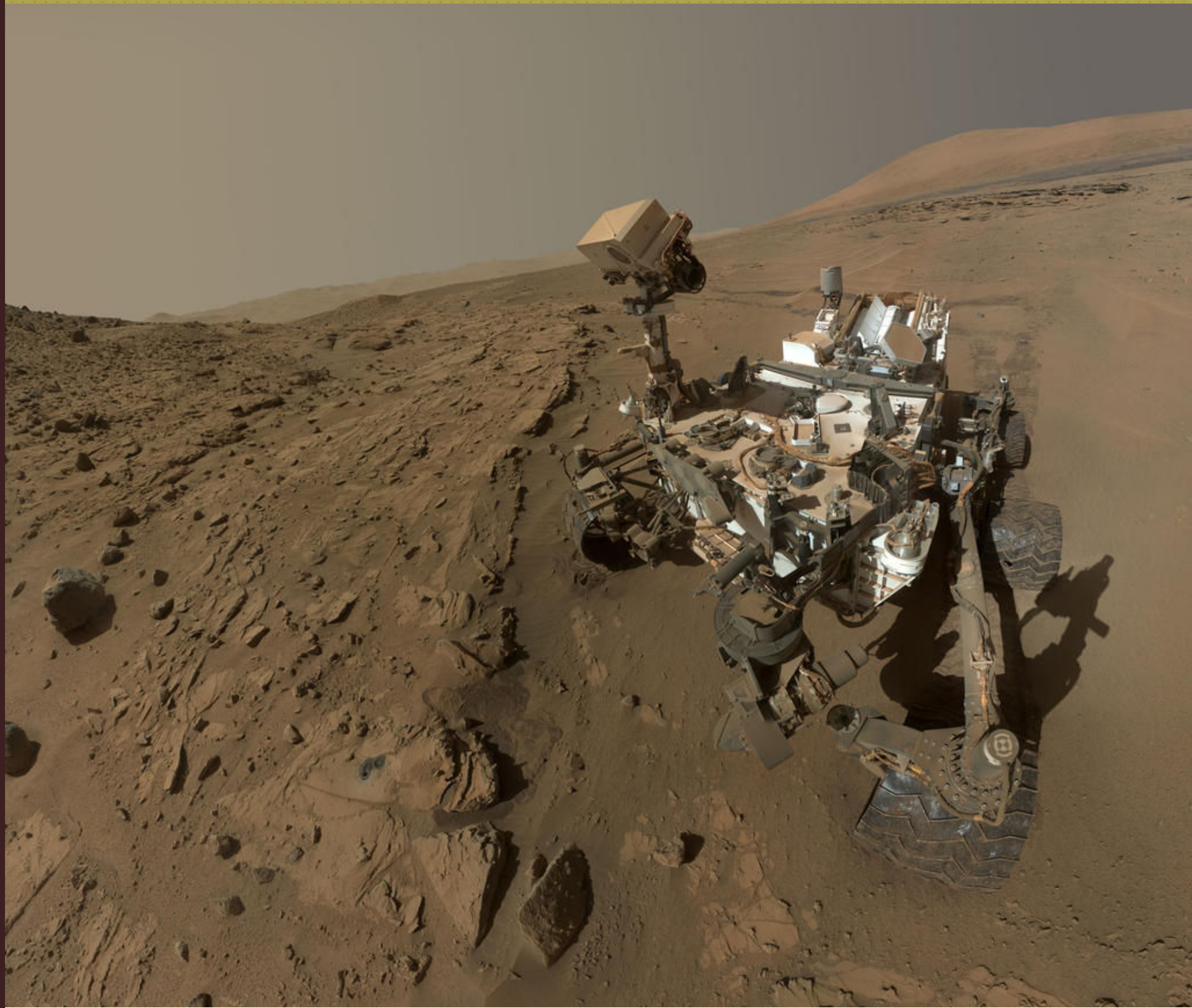

Dynamic System Modeling and Control

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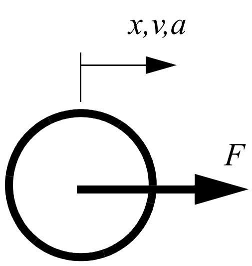
1. Translation

Topic 1.1 *Basic laws of motion.*
Topic 1.2 *Gravity, inertia, springs, dampers, cables and pulleys, drag, friction, FBDs.*
Topic 1.3 *System analysis techniques.*
Topic 1.4 *Design case.*
Objective 1.1 *To be able to develop differential equations that describe translating systems.*

1.1 Introduction

If the velocity and acceleration of a body are both zero then the body will be static. If the applied forces are balanced, and cancel each other out, the body will not accelerate. If the forces are unbalanced then the body will accelerate. If all of the forces act through the center of mass then the body will only translate. Forces that do not act through the center of mass will also cause rotation to occur. This chapter will focus only on translational systems.

The equations of motion for translating bodies are in Figure 1.1. These state simply that velocity is the first derivative of position, equation 1.1, and acceleration is the first derivative of velocity with respect to time, equation 1.2. Conversely, the acceleration can be integrated to find velocity, equation 1.3, and the velocity can be integrated to find position, equation 1.4. Therefore, if we know the acceleration of a body, we can determine the velocity and position. Finally, when a force is applied to a mass, the acceleration can be calculated by dividing the net force by the mass, equation 1.5.



The equations of motion are

$$v(t) = \left(\frac{d}{dt}\right)x(t) \qquad \text{eqn 1.1}$$

$$a(t) = \left(\frac{d}{dt}\right)^2 x(t) = \left(\frac{d}{dt}\right)v(t) \qquad \text{eqn 1.2}$$

or

$$v(t) = \int a(t)dt \qquad \text{eqn 1.3}$$

$$x(t) = \int v(t)dt = \iint a(t)dt \qquad \text{eqn 1.4}$$

$$a(t) = \frac{F(t)}{M} \qquad \text{eqn 1.5}$$

where,

$x(t)$, $v(t)$, $a(t)$ = position, velocity, and acceleration
M = Mass of the body
F = an applied force

Figure 1.1 Velocity and acceleration of a translating mass

An example application of these fundamental laws is shown in Figure 1.2. The initial conditions of the system are supplied (and are normally required to solve this type of problem). These are then used to find the state of the system after a period of time. The solution begins by integrating the acceleration, and using the initial velocity value for the integration constant, equation 1.6. So at t=0 the velocity will be equal to the initial velocity. This is then integrated once more to provide the position of the object, equation 1.7. As before, the initial position is used for the integration constant. This equation is then used to calculate the position after a period of time. Notice that units are used throughout the calculations; this is a good practice for any engineer.

Given an initial (t=0) state of x=5m, v=2m/s, a=$3ms^{-2}$, find the system state 5 seconds later assuming constant acceleration.

The initial conditions for the system at time t=0 are,

$$x_0 = 5m$$
$$v_0 = 2ms^{-1}$$
$$a_0 = 3ms^{-2}$$

Note: units are very important and should normally be carried through all calculations.

The constant acceleration can be integrated to find the velocity as a function of time.

$$v(t) = \int a_0 dt = a_0 t + C = a_0 t + v_0 \qquad \text{eqn 1.6}$$

Note:
$$v(t) = a_0 t + C$$
$$v(0) = a_0(0) + C$$
$$v(0) = C$$

Next, the velocity can be integrated to find the position as a function of time.

$$x(t) = \int v(t) dt = \int (a_0 t + v_0) dt = \frac{a_0}{2} t^2 + v_0 t + x_0 \qquad \text{eqn 1.7}$$

This can then be used to calculate the position of the mass after 5 seconds.

$$x(5) = \frac{a_0}{2} t^2 + v_0 t + x_0 = \frac{3ms^{-2}}{2}(5s)^2 + 2ms^{-1}(5s) + 5m =$$
$$= 37.5m + 10m + 5m = 52.5m$$

Figure 1.2 Example: Calculation for a translating mass, with initial conditions.

1.1 Modeling

When modeling translational systems it is common to break the system into parts. These parts are then described with Free Body Diagrams (FBDs). Common components that must be considered when constructing FBDs are listed below, and are discussed in following sections.

- Springs - resist deflection.
- Dampers and drag - resist motion.
- Friction - opposes relative motion between bodies in contact.
- Cables and pulleys - redirect forces.
- Contact points/joints - transmit forces through up to 3 degrees of freedom.
- Inertia - opposes acceleration and deceleration.
- Gravity and other fields - apply non-contact forces.

Free Body Diagrams

Free Body Diagrams (FBDs) allow us to reduce a complex mechanical system into smaller, more manageable pieces. The forces applied to the FBD can then be summed to provide an equation for the piece. These equations can then be used later to do an analysis of system behavior. These are required elements for any engineering problem involving rigid bodies.

An example of FBD construction is shown in Figure 1.3. In this case there is a mass sitting atop a spring. An FBD can be drawn for the mass. In total there are two obvious forces applied to the mass, gravity pulling the mass downward, and a spring pushing the mass upwards. The FBD for the spring has two forces applied at either end. Notice that the spring force, FR1, acting on the mass, and on the spring have an equal magnitude, but opposite direction.

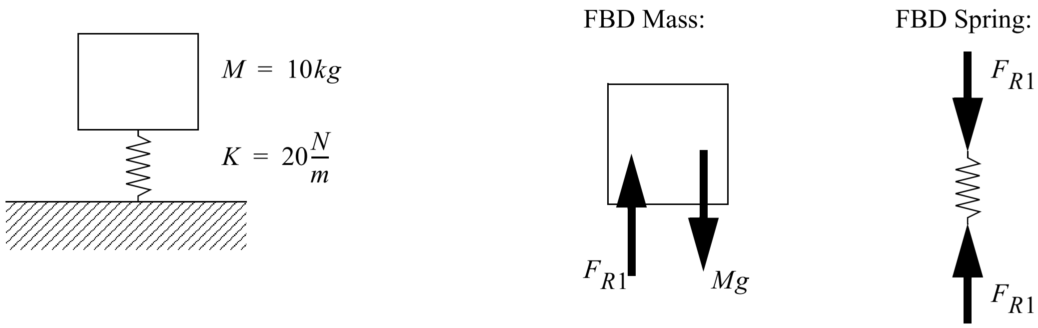


Figure 1.3 Free body diagram example

Mass and Inertia

In a static system the sum of forces is zero and nothing is in motion. In a dynamic system the sum of forces is not zero and the masses accelerate. The resulting imbalance in forces acts on the mass causing it to accelerate. For the purposes of calculation we create a virtual reaction force, called the inertial force. This force is also known as D'Alembert's (pronounced as daa-lamb-bears) force. It can be included in calculations in one of two ways. The first is to add the inertial force to the FBD and then add it into the sum of forces, which will equal zero, equation 1.8. The second method is known as Newton's equation where all of the forces are summed and set equal to the inertial force, as shown in Figure 1.10. The acceleration is proportional to the inertial force and inversely proportional to the mass.

$\sum F - Ma = 0$	D'Alembert's	eqn 1.8
$\sum F = Ma$	Newton's	eqn 1.9

Figure 1.4 D'Alembert's and Newton's equations

An application of Newton's equation to FBDs can be seen in Figure 1.6. In the first case an inertial force is added to the FBD. This force should be in an opposite direction (left here) to the positive direction of the mass (right). When the sum of forces equation is used then the force is added in as a normal force component. In the second case Newton's equation is used so the force is left off the FBD, but added to the final equation. In this case the sign of the inertial force is positive if the assumed positive direction of the mass matches the positive direction for the summation.

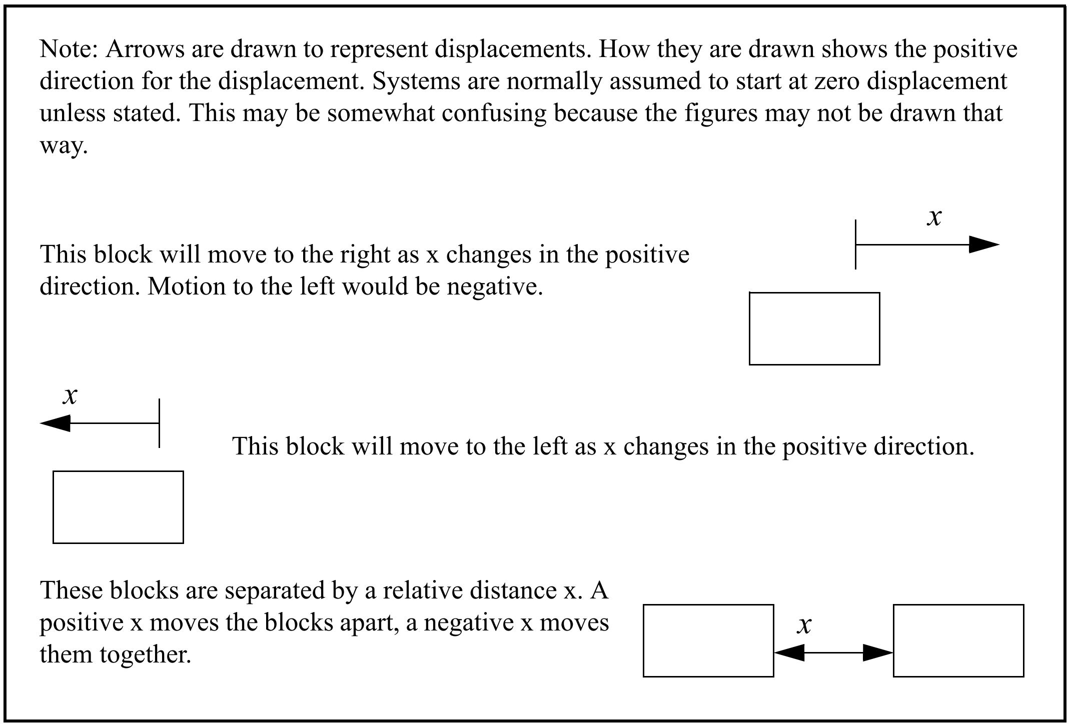
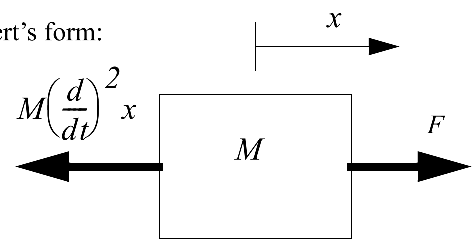


Figure 1.5 Arrow directions and sign conventions

D'Albert's form:

$$Ma = M\left(\frac{d}{dt}\right)^2 x$$



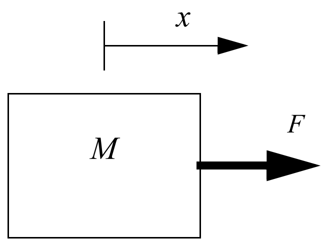
$$\overset{+}{\rightarrow}\sum F_x = F - M\left(\frac{d}{dt}\right)^2 x = 0$$

or

$$\overset{+}{\leftarrow}\sum F_x = -F + M\left(\frac{d}{dt}\right)^2 x = 0$$

Note: If using an inertial force then the direction of the force should be opposite to the positive motion direction for the mass.

Newton's Form:



$$\overset{+}{\rightarrow}\sum F_x = F = M\left(\frac{d}{dt}\right)^2 x$$

or

$$\overset{+}{\leftarrow}\sum F_x = -F = -M\left(\frac{d}{dt}\right)^2 x$$

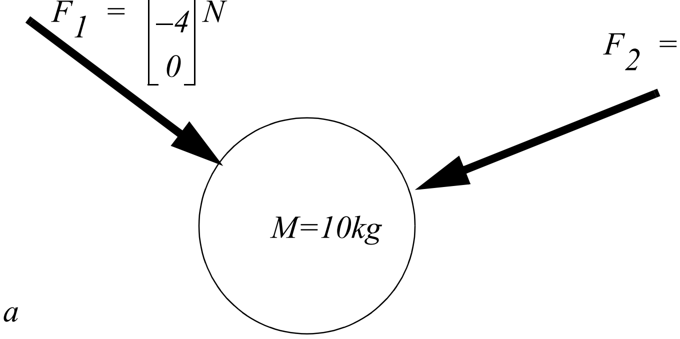
Note: If using Newton's form the sign of the inertial force should be positive if the positive direction for the summation and the mass are the same, otherwise if they are opposite then the sign should be negative.

Figure 1.6 Free body diagram and inertial forces

An example of the application of Newton's equation is shown in Figure 1.7. In this example there are two unbalanced forces applied to a mass. These forces are summed and set equal to the inertial force. Solving the resulting equation results in acceleration values in the ‘x’ and ‘y’ directions. In this example the forces and calculations are done in vector form for convenience and brevity.

If both forces shown act through the center of mass, what is the acceleration of the ball?

$$F_1 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} N \qquad F_2 = \begin{bmatrix} -7 \\ -3 \\ 0 \end{bmatrix} N$$



$$\sum F = F_1 + F_2 = Ma$$

$$\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} N + \begin{bmatrix} -7 \\ -3 \\ 0 \end{bmatrix} N = (10Kg)a$$

$$\therefore a = \left(\frac{1}{10Kg}\right) \begin{bmatrix} -2 \\ -7 \\ 0 \end{bmatrix} N = \begin{bmatrix} -0.2 \\ -0.7 \\ 0 \end{bmatrix} \left(\frac{Kgm}{s^2}\right) \frac{1}{Kg} = \begin{bmatrix} -0.2 \\ -0.7 \\ 0 \end{bmatrix} \frac{m}{s^2}$$

Figure 1.7 Example: Acceleration calculation

```
// A program to sum forces and calculate the acceleration

// define the given forces and mass
F1 = [5, -4, 0];
F2 = [-7, -3, 0];
M = 10;

function foo=Sum() // The sum of the applied forces
        foo = F1 + F2;
endfunction

A = Sum() / M;
printf("The acceleration is ( %f, %f, %f) m/s^2 \n", A(1), A(2), A(3));
printf("The magnitude is |A| = %f m/s^2 \n", norm(A));
```

Figure 1.8 Example: A Scilab calculation

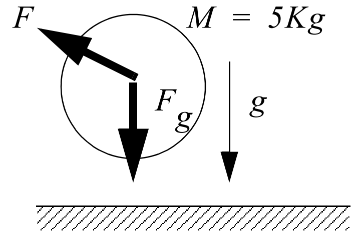
Gravity and Other Fields

Gravity is a weak force of attraction between masses. In our situation we are in the proximity of a large mass (the earth) that produces a large force of attraction. When analyzing forces acting on rigid bodies we add this force to our FBDs. The magnitude of the force is proportional to the mass of the object, with a direction toward the center of the earth (down).

The relationship between mass and force is clear in the metric system with mass having the units Kilograms (kg), and force the units Newtons (N). The relationship between these is the gravitational constant 9.81N/kg, which will vary slightly over the surface of the earth. The Imperial units for force and mass are both the pound (lb.) that often causes confusion. To reduce this confusion the unit for force is normally modified to be, lbf.

An example calculation including gravitational acceleration is shown in Figure 1.9. The 5kg mass is pulled by two forces, gravity and the arbitrary force ‘F’. These forces are described in vector form, with the positive ‘z’ axis pointing upwards. To find the equations of motion the forces are summed. To eliminate the second derivative on the inertia term the equation is integrated twice. The result is a set of three vector equations that describe the x, y and z components of the motion. Notice that the units have been carried through these calculations.

Assume we have a mass that is acted upon by gravity and a second constant force vector. To find the position of the mass as a function of time we first define the gravity vector and position components for the system.



$$g = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{N}{Kg} = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} \qquad X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad F = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad F_g = Mg$$

Next, sum the forces and set them equal to inertial resistance.

$$\sum F = Mg + F = M\left(\frac{d}{dt}\right)^2 X(t)$$

$$5Kg \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = 5Kg\left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Note: When an engineer solves a problem they will always be looking at the equations and unknowns. In this case there are three equations, and there are 9 constants/givens fx, fy, fz, vx0, vy0, xz0, x0, y0 and z0. There are 4 variables/unknowns x, y, z and t. Therefore with 3 equations and 4 unknowns only one value (4-3) is required to find all of the unknown values.

Integrate twice to find the position components.

$$\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}\right] t + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} = \left(\frac{d}{dt}\right) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{1}{2}\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}\right] t^2 + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} t + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0.1f_x t^2 + v_{x_0} t + x_0 \\ 0.1f_y t^2 + v_{y_0} t + y_0 \\ \left(\frac{-9.81}{2} + 0.1f_z\right) t^2 + v_{z_0} t + z_0 \end{bmatrix} m$$

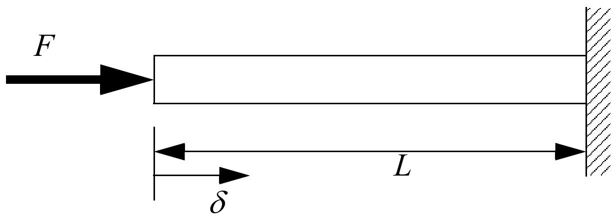
Figure 1.9 Example: Gravity vector calculations

Like gravity, magnetic and electrostatic fields can also apply forces to objects. Magnetic forces are commonly found in motors and other electrical actuators. Electrostatic forces are less common, but may need to be considered for highly charged systems.

Springs

Springs are typically constructed with elastic materials such as metals, and plastics, that will provide an opposing force when deformed. The properties of the spring are determined by the Young's modulus (E) of the material and the geometry of the spring. A primitive spring is shown in Figure 1.10. In this case the spring is a solid member. The relationship between force and displacement is determined by the basic mechanics of materials relationship. In practice springs are more complex, but the parameters (E, A and L) are combined into a more convenient form. This form is known as Hooke's Law.

(Hooke's law)



$$\delta = \left(\frac{L}{EA}\right)F$$

$$F = \left(\frac{EA}{L}\right)\delta$$

$$F = K\delta$$

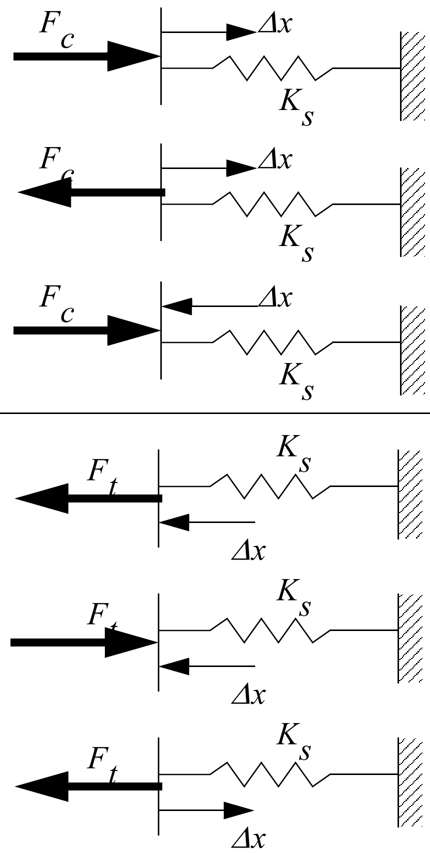
Figure 1.10 A solid member as a spring

Hooke's law does have some limitations that engineers must consider. The basic equation is linear, but as a spring is deformed the material approaches plastic deformation, and the modulus of elasticity will change. In addition the geometry of the object changes, also changing the effective stiffness. Springs are normally assumed to be massless. This allows the inertial effects to be ignored, such as a force propagation delay. In applications with fast rates of change the spring mass may become significant, and they will no longer act as an ideal device.

The cases for tension and compression are shown in Figure 1.11. In the case of compression the spring length has been made shorter than its' normal length. This requires that a compression force be applied. For tension, both the displacement from neutral and the required force reverse direction. It is advisable when solving problems to assume a spring is either in tension or compression, and then select the displacement and force directions accordingly.

$\Delta x = \text{deformed length}$	ASIDE: a spring has a natural or undeformed length. When at this length it is neither in tension or compression

compression as positive:



$$F_c = K_s \Delta x$$

$$F_c = -K_s \Delta x$$

$$F_c = -K_s \Delta x$$

tension as positive:

$$F_t = K_s \Delta x$$

$$F_t = -K_s \Delta x$$

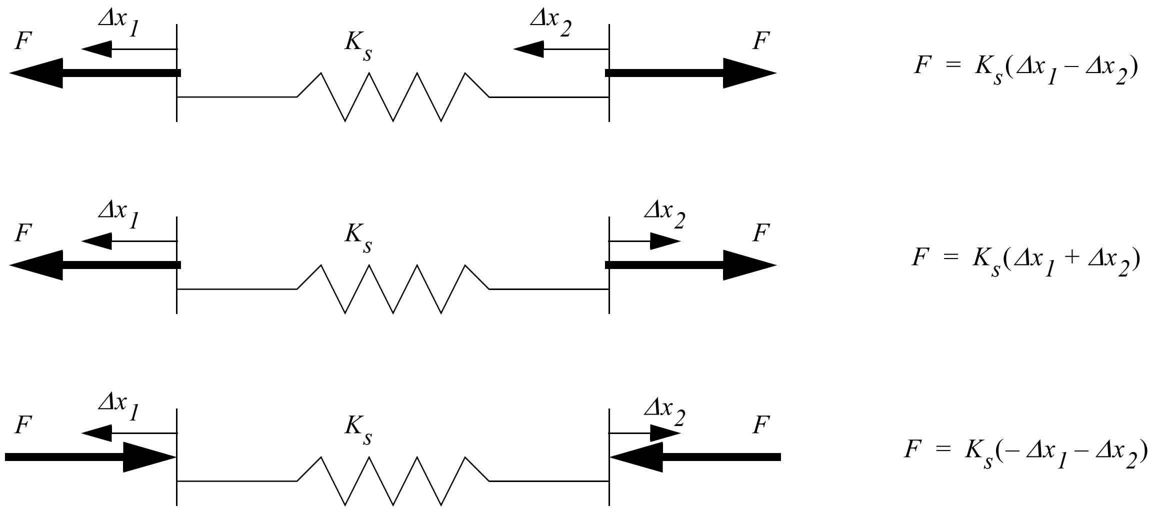
$$F_t = -K_s \Delta x$$

NOTE: the symbols for springs, resistors and inductors are quite often the same or similar. You will need to remember this when dealing with complex systems - and especially in this book where we deal with both types of components.

Figure 1.11 Sign conventions for spring forces and displacements

Previous examples have shown springs with displacements at one end only. In Figure 1.12, springs are shown that have movement at both ends. In these cases the sign of the force applied to the spring is selected with reference to the assumed compression or tension. The primary difference is that care is required to correctly construct the expressions for the tension or compression forces. In all cases the forces on the springs must be assumed and drawn as either tensile or compressive. In the first example the displacement and forces are tensile. The displacement at the left is tensile, so it will be positive, but on the right hand side the displacement is compressive so it is negative. In the second example the force and both displacements are shown as tensile, so the terms are both positive. In the third example the force is shown as compressive, while the displacements are both shown as tensile, therefore both terms are negative.

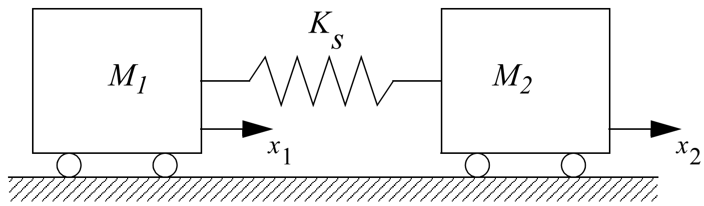
All force arrows are draw assuming the springs are in tension.



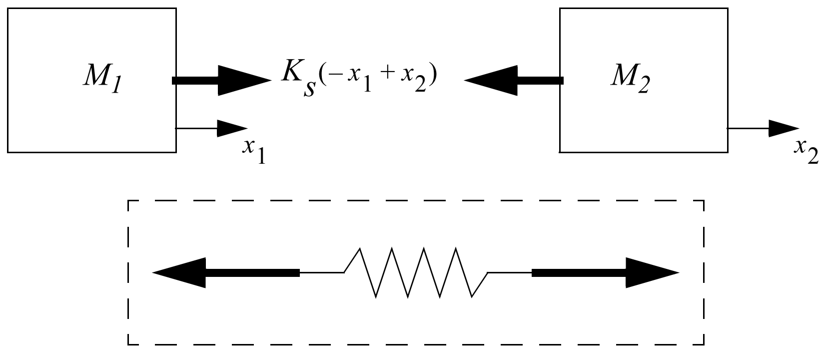
> Aside: it is useful to assume that the spring is either in tension or compression, and then make all decisions based on that assumption.

Figure 1.12 Examples of forces when both sides of a spring can move

Consider the two masses below separated by a spring.



The system can be reduced to free body diagrams assuming the spring is tension.



Note: In this example the spring is assumed to be in tension and the signs of the magnitude are made negative for terms that result in compression for positive changes.

The system can be reduced to free body diagrams assuming the spring is in compression.

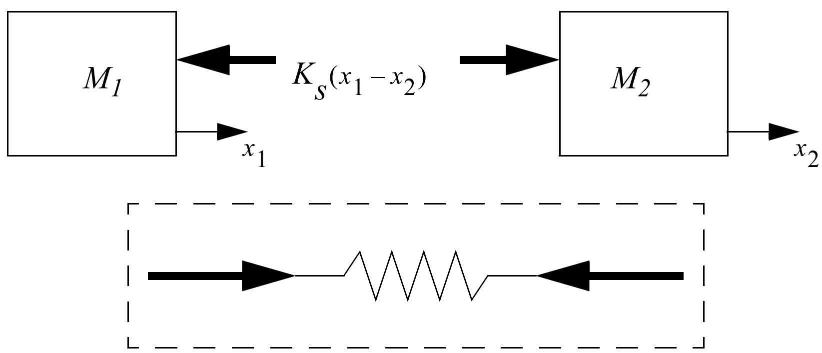


Figure 1.13 Drawing FBDs with interconnecting springs

Sometimes the true length of a spring is important, and the deformation alone is insufficient. In these cases the deformation can be defined as a deformed and undeformed length, as shown in Figure 1.14. In addition to providing forces, springs may be used as energy storage devices.

$$\Delta x = l_1 - l_0$$

where,

Δx = deformation

l_0 = the length when undeformed

l_1 = the length when deformed

$$E_P = \frac{K(\Delta x)^2}{2}$$

where,

K = spring coefficient

E_P = potential energy stored in the spring

Figure 1.14 Using the actual spring length

Consider the springs shown in Figure 1.15. When two springs are combined in this manner they can be replaced with a single equivalent spring. In the parallel spring combination the overall stiffness of the spring would increase. In the series spring combination the overall stiffness would decrease.

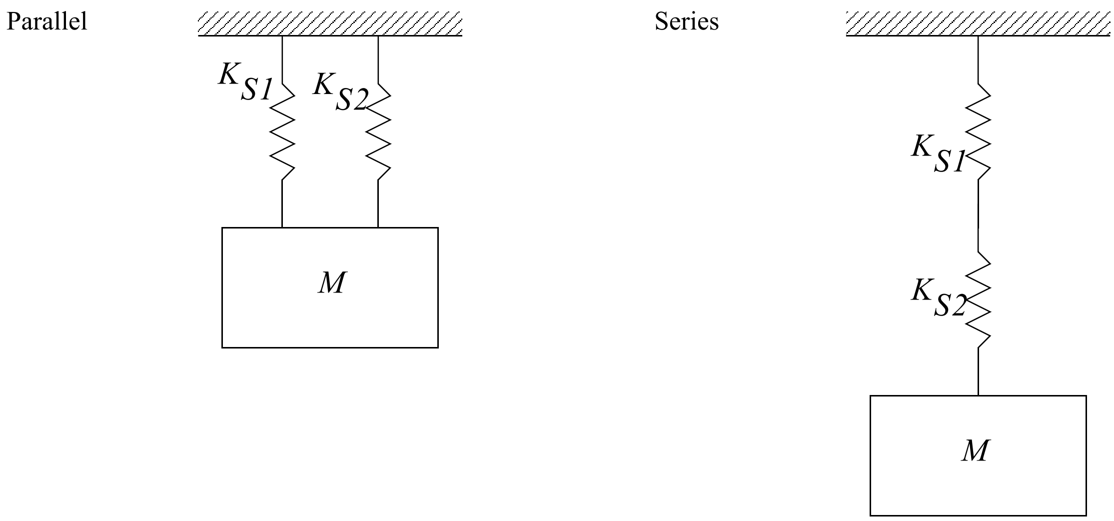
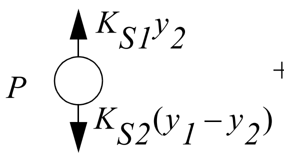


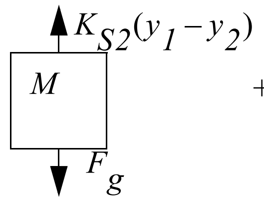
Figure 1.15 Springs in parallel and series (kinematically)

Figure 1.16 shows the calculations required to find a spring coefficient equivalent to the two springs in series. The first step is to draw a FBD for the mass at the bottom, and for a point between the two springs, P. The forces for both of these are then summed. The next process is to combine the two equations to eliminate the height variable created for point P. After this, the equation is rearranged into Hooke's law, and the equivalent spring coefficient is found.

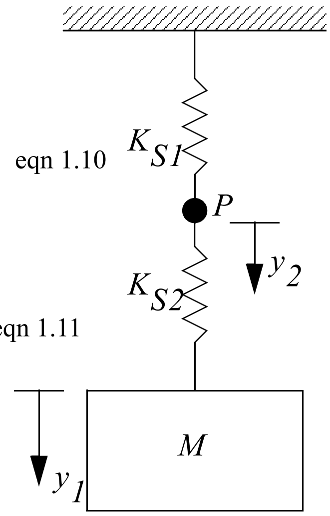
First, draw FBDs for P and M and sum the forces assuming the system is static.



$$+\uparrow\sum F_y = K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0 \qquad \text{eqn 1.10}$$



$$+\uparrow\sum F_y = K_{S2}(y_1 - y_2) - F_g = 0 \qquad \text{eqn 1.11}$$



Next, rearrange the equations to eliminate y2 and simplify,

equation 1.11 becomes

$$K_{S2}(y_1 - y_2) - F_g = 0$$

$$y_1 - y_2 = \frac{F_g}{K_{S2}}$$

$$y_2 = y_1 - \frac{F_g}{K_{S2}} \qquad \text{eqn 1.12}$$

equation 1.10 becomes

$$K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0$$

$$y_2(K_{S1} + K_{S2}) = y_1 K_{S2} \qquad \text{eqn 1.13}$$

next equation 1.12 is substituted into equation 1.13.

$$\left(y_1 - \frac{F_g}{K_{S2}}\right)(K_{S1} + K_{S2}) = y_1 K_{S2}$$

$$y_1 - \frac{F_g}{K_{S2}} = y_1 \frac{K_{S2}}{K_{S1} + K_{S2}}$$

$$F_g = y_1\left(1 - \frac{K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

$$F_g = y_1\left(\frac{K_{S1} + K_{S2} - K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

$$F_g = y_1\left(\frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}\right)$$

Finally, consider the basic spring equation to find the equivalent spring coefficient.

$$K_{equiv} = \frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}$$

Figure 1.16 Example: Calculation of an equivalent spring coefficient for springs in series

Damping and Drag

A damper is a component that resists motion. The resistive force is relative to the rate of displacement. As mentioned before, springs store energy in a system but dampers dissipate energy. Dampers and springs are often used to compliment each other in designs.

Damping can occur naturally in a system, or can be added by design. The physical damper pictured in Figure 1.17 uses a cylinder that contains a fluid. There is a moving rod and piston that can slide within the cylinder. As the piston moves, fluid is forced through a small orifice. When moved slowly the fluid moves easily, but when moved quickly the pressure required to force the fluid through the orifice rises. This rise in pressure results in a higher force of resistance. In ideal terms any motion would result in an opposing force. In reality there is also a break-away force that needs to be applied before motion begins. Other manufacturing variations could also lead to other small differences between dampers. Normally these cause negligible effects.

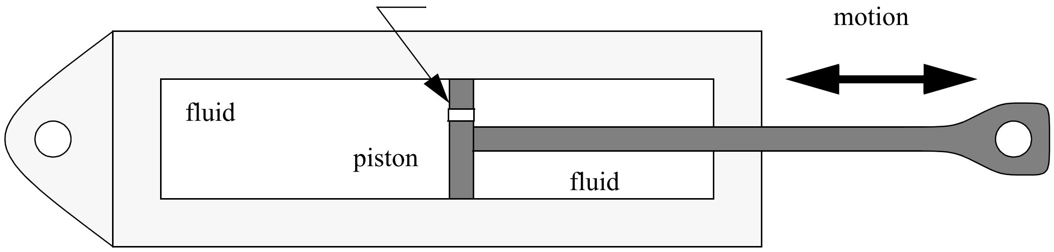
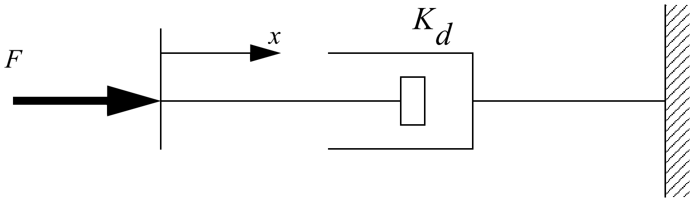


Figure 1.17   A physical damper

The standard symbol for a standard damper is shown in Figure 1.18. The basic equation for an ideal damper in compression is shown in equation 1.14. In this case the force and displacement are both compressive. The force is calculated by multiplying the damper coefficient by the velocity, or first derivative of position. Aside from the use of the first derivative of position, the analysis of dampers in systems is similar to that of springs.

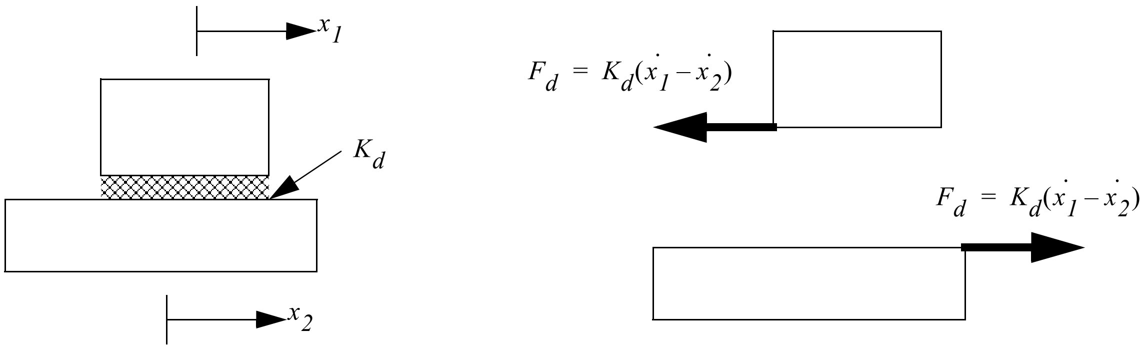


$$F = K_d\left(\frac{d}{dt}\right)x \qquad \text{eqn 1.14}$$

> Aside: The symbol shown is typically used for dampers. It is based on an old damper design called a dashpot. It was constructed using a small piston inside a larger pot filled with oil.

Figure 1.18   An ideal damper

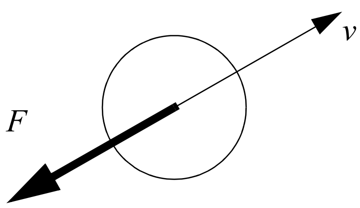
Damping can also occur when there is relative motion between two objects. If the objects are lubricated with a viscous fluid (e.g., oil) then there will be a damping effect. In the example in Figure 1.19 two objects are shown with viscous friction (damping) between them. When the system is broken into free body diagrams the forces are shown to be a function of the relative velocities between the blocks.



Aside: Fluids, such as oils, have a significant viscosity. When these materials are put in shear they resist the motion. The higher the shear rate, the greater the resistance to flow. Normally these forces are small, except at high velocities.

Figure 1.19 Viscous damping between two bodies with relative motion

A damping force is proportional to the first derivative of position (velocity). Aerodynamic drag is proportional to the velocity squared. The equation for drag is shown in Figure 1.20 in vector and scalar forms. The drag force increases as the square of velocity. Normally, the magnitude of the drag force coefficient 'D' is approximated theoretically and/or measured experimentally. The drag coefficient is a function of material type, surface properties, object size, and object geometry.



$$F = -D|v|v$$

Figure 1.20 Aerodynamic drag

1.1 System Reduction

An orderly approach to system analysis can simplify the process of analyzing large systems. The list of steps below is based on general observations of good problem solving techniques.

1. Assign letters/numbers to designate force components (if not already done) - this will allow you to refer to components in your calculations.
2. Define positions and directions for any moving masses. This should include the selection of reference points.
3. Draw free body diagrams for each component, and add forces (inertia is optional).
4. Write equations for each component by summing forces.
5. (next chapter) Combine the equations to eliminate unwanted variables.
6. (next chapter) Develop a final equation that relates input (forcing functions) to outputs (results).

Note: When deriving differential equations, the final value can be checked for errors using unit analysis. This method involves replacing variables with their unit equivalents. All the units should match.

eg.

$$\ddot{x}_2(M_2) + \dot{x}_2(B) + x_2(K_{s2}) + x_1(-K_{s2}) = F$$

$$\therefore \frac{m}{s^2}(Kg) + \frac{m}{s}\left(\frac{Ns}{m}\right) + m\left(\frac{N}{m}\right) + m\left(-\frac{N}{m}\right) = N$$

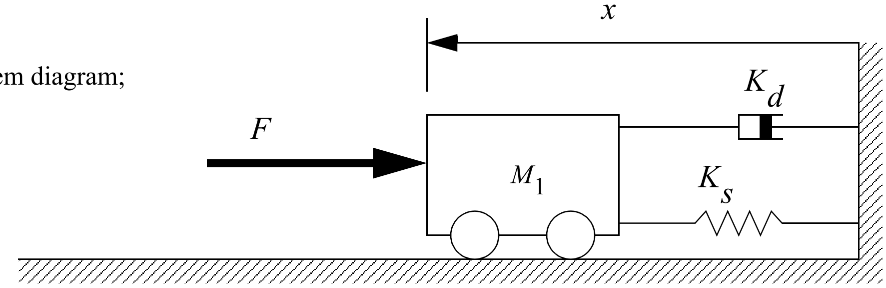
$$\therefore N + (N) + (N) + (-N) = N$$

The units match, so there are no obvious problems.

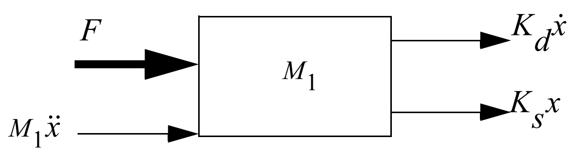
coefficient	units
F	$N = \frac{Kgm}{s^2}$
K_s	$\frac{N}{m}$
K_d	$\frac{Ns}{m}$
M	Kg

Consider the cart in Figure 1.21. On the left is a force that is opposed by a spring and damper on the right. The basic problem definition already contains all of the needed definitions, so no others are required. The FBD for the mass shows the applied force and the reaction forces from the spring and damper. When the forces are summed the inertia is on the right side of the equation in Newton's form. This equation is then rearranged to a second-order non-homogeneous differential equation.

Given the system diagram;



The FBD for the cart is



The forces for the cart are in a single direction and can be summed as,

$$\overset{+}{\leftarrow} \sum F_x = -F - K_d\dot{x} - K_s x = M_1\ddot{x}$$

This equation can be rearranged to a second-order non-homogeneous diff. eqn.

$$\ddot{x} + \frac{K_d}{M_1}\dot{x} + \frac{K_s}{M_1}x = \frac{-F}{M_1}$$

Aside: later on we will solve the differential equations, or use other methods to determine how the system will behave. It is useful to have all of the 'output' variables for the system on the left hand side, and everything else on the other.

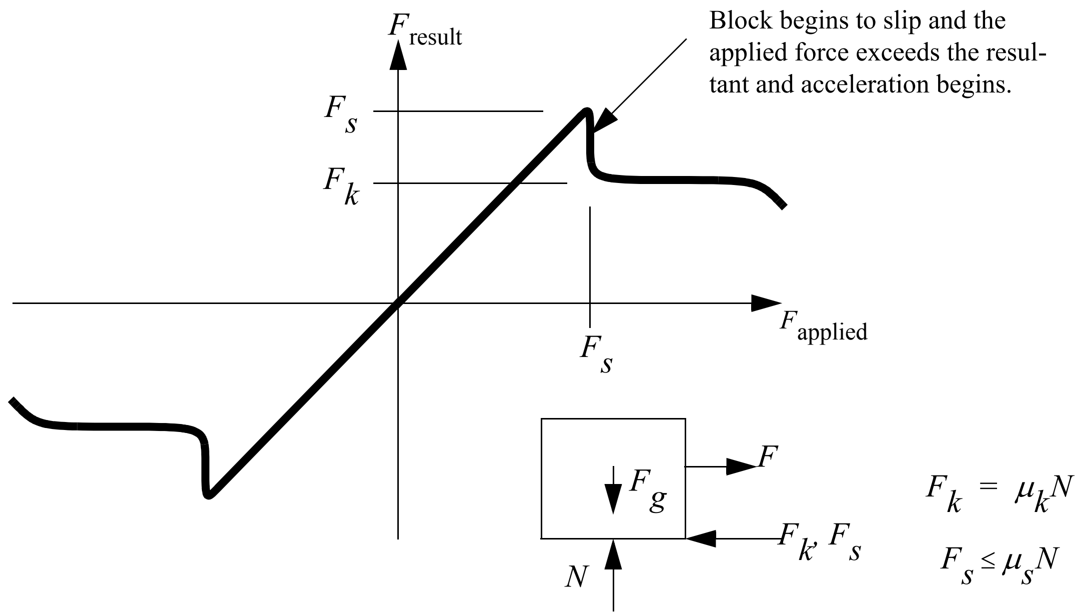
Figure 1.21 Example: A simple translational system

The example in Figure 1.13 shows two masses separated by a spring. In the first example the spring is assumed to be in tension. When x1 becomes positive it will put the spring in compression, so it is made negative, however a positive x2 will put the spring in tension, so it remains positive. In the second case the spring is assumed to be in compression and this time x2 will put it in tension so it is made negative.

Friction

Viscous friction was discussed before, where a lubricant would provide a damping effect between two moving objects. In cases where there is no lubricant, and the touching surfaces are dry, dry coulomb friction may result. In this case the surfaces will stick in place until a maximum force is overcome. After that, the object will begin to slide and a constant friction force will result.

Figure 1.22 shows the classic model for (dry Coulomb) friction. The force on the horizontal axis is the force applied to the friction surfaces while the vertical axis is the resulting friction force. Beneath the slip force the object will stay in place. When the slip force is exceeded the object will begin to move, and the resulting kinetic friction force will be relatively constant. (Note: If the object begins to travel much faster then the kinetic friction force will decrease.) It is common to forget that friction forces are bidirectional, but it always opposes the applied force or motion. The friction force is a function of the coefficient of friction and the normal force across the contact surfaces. The coefficient of friction is a function of the materials, surface texture and surface shape.



Note: When solving problems with friction remember that the friction force will always equal the applied force (not the maximum force) until slip occurs. After that the friction is approximately constant. In addition, the friction forces direction opposes applied forces, and motion.

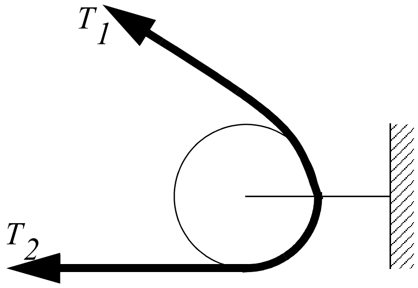
Figure 1.22 Dry friction

Many systems use kinetic friction to dissipate energy from a system as heat, sound and vibration.

Cables And Pulleys

Cables are useful when transmitting tensile forces or displacements. The centerline of the cable becomes the centerline for the force. And, if the force becomes compressive, the cable becomes limp, and will not transmit force. A cable by itself can be represented as a force vector. When used in combination with pulleys, a cable can redirect a force vector or multiply a force.

Typically we assume that a pulley is massless and frictionless (in the rotation chapter we will assume they are not). If this is the case then the tension in the cable on both sides of the pulley are equal, as shown in Figure 1.23.

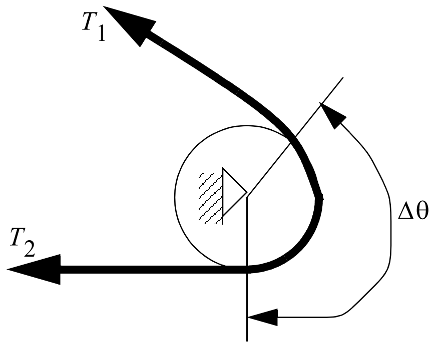


For a massless frictionless pulley

$$T_1 = T_2$$

Figure 1.23 Tension in a cable over a massless frictionless pulley

If we have a pulley that is fixed and cannot rotate, the cable must slide over the surface of the pulley. In this case we can use the coefficient of friction to determine the relative ratio of forces between the sides of the pulley, as shown in Figure 1.24.



Moving toward T1

$$\frac{T_1}{T_2} = e^{\mu_k(\Delta\theta)}$$

Moving toward T2

$$\frac{T_2}{T_1} = e^{\mu_k(\Delta\theta)}$$

Figure 1.24 Friction of a belt over a fixed drum if the belt is moving towards T_1

Although the discussion in this section has focused on cables and pulleys, the theory also applies to belts over drums.

1.2 Systems Examples

A simplified model of an elevator (M1) and a passenger (M2) are shown in Figure 1.25. In this example many of the required variables need to be defined. These are added to the FBDs. Care is also taken to ensure that all forces between bodies are equal in magnitude, but opposite in direction. The wall forces are ignored because they are statically indeterminate, and x-axis force components are irrelevant to the forces in the y-axis.

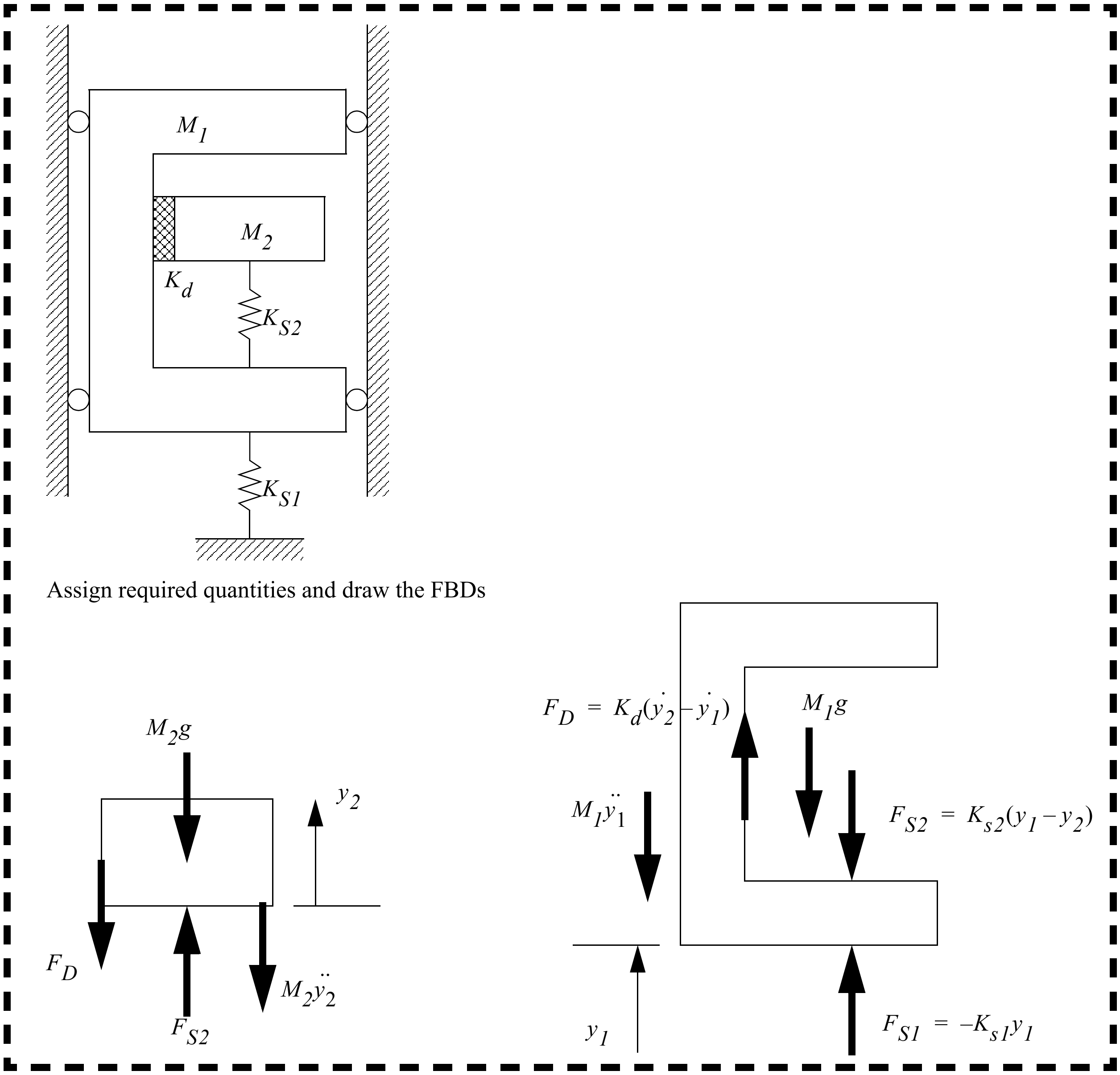


Figure 1.25 Example: A multi-body translating system (an elevator with a passenger)

The forces on the FBDs are summed and the equations are expanded in Figure 1.26.

Now, sum the forces in vector form, and substitute relationships,

$$\sum F_{M_1} = F_{S1} + F_{S2} + M_1 g + F_D = M_1 a_1$$
$$\sum F_{M_2} = -F_{S2} + M_2 g - F_D = M_2 a_2$$

At this point the equations are expanded

$$K_{S1}(-y_1) + K_{S2}(y_2 - y_1) + M_1(-9.81) + K_d(\dot{y}_2 - \dot{y}_1) = M_1 \ddot{y}_1$$
$$-K_{S2}(y_2 - y_1) + M_2(-9.81) - K_d(\dot{y}_2 - \dot{y}_1) = M_2 \ddot{y}_2$$

Figure 1.26 Example: Equations for the elevator

1.3 Other Topics

Designing a system in terms of energy content can allow insights not easily obtained by the methods already discussed. Consider the equations in Figure 1.27. These equations show that the total energy in the system is the sum of kinetic and potential energy, equation 1.15. Kinetic energy is half the product of mass times velocity squared, equation 1.16. Potential energy in translating systems is a force magnitude multiplied by a distance (that force was applied over), equation 1.17. In addition, the power, or energy transfer rate is the force applied multiplied by the velocity, equation 1.18.

$$E = E_P + E_K \qquad \text{eqn 1.15}$$

$$E_K = \frac{Mv^2}{2} \qquad \text{eqn 1.16}$$

$$E_P = Fd = Mgd \qquad \text{eqn 1.17}$$

$$P = Fv = \frac{d}{dt}E \qquad \text{eqn 1.18}$$

Figure 1.27 Energy and power equations for translating masses

1.1 Summary

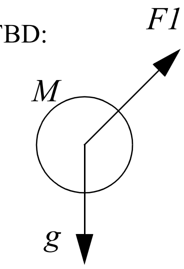
- FBDs are useful for reducing complex systems to simpler parts.
- Equations for translation and rotation can be written for FBDs.
- The equations can be integrated for dynamic cases, or solved algebraically for static cases.

1.2 Problems with Solutions

Problem 1.1 Find the acceleration for the given mass and force.,

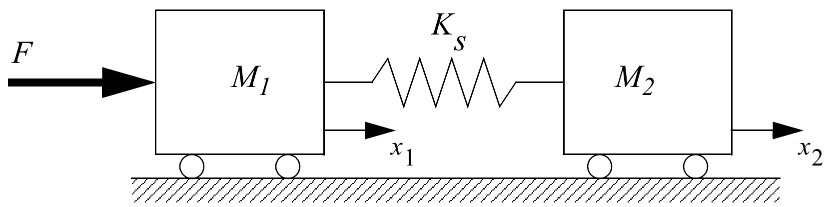
$$F_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} N \qquad g = \begin{bmatrix} 0 \\ -9.81 \\ 0 \end{bmatrix} \frac{N}{Kg} \qquad M = 2Kg$$

FBD:



Problem 1.2 If a spring has a deflection of 6.00 cm for a static load of 200N, what is the spring constant?

Problem 1.3 Draw the FBDs and the equations of motion for the masses.



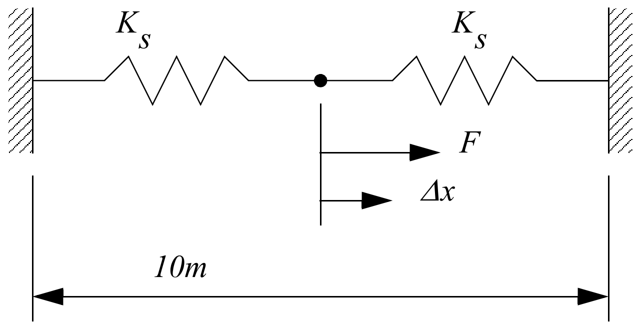
Problem 1.4 Find F assuming the springs are normally 4m long when unloaded.

Given,

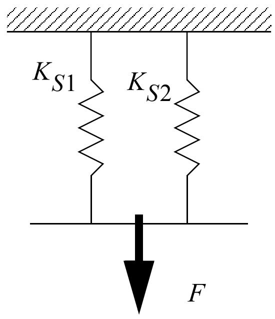
$$K_s = 10\frac{N}{m}$$

$$\Delta x = 0.1m$$

Aside: It can help to draw a FBD of the node between the springs..

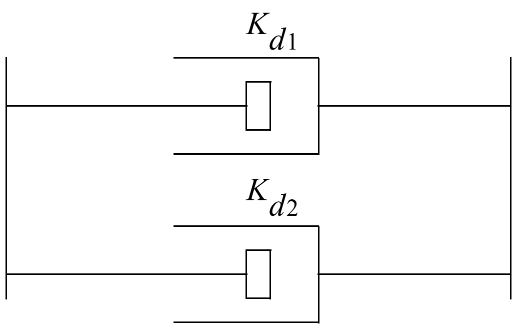


Problem 1.5 Find an equivalent spring coefficient to replace the two springs in parallel

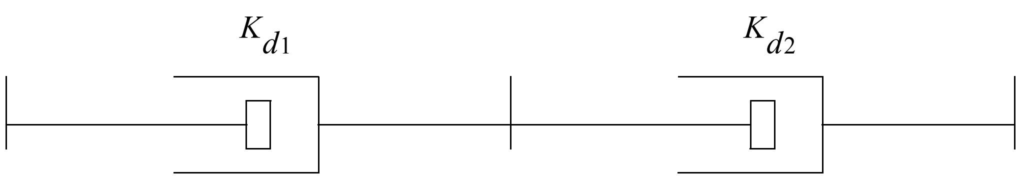


Problem 1.6 Derive the effective damper coefficients for the pairs below from basic principles,

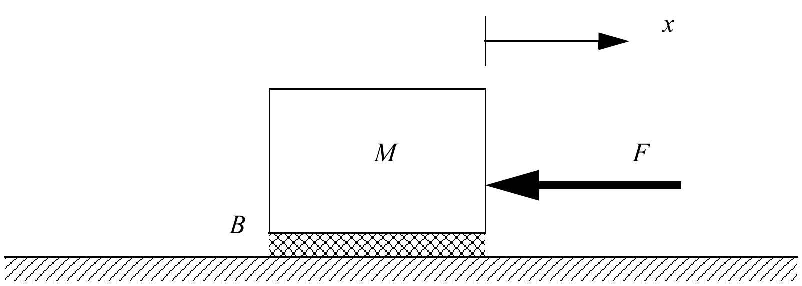
a)



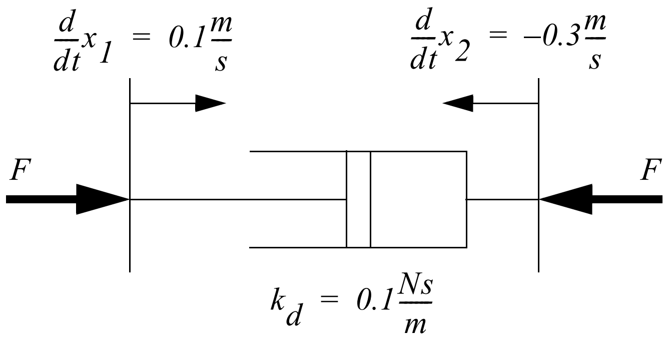
b)



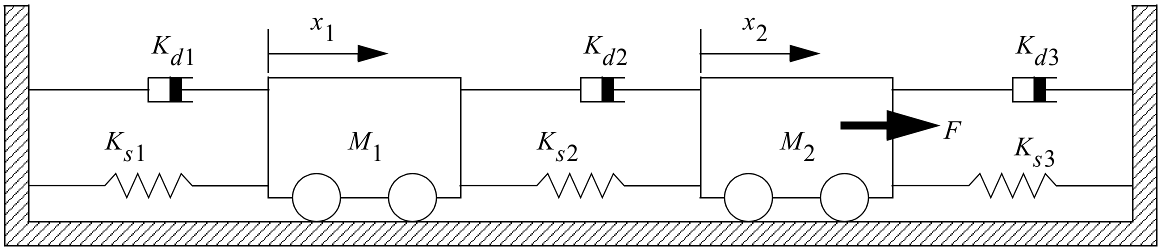
Problem 1.7 Write a differential equation for the mass pictured below.



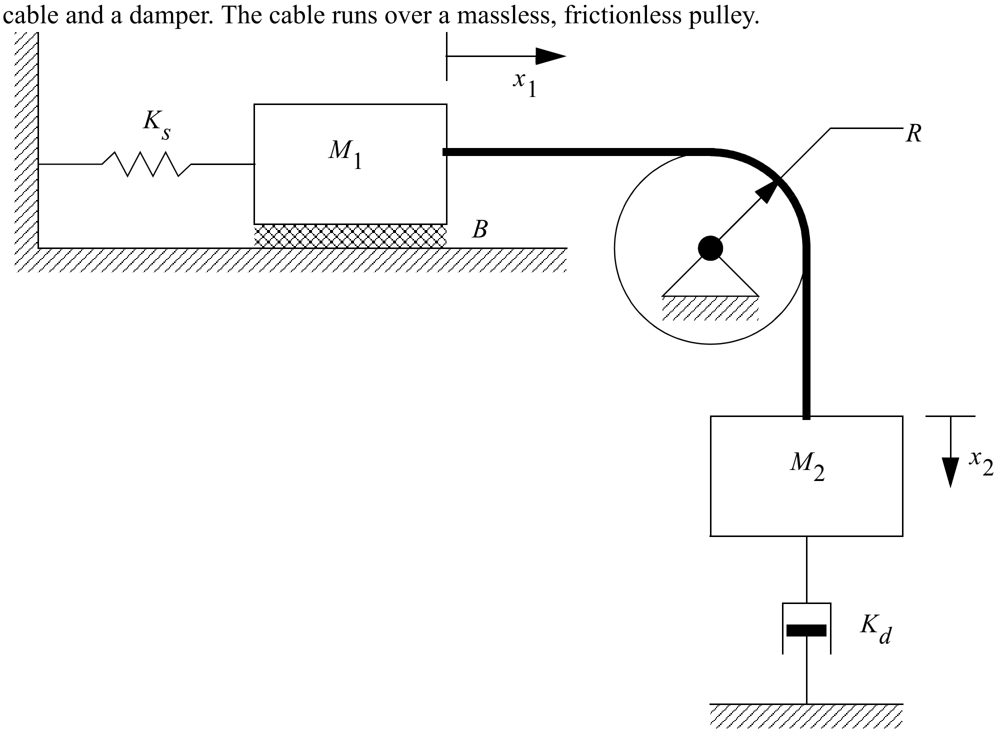
Problem 1.8 The force is acting on the cylinder, resulting in the velocities given below. What is the applied force?



Problem 1.9 Write the differential equations for the system below.



Problem 1.10 Write the differential equations for the system below. In this system the upper mass, M1, is between a spring and a cable and there is viscous damping between the mass and the floor. The suspended mass, M2, is between the cable and a damper. The cable runs over a massless, frictionless pulley.



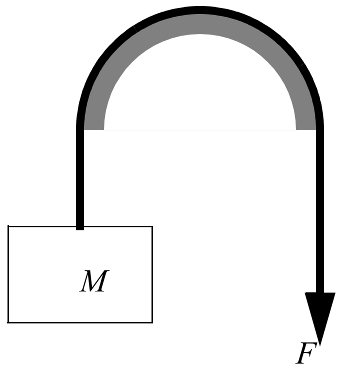
Problem 1.11 *Find F to start the mass moving up, and then the force required to maintain a low velocity downwards motion of*

the mass.

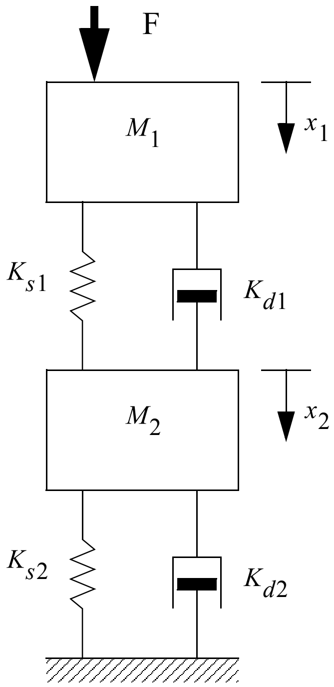
Given,

$$\mu_s = 0.35 \qquad \mu_k = 0.2$$

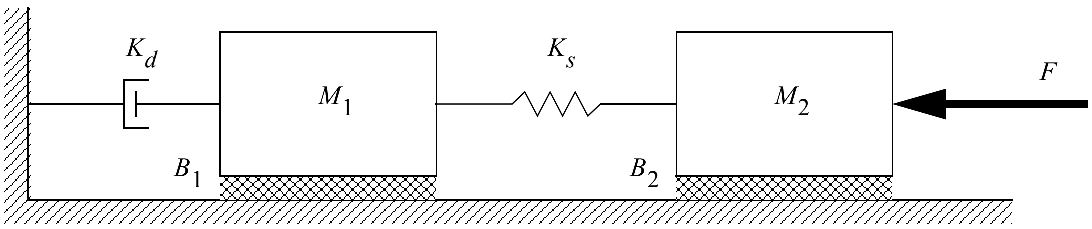
$$M = 1Kg$$



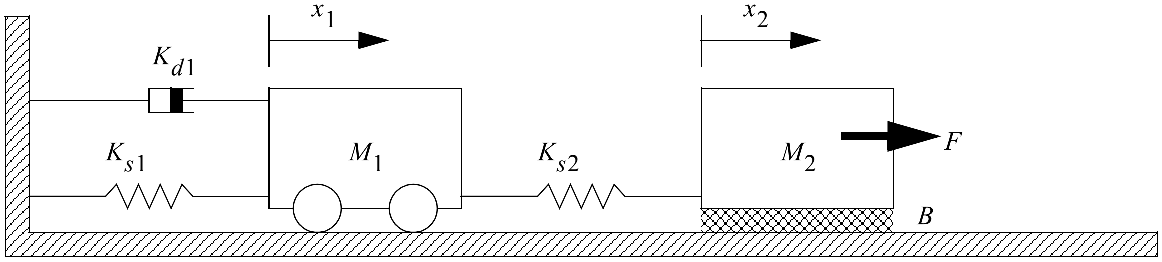
Problem 1.12 Write the differential equations for the system below.



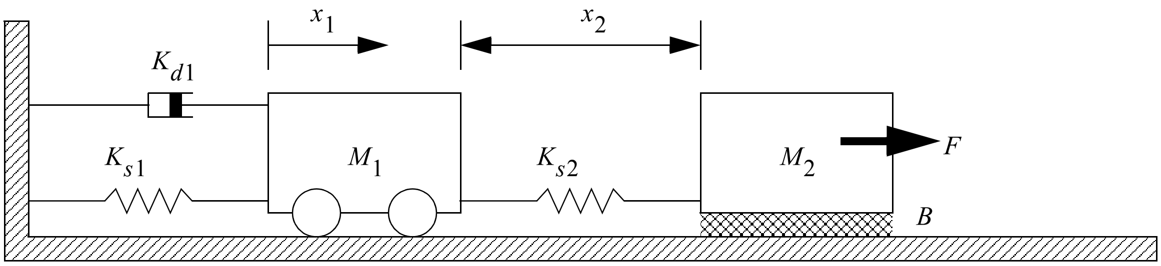
Problem 1.13 Write the differential equations for the system given below.



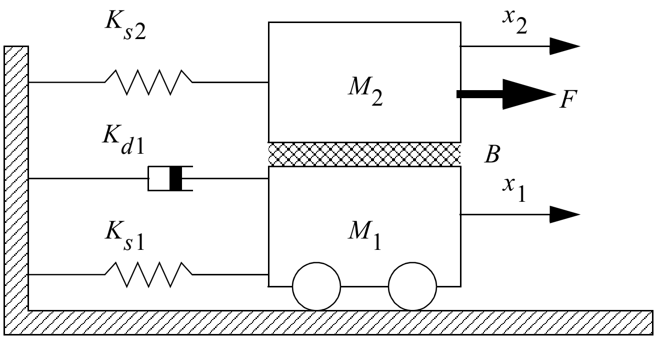
Problem 1.14 Write the differential equations for the system below.



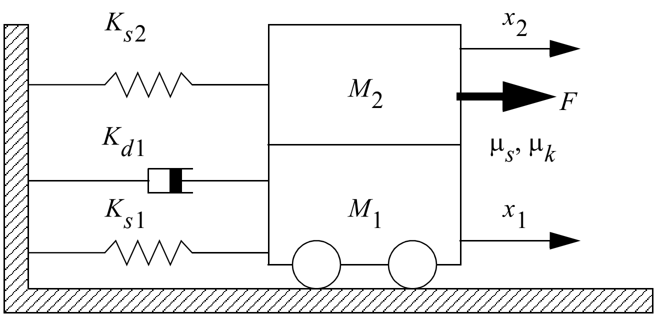
Problem 1.15 Write the differential equations for the system below. (Note: This problem differs because the x2 dimension is relative, not absolute.)



Problem 1.16 Write the differential equations for the system below.

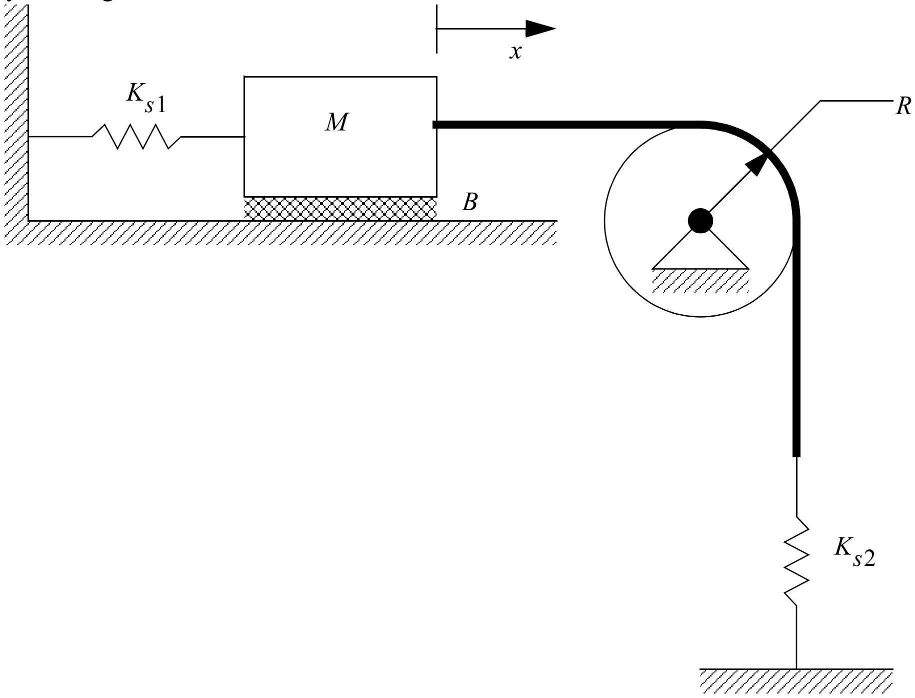


Problem 1.17 Write the differential equations for the system below.

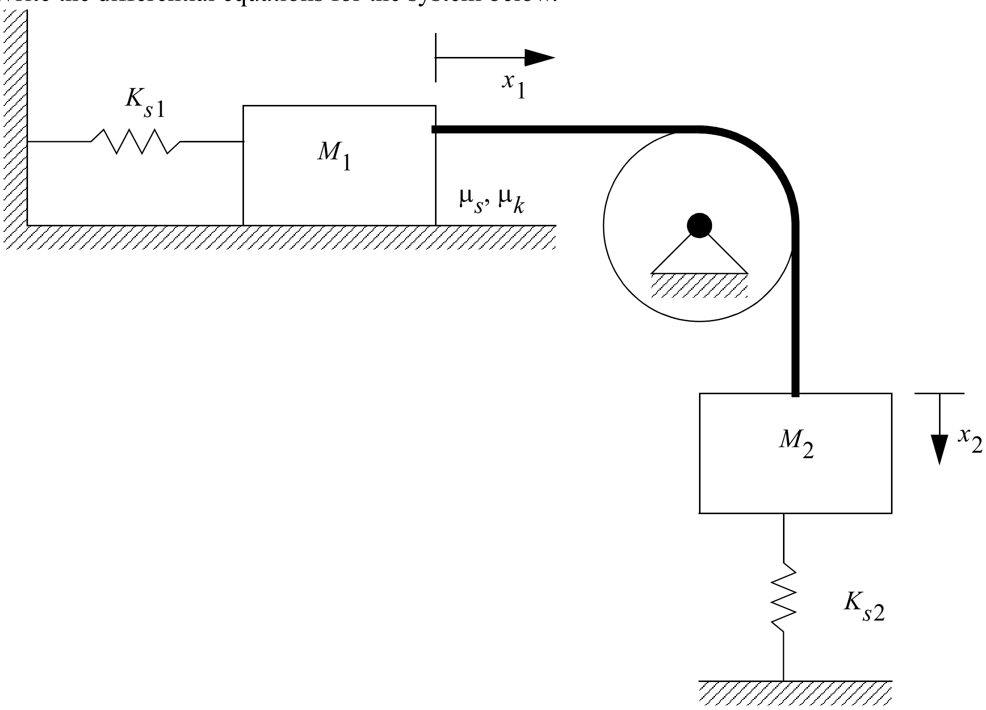


Problem 1.18 Write the differential equations for the system below. Assume that the pulley is massless and frictionless and that

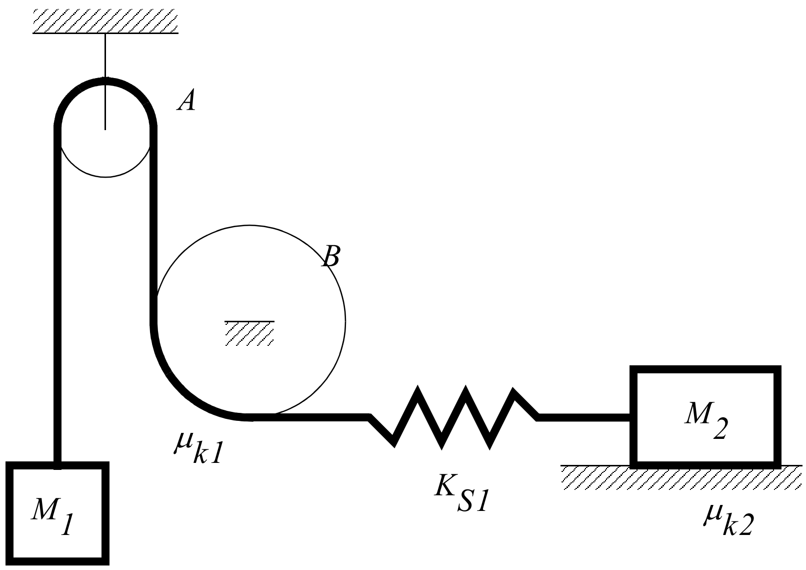
the system begins undeflected.



Problem 1.19 Write the differential equations for the system below.



Problem 1.20 Write the differential equations for the system below. Assume it is in motion.



1.3 Problem Solutions

Answer 1.1

$$a = \begin{bmatrix} 1.5 \\ -7.81 \\ 0 \end{bmatrix} \frac{m}{s^2}$$

Answer 1.2 Ks = 33.3 N/cm = 3330 N/m

Answer 1.3

$$\ddot{x}_1(M_1) + x_1(K_s) + x_2(-K_s) = F$$

$$x_1(K_s) + \ddot{x}_2(-M_2) + x_2(-K_s) = 0$$

Answer 1.4 F = 2N

Answer 1.5 Consider that when an object has no mass, the force applied to one side of the spring will also be applied to the other. The only factor that changes is displacement.

$$F = (K_{S1} + K_{S2})y$$

Answer 1.6

a) $$K_{eq} = K_{d1} + K_{d2}$$

b) $$K_{eq} = \frac{K_{d1}K_{d2}}{K_{d1} + K_{d2}}$$

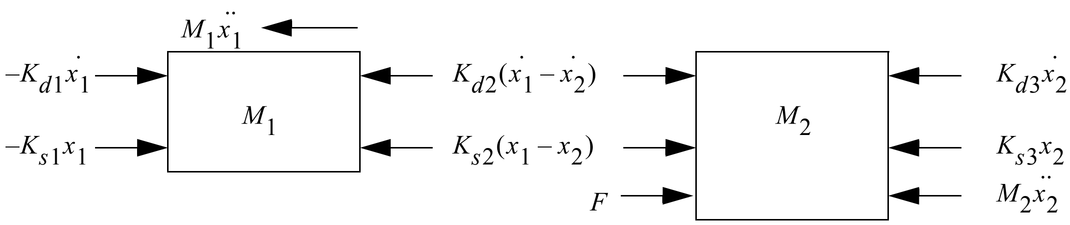
Answer 1.7

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) = -\frac{F}{M}$$

Answer 1.8 F = -0.02N

Answer 1.9

FBDs:



For M1:

$$\xrightarrow{+}\ \sum F = -K_{d1}\dot{x}_1 - K_{s1}x_1 - K_{d2}(\dot{x}_1-\dot{x}_2) - K_{s2}(x_1-x_2) = M_1\ddot{x}_1$$

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1}+K_{d2}) + x_1(K_{s1}+K_{s2}) + \dot{x}_2(-K_{d2}) + x_2(-K_{s2}) = 0$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}+K_{d2}}{M_1}\right) + x_1\left(\frac{K_{s1}+K_{s2}}{M_1}\right) + \dot{x}_2\left(\frac{-K_{d2}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

For M2:

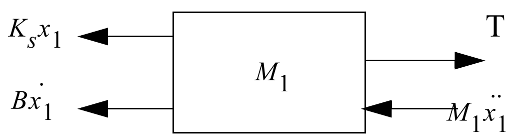
$$\xrightarrow{+}\ \sum F = K_{d2}(\dot{x}_1-\dot{x}_2) + K_{s2}(x_1-x_2) + F - K_{d3}\dot{x}_2 - K_{s3}x_2 = M_2\dot{x}_2$$

$$\ddot{x}_2(M_2) + \dot{x}_2(K_{d2}+K_{d3}) + x_2(K_{s2}+K_{s3}) + \dot{x}_1(-K_{d2}) + x_1(-K_{s2}) = F$$

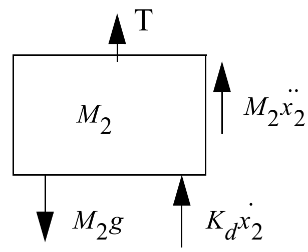
$$\ddot{x}_2 + \dot{x}_2\left(\frac{K_{d2}+K_{d3}}{M_2}\right) + x_2\left(\frac{K_{s2}+K_{s3}}{M_2}\right) + \dot{x}_1\left(\frac{-K_{d2}}{M_2}\right) + x_1\left(\frac{-K_{s2}}{M_2}\right) = \frac{F}{M_2}$$

Answer 1.10

FBD M1:



FBD M2:



$$\xrightarrow{+}\ \sum F = -K_sx_1 - B\dot{x}_1 + T = M_1\ddot{x}_1$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{B}{M_1}\right) + x_1\left(\frac{K_s}{M_1}\right) = \frac{T}{M_1}$$

$$+\uparrow\ \sum F = T + K_d\dot{x}_2 - M_2g = -M_2\ddot{x}_2$$

$$\ddot{x}_2(-M_2) + \dot{x}_2(-K_d) = T - M_2g$$

$$\ddot{x}_1(M_1) + \dot{x}_1(B) + x_1(K_s) = T \qquad \ddot{x}_2 + \dot{x}_2\left(\frac{K_d}{M_2}\right) = \frac{T-M_2g}{-M_2}$$

For T: if T <= 0 then T = 0N
if T > 0 x1 = x2

Answer 1.11

$$F_{up} = 9.81e^{0.35(\pi)}N$$

$$F_{down} = \frac{9.81N}{e^{0.2(\pi)}}$$

Answer 1.12

(assuming no gravity

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1}) + x_1(K_{s1}) + \dot{x}_2(-K_{d1}) + x_2(-K_{s1}) = F$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + \dot{x}_2\left(\frac{-K_{d1}}{M_1}\right) + x_2\left(\frac{-K_{s1}}{M_1}\right) = \frac{F}{M_1}$$

$$\ddot{x}_2(M_2) + \dot{x}_2(K_{d1} + K_{d2}) + x_2(K_{s1} + K_{s2}) + \dot{x}_1(-K_{d1}) + x_1(-K_{s1}) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{K_{d1} + K_{d2}}{M_2}\right) + x_2\left(\frac{K_{s1} + K_{s2}}{M_2}\right) + \dot{x}_1\left(\frac{-K_{d1}}{M_2}\right) + x_1\left(\frac{-K_{s1}}{M_2}\right) = 0$$

Answer 1.13

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_d + B_1}{M_1}\right) + x_1\left(\frac{K_s}{M_1}\right) + x_2\left(\frac{-K_s}{M_1}\right) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B_2}{M_2}\right) + x_2\left(\frac{K_s}{M_2}\right) + x_1\left(\frac{-K_s}{M_2}\right) = \frac{-F}{M_2}$$

Answer 1.14

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1} + K_{s2}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_{s2}}{M_2}\right) + x_1\left(\frac{-K_{s2}}{M_2}\right) = \frac{F}{M_2}$$

Answer 1.15

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{B}{M_2}\right) + \ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_s}{M_2}\right) = \frac{F}{M_2}$$

Answer 1.16

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1} + B}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + \dot{x}_2\left(\frac{-B}{M_1}\right) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_{s2}}{M_2}\right) + \dot{x}_1\left(\frac{-B}{M_2}\right) = \frac{F}{M_2}$$

Answer 1.17

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1}) + x_1(K_{s1}) = F_F$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) = \frac{F_F}{M_1}$$

$$\ddot{x}_2(M_2) + x_2(K_{s2}) = F - F_F$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) = \frac{F - F_F}{M_2}$$

where,

$$|F_F| \le \mu_s M_2 g \qquad \text{if} \qquad \dot{x}_1 = \dot{x}_2$$

$$F_F = \mu_k M_2 g\left(\frac{\dot{x}_1 - \dot{x}_2}{|\dot{x}_1 - \dot{x}_2|}\right) \qquad \text{if} \qquad \dot{x}_1 \ne \dot{x}_2$$

Answer 1.18

$$if(x \ge 0) \qquad T = 0$$

$$if(x < 0) \qquad T = -K_{s2}x$$

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) + x\left(\frac{K_{s1}}{M}\right) = \frac{T}{M}$$

Answer 1.19

$$if((x_2 - x_1) < 0) \qquad T = 0$$

$$if(T > 0) \qquad x_1 = x_2$$

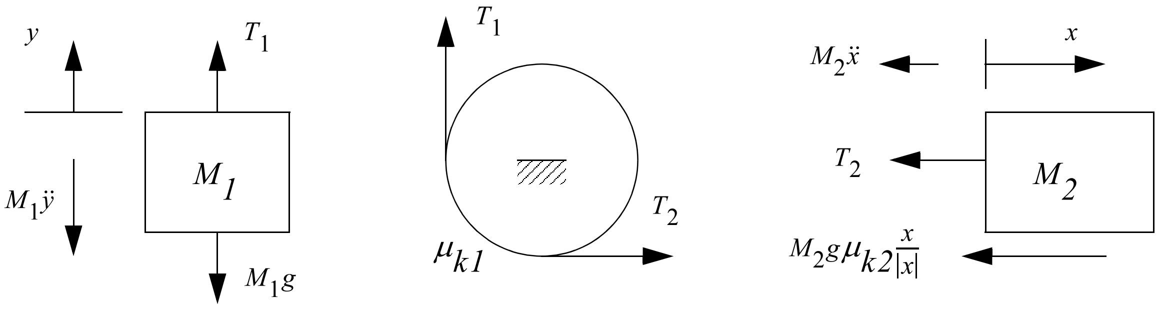
$$if(\dot{x}_1 = 0) \qquad |F_F| \le M_1 g \mu_s$$

$$if(\dot{x}_1 \ne 0) \qquad F_F = M_1 g \mu_k \frac{\dot{x}_1}{|\dot{x}_1|}$$

$$\ddot{x}_1 + x_1\left(\frac{K_{s1}}{M_1}\right) = \frac{T - F_F}{M_1}$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) = \frac{-T}{M_2} + g$$

Answer 1.20



$$\sum F_{M1} = T_1 - M_1 g - M_1 \ddot{y} = 0$$

$$\boxed{M_1 \ddot{y} = T_1 - M_1 g}$$

$$\sum F_{M2} = -M_2 \ddot{x} - T_2 - M_2 g \mu_{k2} \frac{x}{|x|} = 0$$

$$\boxed{M_2 \ddot{x} = -T_2 - M_2 g \mu_{k2} \frac{x}{|x|}}$$

$if((x - y) > 0)$

$\quad T_2 = K_{S1}(x - y)$

$\quad if(\dot{x} > 0)$

$$\frac{T_2}{T_1} = e^{\frac{\pi}{2}\mu_{k1}}$$

$\quad else$

$$\frac{T_1}{T_2} = e^{\frac{\pi}{2}\mu_{k1}}$$

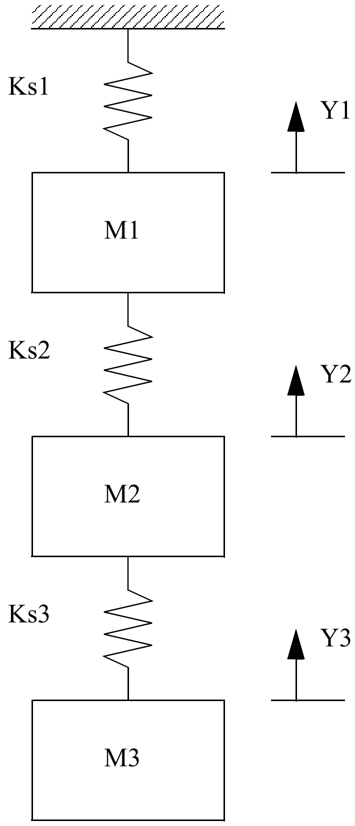
$else$

$\quad T_1 = 0$

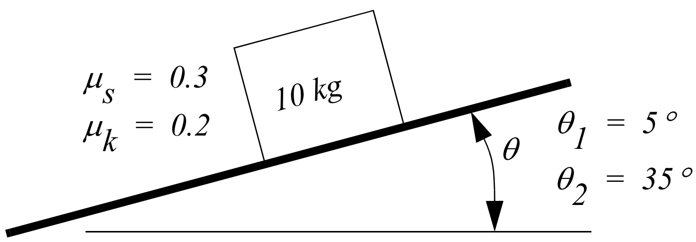
$\quad T_2 = 0$

1.4 Problems Without Solutions

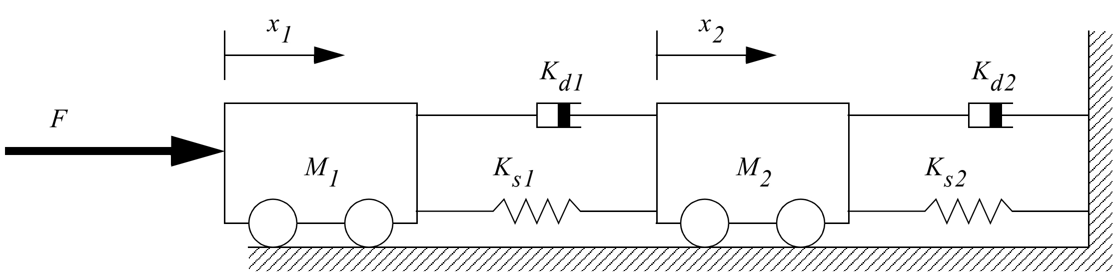
Problem 1.21 Draw free body diagrams for the following system, and then develop differential equations.



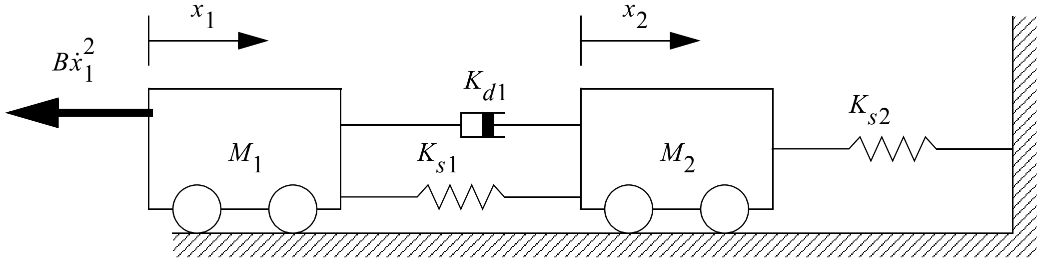
Problem 1.22 Find the acceleration of the block for both angles indicated.



Problem 1.23 Develop the equation relating the input force to the motion (in terms of *x*) of the left hand cart for the problem below.



Problem 1.24 Write the differential equations for the systems below.



2. Analysis of Differential Equations

Topic 2.1 *First and second-order homogeneous differential equations.*
Topic 2.2 *Non-homogeneous differential equations.*
Topic 2.3 *First and second-order responses.*
Topic 2.4 *Non-linear system elements.*
Topic 2.5 *Design case.*
Objective 2.1 *To develop explicit equations that describe a system response.*
Objective 2.2 *To recognize first and second-order equation forms.*

In the previous chapter we derived differential equations of motion for translating systems. These equations can be used to analyze the behavior of the system and make design decisions. The most basic method is to select a standard input type (a forcing function) and initial conditions, and then solve the differential equation. It is also possible to estimate the system response without solving the differential equation as will be discussed later.

Figure 2.1 shows an abstract description of a system. The basic concept is that the system changes the inputs to outputs. Say, for example, that the system to be analyzed is an elevator. Inputs to the system would be the mass of human riders and desired elevator height. The output response of the system would be the actual height of the elevator. For analysis, the system model could be developed using differential equations for the motor, elastic lift cable, mass of the car, etc. A basic test would involve assuming that the elevator starts at the ground floor and must travel to the top floor. Using assumed initial values and input functions the differential equation could be solved to get an explicit equation for elevator height. This output response can then be used as a guide to modify design choices (parameters). In practice, many of the assumptions and tests are mandated by law or by groups such as Underwriters Laboratories (UL), Canadian Standards Association (CSA) and the European Commission (CE).

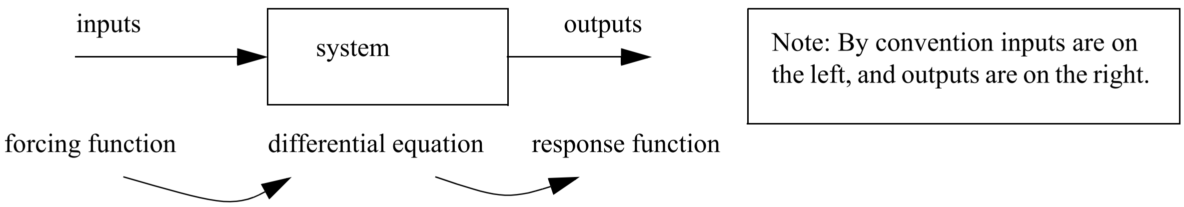


Figure 2.1 A system with an input and output response

There are several standard input types used to test a system. These are listed below in order of relative popularity with brief explanations.

- Step - a sudden change of input, from off to on and on to off, such as very rapidly changing a desired speed from 0Hz to 50Hz. These may repeat.
- Ramp - a continuously increasing input, such as a motor speed that increases constantly at 10Hz per minute.
- Sinusoidal - a cyclic input that varies continuously, such as wave height that is continually oscillating at 1Hz.
- Parabolic - an exponentially increasing input, such as a motor speed that is 2Hz at 1 second, 4rad/s at 2 seconds, 8rad/s at 3 seconds, etc.

After the system has been modeled, an input type has been chosen, and the initial conditions have been selected, the system can be analyzed to determine its behavior. The most fundamental technique is to integrate the differential equation(s) for the system.

2.1 Explicit Solutions

Solving a differential equation with initial conditions will result in an explicit solution. This equation provides the general response as a function of time, but it can also be used to find frequencies and other characteristics of interest. This section will review techniques used to integrate first and second-order homogeneous differential equations. These equations correspond to systems without inputs, also called unforced systems. Non-homogeneous differential equations will also be reviewed.

The basic types of differential equations are shown in Figure 2.2. Each of these equations is linear. On the left hand side is the integration variable ‘x’. If the right hand side is zero, then the equation is homogeneous. Each of these equations is linear because each of the terms on the left hand side is simply multiplied by a linear coefficient.

$A\dot{x} + Bx = 0$	first-order homogeneous
$A\dot{x} + Bx = Cf(t)$	first-order non-homogeneous
$A\ddot{x} + B\dot{x} + Cx = 0$	second-order homogeneous
$A\ddot{x} + B\dot{x} + Cx = Df(t)$	second-order non-homogeneous
$A\dddot{x} + B\ddot{x} + Cx = 0$	third-order homogeneous
$A\dddot{x} + B\dot{x} + Cx = Df(t)$	third-order non-homogeneous

Figure 2.2 Standard differential equation forms

A general solution for a first-order homogeneous differential equation is given in Figure 2.3. The solution begins with the solution of the homogeneous equation where a general form is ‘guessed’. Substitution leads to finding the value of the coefficient ‘Y’. Following this, the initial conditions for the equation are used to find the value of the coefficient ‘X’. Notice that the final equation will begin at the initial displacement, but approach zero as time goes to infinity. The e-to-the-x behavior is characteristic for a first-order response.

Given the general form of a first-order homogeneous equation,

$$A\dot{x} + Bx = 0 \qquad \textit{and} \qquad x(0) = x_0$$

initial condition

Guess a solution form and solve.

$$x = Xe^{-Yt} \qquad \dot{x} = -YXe^{-Yt}$$

$$A(-YXe^{-Yt}) + B(Xe^{-Yt}) = 0$$

$$A(-Y) + B = 0$$

$$Y = \frac{B}{A}$$

Therefore the general form is,

$$x_h = Xe^{-\frac{B}{A}t}$$

Note: The general form below is useful for finding almost all homogeneous equations

$$x_h(t) = Xe^{-Yt}$$

Therefore the final equation is,

$$x_h = Xe^{-\frac{B}{A}t}$$

$$x_0 = Xe^{-\frac{B}{A}0}$$

$$x_0 = X$$

Next, use the initial conditions to find the remaining unknowns.

$$x(t) = x_0 e^{-\frac{B}{A}t}$$

Figure 2.3 Example: General solution of a first-order homogeneous equation

The general solution to a second-order homogeneous equation is shown in Figure 2.4. The solution begins with a guess of the homogeneous solution, and the solution of a quadratic equation. There are three possible cases that result from the solution of the quadratic equation: different but real roots; two identical real roots; or two complex roots. The three cases result in three different forms of solutions, as shown. The complex result is the most notable because it results in sinusoidal oscillations. It is not shown, but after the homogeneous solution has been found, the initial conditions need to be used to find the remaining coefficient values.

As mentioned above, a complex solution when solving the homogeneous equation results in a sinusoidal oscillation, as proven in Figure 2.5. The most notable part of the solution is that there is both a frequency of oscillation and a phase shift. This form is very useful for analyzing the frequency response of a system, as will be seen in a later chapter.

Given,

$$A\ddot{x} + B\dot{x} + Cx = 0 \qquad x(0) = x_0 \qquad and \qquad \dot{x}(0) = v_0$$

Guess a general equation form and substitute it into the differential equation,

$$x_h = Xe^{Yt} \qquad \dot{x}_h = YXe^{Yt} \qquad \ddot{x}_h = Y^2Xe^{Yt}$$

$$A(Y^2Xe^{Yt}) + B(YXe^{Yt}) + C(Xe^{Yt}) = 0$$

$$A(Y^2) + B(Y) + C = 0$$

$$Y = \frac{-B \pm \sqrt{(B)^2 - 4(AC)}}{2A} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note: There are three possible outcomes of finding the roots of the equations: two different real roots, two identical real roots, or two complex roots. Therefore there are three fundamentally different results.

If the values for Y are both real, but different, the general form is,

$$Y = R_1, R_2 \qquad x_h = X_1e^{R_1t} + X_2e^{R_2t}$$

Note: The initial conditions are then used to find the values for X_1 and X_2.

If the value are identical,

$$Y = R_1, R_1 \qquad x_h = X_1e^{R_1t} + X_2te^{R_1t}$$

the initial conditions are then used to find the values for X_1 and X_2.

If the values for Y are complex, the general form is,

$$Y = \sigma \pm \omega j \qquad x_h = X_3e^{\sigma t}\cos(\omega t + X_4)$$

the initial conditions are then used to find the values of X_3 and X_4.

Note: It is most likely that when you saw differential equations in your applied math class the left hand form (with sin AND cos) was used. However this is not useful for engineering analysis. The right hand form is equivalent (see later in the chapter) and will be the only form acceptable for a final answer.

$$x_h = A\cos\omega t + B\sin\omega t = X_3e^{\sigma t}\cos(\omega t + X_4)$$

Figure 2.4 Example: Solution of a second-order homogeneous equation

Consider the situation where the results of a homogeneous solution are the complex conjugate pair.

$$Y = R \pm Cj$$

This gives the general result, as shown below:

$$x = X_1 e^{(R + Cj)t} + X_2 e^{(R - Cj)t}$$

$$x = X_1 e^{Rt} e^{Cjt} + X_2 e^{Rt} e^{-Cjt}$$

$$x = e^{Rt}\left(X_1 e^{Cjt} + X_2 e^{-Cjt}\right)$$

$$x = e^{Rt}(X_1(cos(Ct) + j sin(Ct)) + X_2(cos(-Ct) + j sin(-Ct)))$$

$$x = e^{Rt}(X_1(cos(Ct) + j sin(Ct)) + X_2(cos((Ct) - j sin(Ct))))$$

$$x = e^{Rt}((X_1 + X_2) cos(Ct) + j(X_1 - X_2) sin(Ct))$$

$$x = e^{Rt}((X_1 + X_2) cos(Ct) + j(X_1 - X_2) sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}((X_1 + X_2) cos(Ct) + j(X_1 - X_2) sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - \left(X_1^2 + -2X_1X_2 + X_2^2\right)}}{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - \left(X_1^2 + -2X_1X_2 + X_2^2\right)}}((X_1 + X_2) cos(Ct) + j(X_1 - X_2) sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{4X_1X_2}}{\sqrt{4X_1X_2}}((X_1 + X_2) cos(Ct) + j(X_1 - X_2) sin(Ct))$$

$$x = e^{Rt}\sqrt{4X_1X_2}\left(\frac{(X_1 + X_2)}{\sqrt{4X_1X_2}} cos(Ct) + j\frac{(X_1 - X_2)}{\sqrt{4X_1X_2}} sin(Ct)\right)$$

$$x = e^{Rt}\sqrt{4X_1X_2}\, cos\left(Ct + atan\left(\frac{(X_1 + X_2)}{(X_1 - X_2)}\right)\right)$$

$$x = e^{Rt} X_3 cos(Ct + X_4) \qquad \text{where,} \qquad X_3 = \sqrt{4X_1X_2}$$

frequency ↗ ↖ phase shift

$$X_4 = atan\left(\frac{(X_1 + X_2)}{(X_1 - X_2)}\right)$$

Figure 2.5 Example: Phase shift solution for a second-order homogeneous differential equation

Note: Occasionally a problem solution might consist of both a sine and cosine term with the same frequency. These should normally be combined to a single term with a phase shift as shown below.

Recall the double angle formula,

$$\sin(\omega t + \theta) = \sin\omega t\cos\theta + \sin\theta\cos\omega t$$

This can be written in a more common form,

$$A(\sin\omega t\cos\theta + \sin\theta\cos\omega t) = A\sin(\omega t + \theta)$$

$$A\cos\theta\sin\omega t + A\sin\theta\cos\omega t = A\sin(\omega t + \theta)$$

$$X\sin\omega t + Y\cos\omega t = A\sin(\omega t + \theta)$$

where,

$$X = A\cos\theta$$
$$Y = A\sin\theta$$

$$A = \frac{X}{\cos\theta} = \frac{Y}{\sin\theta}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{Y}{X}$$

$$\boxed{\theta = \operatorname{atan}\left(\frac{Y}{X}\right)}$$

$$\boxed{A = \sqrt{X^2 + Y^2}}$$

Consider the example,

$$3\sin 5t + 4\cos 5t = \sqrt{3^2 + 4^2}\sin\left(5t + \operatorname{atan}\left(\frac{4}{3}\right)\right) = 5\sin(5t + 0.9273)$$

Figure 2.6 Example: Phase shift solution form

The methods for solving non-homogeneous differential equations builds upon the methods used for the solution of homogeneous equations. This process adds a step to find the particular solution of the equation. An example of the solution of a first-order non-homogeneous equation is shown in Figure 2.7. To find the homogeneous solution the non-homogeneous part of the equation is set to zero. To find the particular solution the final form must be guessed. This is then substituted into the equation, and the values of the coefficients are found. Finally the homogeneous and particular solutions are added to get the final equation. The overall response of the system can be obtained by adding the homogeneous and particular parts. This is acceptable because the equations are linear, and the principle of superposition applies. The homogeneous equation deals with the response to initial conditions, and the particular solution deals with the response to forced inputs.

Generally,

$$A\dot{x} + Bx = Cf(t) \qquad x(0) = x_0$$

First, find the homogeneous solution as before, in Figure 2.3.

$$x_h = x_0 e^{-\frac{B}{A}t}$$

Next, guess the particular solution by looking at the form of 'f(t)'. This step is highly subjective, and if an incorrect guess is made, it will be unsolvable. When this happens, just make another guess and repeat the process. An example is given below. In the case below the guess should be similar to the exponential forcing function.

For example, if we are given

$$6\dot{x} + 2x = 5e^{4t}$$

A reasonable guess for the particular solution is,

$$x_p = C_1 e^{4t} \qquad \dot{x}_p = 4C_1 e^{4t}$$

Substitute these into the differential equation and solve for A.

$$6\left(4C_1 e^{4t}\right) + 2\left(C_1 e^{4t}\right) = 5e^{4t}$$

$$24C_1 + 2C_1 = 5 \qquad \therefore C_1 = \frac{5}{26}$$

Combine the particular and homogeneous solutions.

$$x = x_p + x_h = \frac{5}{26}e^{4t} + x_0 e^{-\frac{6}{2}t}$$

Figure 2.7 Example: Solution of a first-order non-homogeneous equation

The method for finding a particular solution for a second-order non-homogeneous differential equation is shown in Figure 2.8. In this example the forcing function is sinusoidal, so the particular result should also be sinusoidal. The final result is converted into a phase shift form.

Generally,

$$A\ddot{x} + B\dot{x} + Cx = Df(t) \qquad x(0) = x_0 \qquad and \qquad \dot{x}(0) = v_0$$

1. Find the homogeneous solution as before.

$$x_h = X_3 e^{\sigma t} cos(\omega t + X_4)$$
$$or$$
$$x_h = X_1 e^{\sigma t} + X_2 t e^{\sigma t}$$
$$or$$
$$x_h = X_1 e^{\sigma_1 t} + X_2 e^{\sigma_2 t}$$

2. Guess the particular solution by looking at the form of ‘f(t)’. This step is highly subjective, and if an incorrect guess is made it will be unsolvable. When this happens, just make another guess and repeat the process. For the purpose of illustration an example is given below. In the case below it should be similar to the sine function.

For example, if we are given

$$2\ddot{x} + 6\dot{x} + 2x = 2sin(3t + 4)$$

A reasonable guess is made, and the first and second derivatives are written.

$$x_p = A sin(3t) + B cos(3t)$$
$$\dot{x}_p = 3A cos(3t) - 3B sin(3t)$$
$$\ddot{x}_p = -9A sin(3t) - 9B cos(3t)$$

Substitute these into the differential equation ans solve for A and B.

$$2(-9A sin(3t) - 9B cos(3t)) + 6(3A cos(3t) - 3B sin(3t)) +$$
$$2(A sin(3t) + B cos(3t)) = 2 sin(3t + 4)$$

$$(-18A - 18B + 2A) sin(3t) + (-18B + 18A + 2B) cos(3t) = 2 sin(3t + 4)$$

$$(-16A - 18B) sin(3t) + (18A - 16B) cos(3t) = 2(sin 3t cos 4 + cos 3t sin 4)$$

$$(-16A - 18B) sin(3t) + (18A - 16B) cos(3t) = (2 cos 4) sin(3t) + (2 sin 4) cos(3t)$$

$$-16A - 18B = 2 cos 4 \qquad 18A - 16B = 2 sin 4$$

$$\begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 cos 4 \\ 2 sin 4 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix}^{-1} \begin{bmatrix} -1.307 \\ -1.514 \end{bmatrix} = \begin{bmatrix} -0.0109 \\ 0.0823 \end{bmatrix}$$

Next, rearrange the equation to phase shift form.

$$x_p = -0.0109 sin(3t) + 0.0823 cos(3t)$$
$$x_p = \sqrt{-0.0109^2 + 0.0823^2}\, sin\left(3t + atan\left(\frac{0.0823}{-0.0109}\right)\right)$$

3. Use the initial conditions to determine the coefficients in the homogeneous solution.

Figure 2.8 Example: Solution of a second-order non-homogeneous equation

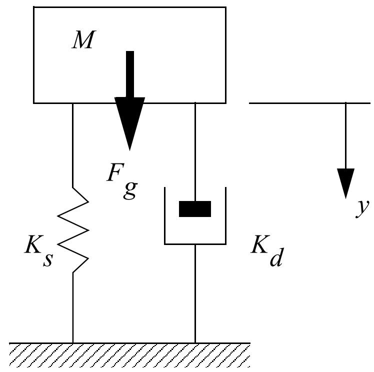
When guessing particular solutions, the forms in Figure 2.9 can be helpful.

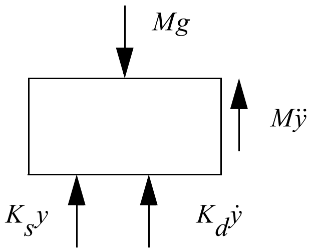
Forcing Function	Guess
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} *or* Cxe^{Ax}
$B\,sin(Ax)$ or $B\,cos(Ax)$	$C\,sin(Ax) + D\,cos(Ax)$ or $Cx\,sin(Ax) + xD\,cos(Ax)$

Figure 2.9 General forms for particular solutions

An example of a second-order system is shown in Figure 2.10. As expected, it begins with a FBD and summation of forces. This is followed with the general solution of the homogeneous equation. Real roots are assumed thus allowing the problem solution to continue in Figure 2.11.

Assume the system illustrated to the right starts from rest at a height 'h'. At time 't=0' the system is released and allowed to move.





$$+\uparrow \quad \sum F_y = -Mg + K_s y + K_d \dot{y} = -M\ddot{y}$$

$$M\ddot{y} + K_d\dot{y} + K_s y = Mg$$

Find the homogeneous solution.

$$y_h = e^{At} \qquad \dot{y}_h = Ae^{At} \qquad \ddot{y}_h = A^2 e^{At}$$

$$M\ddot{y} + K_d\dot{y} + K_s y = 0$$

$$M\left(A^2 e^{At}\right) + K_d(Ae^{At}) + K_s(e^{At}) = 0$$

$$MA^2 + K_d A + K_s = 0$$

$$A = \frac{-K_d \pm \sqrt{K_d^2 - 4MK_s}}{2M}$$

Let us assume that the values of M, K_d and K_s lead to the case of two different positive roots. This would occur if the damper value was much larger than the spring and mass values. Thus,

$$A = R_1, R_2$$

$$y_h = C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

Figure 2.10 Example: Second-order system

The solution continues by assuming a particular solution and calculating values for the coefficients using the initial conditions in Figure 2.11. The final result is a second-order system that is overdamped, with no oscillation.

Next, find the particular solution.

$$y_p = C \qquad \dot{y}_h = 0 \qquad \ddot{y}_h = 0$$

$$M(0) + K_d(0) + K_s(C) = Mg$$

$$C = \frac{Mg}{K_s}$$

Now, add the homogeneous and particular solutions and solve for the unknowns using the initial conditions.

$$y(t) = y_p + y_h = \frac{Mg}{K_s} + C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

$$y(0) = h \qquad y'(0) = 0$$

$$h = \frac{Mg}{K_s} + C_1 e^0 + C_2 e^0$$

$$C_1 + C_2 = h - \frac{Mg}{K_s}$$

$$y'(t) = R_1 C_1 e^{R_1 t} + R_2 C_2 e^{R_2 t}$$

$$0 = R_1 C_1 e^0 + R_2 C_2 e^0$$

$$0 = R_1 C_1 + R_2 C_2 \qquad C_1 = \frac{-R_2}{R_1} C_2$$

$$-\frac{R_2}{R_1} C_2 + C_2 = h - \frac{Mg}{K_s}$$

$$C_2 = \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) \qquad C_1 = \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right)$$

Now, combine the solutions and solve for the unknowns using the initial conditions.

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_2 t}$$

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{R_2}{R_1 - R_2}\right) e^{R_2 t}$$

Figure 2.11 Example: Second-order system (continued)

Given

$$A\sin(\omega t + \theta) \qquad \text{(the desired final form)}$$

$$A(\cos\omega t \sin\theta + \sin\omega t \cos\theta)$$

$$(A\sin\theta)\cos\omega t + (A\cos\theta)\sin\omega t$$

$$B\cos\omega t + C\sin\omega t \qquad \text{(the form we will start with)}$$

where,

$$B = A\sin\theta$$

$$C = A\cos\theta$$

To find theta,

$$\frac{B}{C} = \frac{A\sin\theta}{A\cos\theta} = \tan\theta$$

$$\theta = \operatorname{atan}\left(\frac{B}{C}\right)$$

To find A (method #1)

$$A = \frac{B}{\sin\theta} = \frac{C}{\cos\theta}$$

To find A, (method #2)

$$A = \sqrt{B^2 + C^2}$$

For example,

$$3\cos 5t + 4\sin 5t$$

$$\sqrt{3^2 + 4^2}\sin\left(5t + \operatorname{atan}\frac{3}{4}\right)$$

$$5\sin(5t + 0.6435)$$

Figure 2.12 Proof for conversion to phase form

Note: Unless initial conditions are provided, normally assume that they are all zero. As a student this is beneficial because it simplifies problem solutions. This is normally used for professional system analysis to determine the reaction when a system is initially turned on. Common terms to indicate that the conditions are zero include “the system starts at rest”, “initially the system is off”, “the system starts undeflected”.

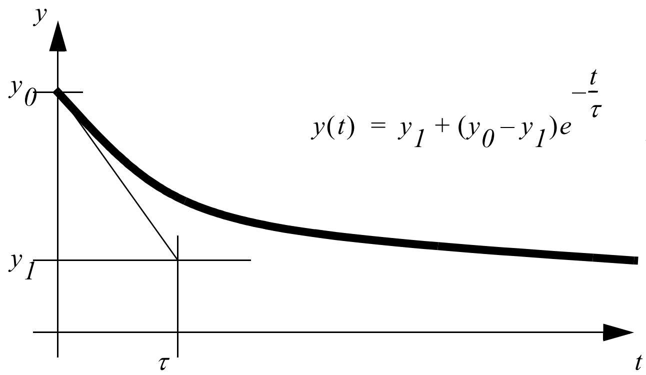
2.2 Responses

Solving differential equations tends to yield one of two basic equation forms. The e-to-the-negative-t forms are the first-order responses and slowly decay over time. They never naturally oscillate, and only oscillate if forced to do so. The second-order forms may include natural oscillation.

First-Order

A first-order system is described with a first-order differential equation. The response function for these systems is natural decay or growth as shown in Figure 2.13. The time constant for the system can be found directly from the differential equation. It is a measure of how quickly the system responds to a change. When an input to a system has changed, the system output will be approximately 63% of the way to its final value when the elapsed time equals the time constant. The initial and final values of the function can be determined algebraically to find the first-order response with little effort.

If we have experimental results for a system, we can calculate the time constant, initial and final values. The time constant can be found two ways, one by extending the slope of the first (linear) part of the curve until it intersects the final value line. That time at the intersection is the time constant. The other method is to look for the time when the output value has shifted 63.2% of the way from the initial to final values for the system. Assuming the change started at t=0, the time at this point corresponds to the time constant.

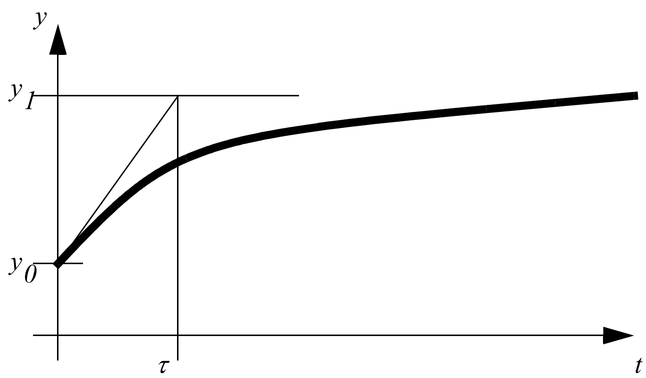


$$y(t) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

$$\dot{y} + \frac{1}{\tau}y = f(t)$$

time constant

OR



Note: The time will be equal to the time constant when the value is 63.2% of the way to the final value, as shown below.

$$y(\tau) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

$$y(\tau) = y_1 + (y_0 - y_1)e^{-1}$$

$$y(\tau) = y_1 + (y_0 - y_1)0.368$$

$$y(\tau) = 1 + (0 - 1)0.368$$

$$y(\tau) = 0.632$$

Note: The time constant can also be found using the asymptote of the base (not the middle) of the first order curve. In the simple proof that follows a straight line is extended from the base of the curve to the steady state asymptote.

$$y(\tau) \approx y_0 + \tau\frac{d}{dt}y(0) = y_0 + \tau(y_0 - y_1)\left(-\frac{1}{\tau}\right)e^{-\frac{0}{\tau}} = y_0 + \tau(y_0 - y_1)\left(-\frac{1}{\tau}\right)e^{-\frac{0}{\tau}} = y_1$$

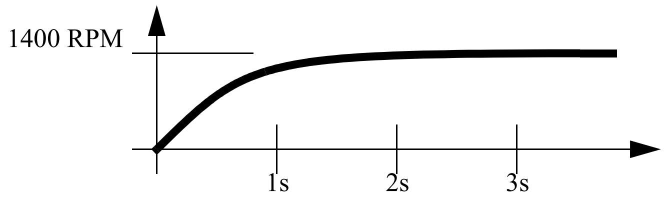
Note: Given a first order system the general form can be found knowing the input, steady state output, and time constant. Consider the example below with an input of F and an output of x.

$$\dot{x} + \left(\frac{1}{\tau}\right)x = \left(\frac{x_{ss}}{\tau F_{ss}}\right)F$$

Figure 2.13 Typical first-order responses

The example in Figure 2.14 calculates the coefficients for a first-order differential equation given a graphical output response to an input. The differential equation is for a permanent magnet DC motor, and will be examined in a later chapter. If we consider the steady state when the speed is steady at 1400RPM, the first derivative will be zero. This simplifies the equation and allows us to calculate a value for the parameter K in the differential equation. The time constant can be found by drawing two lines over the data curve. The first line is asymptotic to the start of the motor curve, and the second line is asymptotic to the steady state speed. The time point where the lines intersect is the time constant. This example results in an approximate time constant of 0.8 s. This can then be used to calculate the remaining coefficient. Some additional numerical calculation leads to the final differential equation as shown.

For the motor, use the differential equation and the speed curve when Vs=10V is applied:



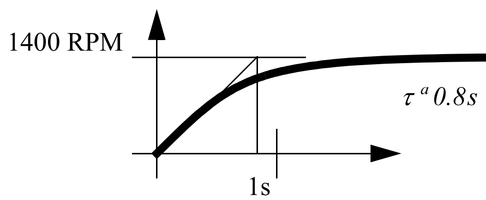
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\left(\frac{K^2}{JR}\right) = \frac{1}{0.8s}$$

$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

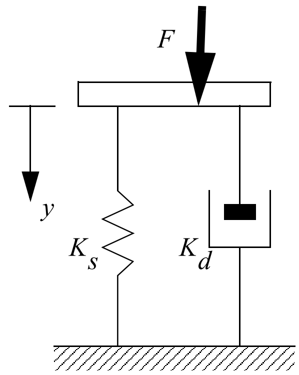
$$\frac{K}{JR} = 18.328$$

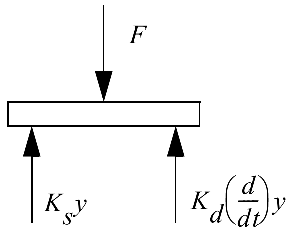
$$\dot{\omega} + \frac{1}{0.8}\omega = 18.328V_s$$

Figure 2.14 Example: Finding an equation using experimental data

A simple mechanical example is given in Figure 2.15. The modeling starts with a FBD and a sum of forces. After this, the homogeneous solution is calculated without the non-homogeneous terms. Next, the particular solution is calculated using the complete differential equation. The homogeneous and particular solutions are added for the overall systems response. The initial conditions are used to find the remaining unknown coefficients.

Find the response to the applied force if the force is applied at t=0s.
Assume the system is initially deflected a height of *h*.





$$+\uparrow \quad \sum F_y = -F + K_s y + K_d\left(\frac{d}{dt}\right)y = 0$$

$$K_d \dot{y} + K_s y = F$$

Find the homogeneous solution.

$$y_h = Ae^{Bt} \qquad \dot{y}_h = ABe^{Bt}$$

$$K_d(ABe^{Bt}) + K_s(Ae^{Bt}) = 0$$

$$K_d B + K_s = 0$$

$$B = \frac{-K_s}{K_d}$$

Next, find the particular solution.

$$y_p = C \qquad \dot{y}_p = 0$$

$$K_d(0) + K_s(C) = F \qquad \backslash C = \frac{F}{K_s}$$

Combine the solutions, and find the remaining unknown.

$$y(t) = y_p + y_h = Ae^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

$$y(0) = h$$

$$h = Ae^{0} + \frac{F}{K_s} \qquad \backslash A = h - \frac{F}{K_s}$$

The final solution is,

$$y(t) = \left(h - \frac{F}{K_s}\right)e^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

Figure 2.15 Example: First-order system analysis

A first-order system tends to be passive, meaning it doesn't deliver energy or power. A first-order system will not oscillate unless the input forcing function is also oscillating. The output response lags the input and the delay is determined by the system's time constant.

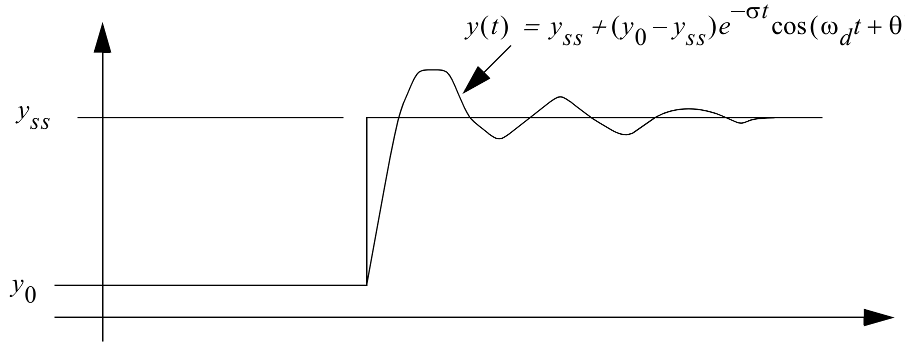
Second-Order

A second-order system response typically contains two first-order responses, or a first-order response and a sinusoidal component. A typical sinusoidal second-order response is shown in Figure 2.16. Notice that the coefficients of the differential equation include a damping factor and a natural frequency. These can be used to develop the final response, given the initial conditions and forcing function. Notice that the damped frequency of oscillation is the actual frequency of oscillation. The damped frequency will be lower than the natural frequency when the damping factor is between 0 and 1. If the damping factor is greater than one the damped frequency becomes negative, and the system will not oscillate because it is overdamped.

A second-order system, and a typical response to a stepped input.

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = F(t) \longrightarrow \sigma = \zeta\omega_n \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$y(t) = y_{ss} + (y_0 - y_{ss})e^{-\sigma t}\cos(\omega_d t + \theta)$$



ω_n Natural frequency of system - Approximate frequency of response system oscillations.

ζ Damping factor of system - If < 1 underdamped, and system will oscillate. If =1 critically damped. If > 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.

ω_d The actual frequency of oscillation - It is below the natural frequency because of the damping.

θ The phase shift represents a lag between the input action and output response. At low frequencies this is often zero.

Figure 2.16 The general form for a second-order system

When only the damping factor is increased, the frequency of oscillation, and overall response time will slow, as seen in Figure 2.17. When the damping factor is 0 the system will oscillate indefinitely. Critical damping occurs when the damping factor is 1. At this point both roots of the differential equation are equal. The system will not oscillate if the damping factor is greater than or equal to 1.

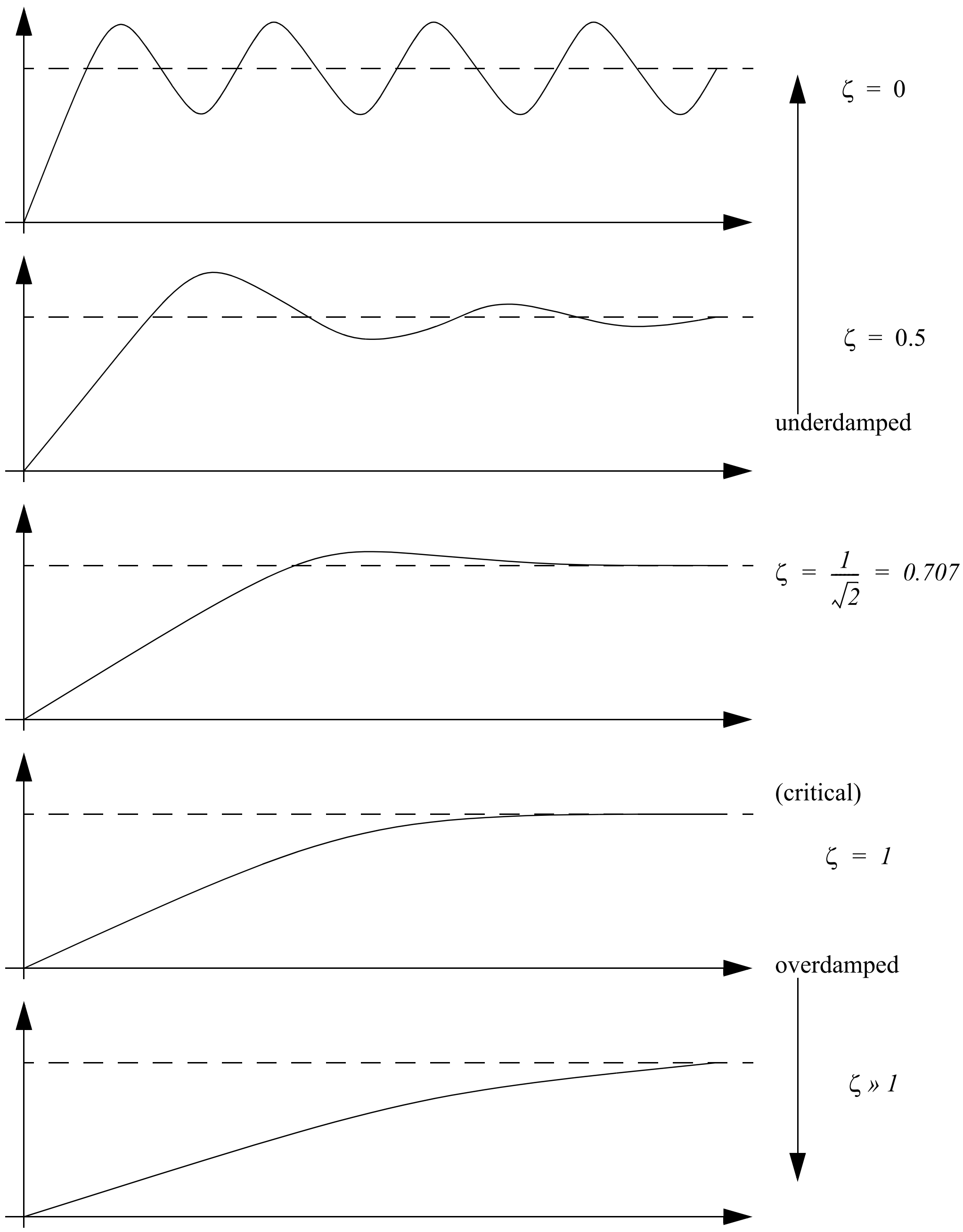
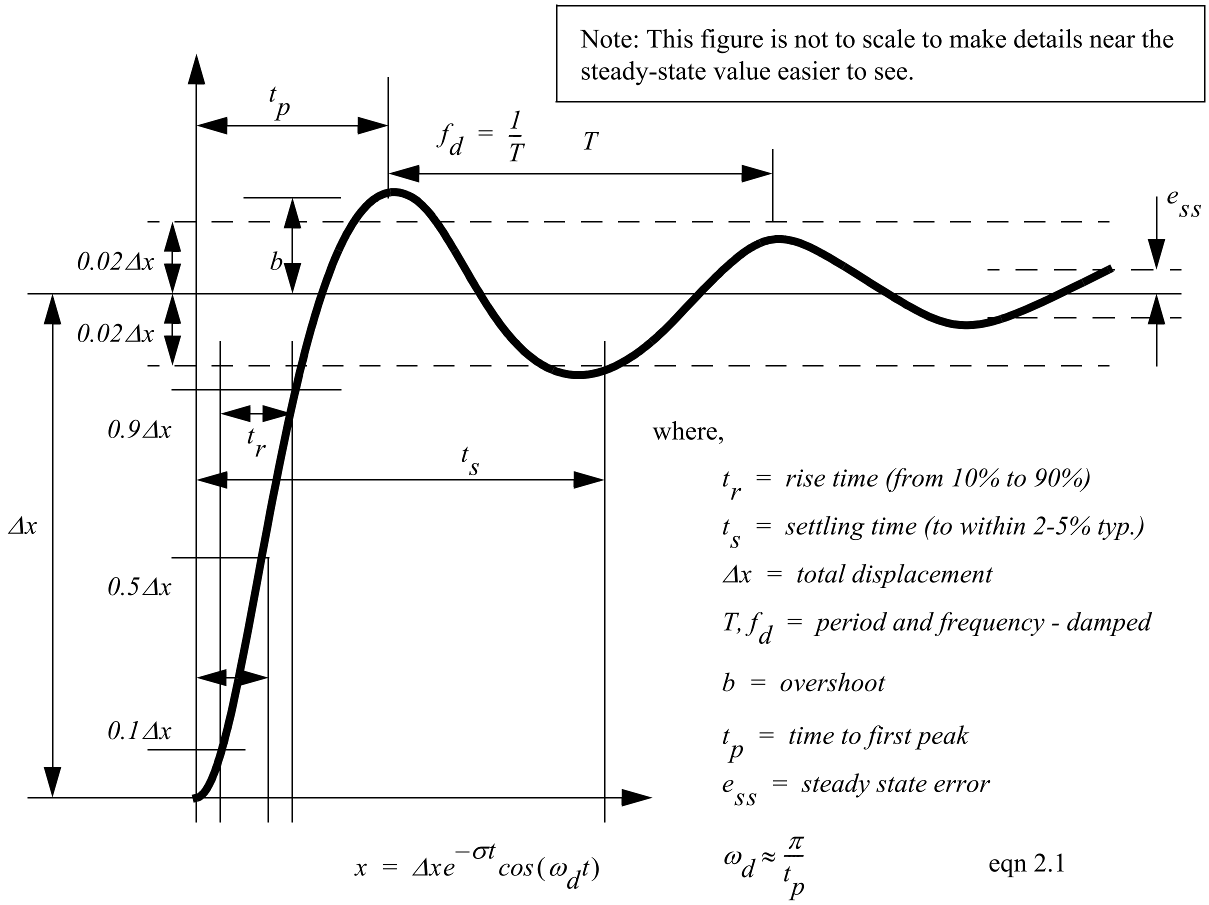


Figure 2.17 The effect of the damping factor

When observing second-order systems it is more common to use more direct measurements of the response. Some of these measures are shown in Figure 2.18. The rise time is the time it takes to go from 10% to 90% of the total displacement, and is comparable to a first order time constant. The settling time indicates how long it takes for the system to pass within a tolerance band around the final value. The permissible zone shown is 2%, but if it were larger the system would have a shorter settling time. The period of oscillation can be measured directly as the time between peaks of the oscillation; the inverse is the damped frequency. (Note: don’t forget to convert to radians.) The damped frequency can also be found using the time to the first peak, as half the period. The overshoot is the height of the first peak. Using the time to the first peak, and the overshoot the damping factor can be found.



where,

t_r = *rise time (from 10% to 90%)*

t_s = *settling time (to within 2-5% typ.)*

Δx = *total displacement*

T, f_d = *period and frequency - damped*

b = *overshoot*

t_p = *time to first peak*

e_{ss} = *steady state error*

$$\omega_d \approx \frac{\pi}{t_p} \quad \text{eqn 2.1}$$

$$\sigma = \zeta \omega_n \quad \text{eqn 2.2}$$

$$\frac{b}{\Delta x} = e^{-\sigma t_p} \quad \text{eqn 2.3}$$

$$\zeta = \frac{1}{\sqrt{\left(\frac{\pi}{t_p \sigma}\right)^2 + 1}} \quad \text{eqn 2.4}$$

Figure 2.18 Characterizing a second-order response (not to scale)

Note: We can calculate these relationships using the complex homogeneous form, and the generic second order equation form.

$$A^2 + 2\zeta\omega_n A + \omega_n^2 = 0$$

$$A = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \sigma \pm j\omega_d$$

$$\frac{-2\zeta\omega_n}{2} = \sigma = -\zeta\omega_n \qquad \omega_n = \frac{\sigma}{-\zeta} \qquad \text{eqn 2.5}$$

$$\frac{\sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = j\omega_d$$

$$4\zeta^2\omega_n^2 - 4\omega_n^2 = 4(-1)\omega_d^2$$

$$\omega_n^2 - \zeta^2\omega_n^2 = \omega_d^2 \qquad \omega_n\sqrt{1-\zeta^2} = \omega_d \qquad \text{eqn 2.6}$$

$$\frac{\sigma^2}{\zeta^2} - \zeta^2\frac{\sigma^2}{\zeta^2} = \omega_d^2$$

$$\frac{1}{\zeta^2} = \frac{\omega_d^2}{\sigma^2} + 1 \qquad \zeta = \frac{1}{\sqrt{\frac{\omega_d^2}{\sigma^2} + 1}} \qquad \text{eqn 2.7}$$

The time to the first peak can be used to find the approximate decay constant

$$x(t) = C_1 e^{-\sigma t}\cos(\omega_d t + C_2)$$

$$\omega_d = \frac{\pi}{t_p} \qquad \text{eqn 2.8}$$

$$b \approx \Delta x e^{-\sigma t_p}(1)$$

$$\sigma = -\frac{\ln\left(\frac{b}{\Delta x}\right)}{t_p} \qquad \text{eqn 2.8}$$

Figure 2.19 Second order relationships between damped and natural frequency

Other Responses

First-order systems have e-to-the-t type responses. Second-order systems add another e-to-the-t response or a sinusoidal excitation. As we move to higher order linear systems we typically add more e-to-the-t terms, and/or more sinusoidal terms. A possible higher order system response is seen in Figure 2.20. The underlying function is a first-order response that drops at the beginning, but levels out. There are two sinusoidal functions superimposed, one with about one period showing, the other with a much higher frequency.

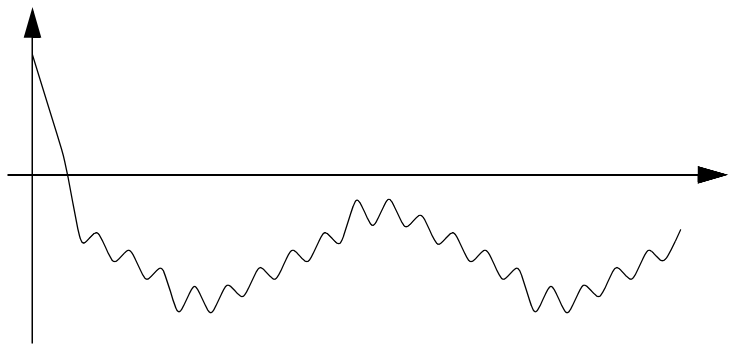


Figure 2.20 An example of a higher order system response

The basic techniques used for solving first and second-order differential equations can be applied to higher order differential equations, although the solutions will start to become complicated for systems with much higher orders. The example in Figure 2.21 shows a fourth order differential equation. In this case the resulting homogeneous solution yields four roots. This result is two real roots, and a complex pair. The two real roots result in e-to-the-t terms, while the complex pair results in a damped sinusoid. The particular solution is relatively simple to find in this example because the non-homogeneous term is a constant.

Given the homogeneous differential equation

$$\left(\frac{d}{dt}\right)^4 x + 13\left(\frac{d}{dt}\right)^3 x + 34\left(\frac{d}{dt}\right)^2 x + 42\left(\frac{d}{dt}\right)x + 20x = 5$$

Guess a solution for the homogeneous equation,

$$x_h = e^{At}$$

$$\frac{d}{dt}x_h = Ae^{At} \qquad \left(\frac{d}{dt}\right)^2 x_h = A^2 e^{At} \qquad \left(\frac{d}{dt}\right)^3 x_h = A^3 e^{At} \qquad \left(\frac{d}{dt}\right)^4 x_h = A^4 e^{At}$$

Substitute the values into the differential equation and find a value for the unknown.

$$A^4 e^{At} + 13A^3 e^{At} + 34A^2 e^{At} + 42Ae^{At} + 20e^{At} = 0$$

$$A^4 + 13A^3 + 34A^2 + 42A + 20 = 0$$

$$A = -1, -10, -1-j, -1+j$$

$$x_h = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t} \cos(t + C_4)$$

Guess a particular solution, and then solve for the coefficient.

$$x_p = A \qquad \frac{d}{dt}x_p = 0 \qquad \left(\frac{d}{dt}\right)^2 x_p = 0 \qquad \left(\frac{d}{dt}\right)^3 x_p = 0 \qquad \left(\frac{d}{dt}\right)^4 x_p = 0$$

$$0 + 13(0) + 34(0) + 42(0) + 20A = 5 \qquad A = 0.25$$

Figure 2.21 Example: Solution of a higher order differential equation

The example is continued in Figure 2.22 and Figure 2.23 where the initial conditions are used to find values for the coefficients in the homogeneous solution.

Solve for the unknowns, assuming the system starts at rest and undeflected.

$$x(t) = C_1e^{-t} + C_2e^{-10t} + C_3e^{-t}cos(t + C_4) + 0.25$$

$$0 = C_1 + C_2 + C_3cos(C_4) + 0.25 \qquad \text{eqn 2.9}$$

$$C_3cos(C_4) = -C_1 - C_2 - 0.25 \qquad \text{eqn 2.10}$$

$$\frac{d}{dt}x_h(t) = -C_1e^{-t} - 10C_2e^{-10t} - C_3e^{-t}cos(t + C_4) - C_3e^{-t}sin(t + C_4)$$

$$0 = -C_1 - 10C_2 - C_3cos(C_4) - C_3sin(C_4) \qquad \text{eqn 2.11}$$

Equations (1) and (3) can be added to get the simplified equation below.

$$0 = -C_1 - 10C_2 - (-C_1 - C_2 - 0.25) - C_3sin(C_4)$$

$$0 = -9C_2 - C_3sin(C_4) + 0.25$$

$$C_3sin(C_4) = -9C_2 + 0.25 \qquad \text{eqn 2.12}$$

$$\left(\frac{d}{dt}\right)^2 x_h(t) = C_1e^{-t} + 100C_2e^{-10t} + C_3e^{-t}cos(t + C_4) + C_3e^{-t}sin(t + C_4) + C_3e^{-t}sin(t + C_4) - C_3e^{-t}cos(t + C_4)$$

$$\left(\frac{d}{dt}\right)^2 x_h(t) = C_1e^{-t} + 100C_2e^{-10t} + 2C_3e^{-t}sin(t + C_4)$$

$$0 = C_1 + 100C_2 + C_3cos(C_4) + C_3sin(C_4) + C_3sin(C_4) - C_3cos(C_4)$$

$$0 = C_1 + 100C_2 + 2C_3sin(C_4) \qquad \text{eqn 2.13}$$

Equations (4) and (5) can be combined.

$$0 = C_1 + 100C_2 + 2(-9C_2 + 0.25)$$

$$0 = C_1 + 82C_2 + 0.5 \qquad \text{eqn 2.14}$$

$$\left(\frac{d}{dt}\right)^3 x_h(t) = -C_1e^{-t} + (-1000)C_2e^{-10t} - 2C_3e^{-t}sin(t + C_4) + 2C_3e^{-t}cos(t + C_4)$$

$$0 = -C_1 + (-1000)C_2 - 2C_3\sin(C_4) + 2C_3\cos(C_4) \qquad \text{eqn 2.15}$$

Figure 2.22 Example: Solution of a higher order differential equation

Equations (2) and (4) are substituted into equation (7).

$$0 = -C_1 + (-1000)C_2 - 2(-9C_2 + 0.25) + 2(-C_1 - C_2 - 0.25)$$

$$0 = -3C_1 + (-984)C_2 - 1$$

$$C_1 = \left(-\frac{984}{3}\right)C_2 - \frac{1}{3} \qquad \text{eqn 2.16}$$

Equations (6) and (8) can be combined.

$$0 = \left(\left(-\frac{984}{3}\right)C_2 - \frac{1}{3}\right) + 82C_2 + 0.5$$

$$0 = (-246)C_2 + 0.166667 \qquad C_2 = 0.0006775$$

$$C_1 = \left(-\frac{984}{3}\right)\left(\frac{1}{1476}\right) - \frac{1}{3} \qquad C_1 = -0.5555$$

Equations (2) and (4) can be combined.

$$\frac{C_3 sin(C_4)}{C_3 cos(C_4)} = \frac{-9C_2 + 0.25}{-C_1 - C_2 - 0.25}$$

$$tan(C_4) = \frac{-9(0.0006775) + 0.25}{-(-0.5555) - (0.0006775) - 0.25} \qquad C_4 = 0.6748$$

Equation (4) can be used the find the remaining unknown.

$$C_3 sin(0.6748) = -9(0.0006775) + 0.25 \qquad C_3 = 0.3904$$

The final response function is,

$$x(t) = (-0.5555)e^{-t} + (0.0006775)e^{-10t} + (0.3904)e^{-t} cos(t + 0.3904) + 0.25$$

Figure 2.23 Example: Solution of a higher order differential equation (cont'd)

In some cases we will have systems with multiple differential equations, or non-linear terms. In these cases explicit analysis of the equations may not be feasible. In these cases we may use other techniques, such as numerical integration, which will be covered in later chapters.

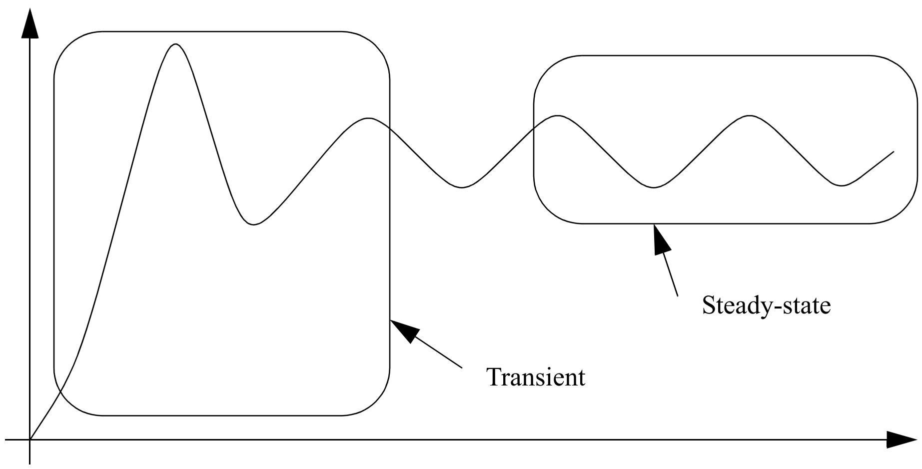
2.1 Response Analysis

Up to this point we have mostly discussed the process of calculating the system response. As an engineer, obtaining the response is important, but evaluating the results is more important. The most critical design consideration is system stability. In most cases a system should be inherently stable in all situations, such as a car "cruise control". In other cases an unstable system may be the objective, such as an explosive device. Simple methods for determining the stability of a system are listed below:

1. If a step input causes the system to go to infinity, it will be inherently unstable.
2. A ramp input might cause the system to go to infinity; if this is the case, the system might not respond well to constant change.
3. If the response to a sinusoidal input grows with each cycle, the system is probably resonating, and will become unstable.

Beyond establishing the stability of a system, we must also consider general performance. This includes the time constant for a first-order system, or damping factor and natural frequency for a second-order system. For example, assume we have designed an elevator that is a second-order system. If it is under damped the elevator will oscillate, possibly leading to motion sickness, or worse. If the elevator is over damped it will take longer to get to floors. If it is critically damped it will reach the floors quickly, without overshoot.

Engineers distinguish between initial setting effects (transient) and long term effects (steady-state). The transient effects are closely related to the homogeneous solution to the differential equations and the initial conditions. The steady-state effects occur after some period of time when the system is acting in a repeatable or non-changing form. Figure 2.24 shows a system response. The transient effects at the beginning include a quick rise time and an overshoot. The steady-state response settles down to a constant amplitude sine wave.



Note: the transient response is predicted with the homogeneous solution. The steady state response in mainly predicted with the particular solution, although in some cases the homogeneous solution might have steady state effects, such as a non-decaying oscillation.

Figure 2.24 A system response with transient and steady-state effects

2.2 Non-Linear Systems

Non-linear systems cannot be described with a linear differential equation. A basic linear differential equation has coefficients that are constant, and the derivatives are all first order. Examples of non-linear differential equations are shown in Figure 2.25.

$$\dot{x} + x^2 = 5$$

$$\dot{x}^2 + x = 5$$

$$\dot{x} + log(x) = 5$$

$$\dot{x} + (5t)x = 5$$

Note: the sources of non-linearity are circled.

Figure 2.25 Examples of non-linear differential equations

Examples of system conditions that lead to non-linear solutions are,

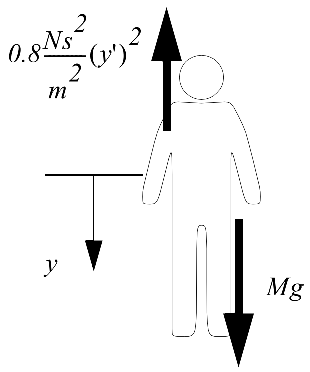
- Aerodynamic drag.
- Forces that are a squared function of distance.
- Devices with non-linear responses.

Explicitly solving non-linear differential equations can be difficult, and will typically involve complex solutions for simple problems.

Non-Linear Differential Equations

A non-linear differential equation is presented in Figure 2.26. It involves a person ejected from an aircraft with a drag force coefficient of 0.8. (Note: This coefficient is calculated using the drag coefficient and other properties such as the speed of sound and cross sectional area.) The FBD shows the sum of forces, and the resulting differential equation. The velocity squared term makes the equation non-linear, and so it cannot be analyzed with the previous methods. In this case the terminal velocity is calculated by setting the acceleration to zero. This results in a maximum speed of 126 kph.

Consider the differential equation for a 100kg human ejected from an airplane. The aerodynamic drag will introduce a squared variable, therefore making the equation non-linear.



$$\sum F_y = 0.8(\dot{y})^2 - Mg = -M\ddot{y}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 100kg9.81\frac{N}{kg}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 981N$$

$$100kg\ddot{y} + 0.8kg\frac{m}{s^2}\frac{s^2}{m^2}(\dot{y})^2 = 981kg\frac{m}{s^2}$$

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$\ddot{y} + 8\times10^{-3}m^{-1}(\dot{y})^2 = 9.81ms^{-2}$$

The terminal velocity can be found be setting the acceleration to zero.

$$(0) + 8\times10^{-3}m^{-1}(\dot{y})^2 = 9.81ms^{-2}$$

$$\dot{y} = \sqrt{\frac{9.81ms^{-2}}{8\times10^{-3}m^{-1}}} = \sqrt{\frac{9.81}{8\times10^{-3}}m^2s^{-2}} = 35.0\frac{m}{s} = 126\frac{km}{h}$$

Figure 2.26 Example: Development of a non-linear differential equation

The equation can also be solved using explicit integration, as shown in Figure 2.27. In this case the equation is separated and rearranged to isolate the ‘v’ terms on the left, and time on the right. The term is then integrated in Figure 2.28 and Figure 2.29. The final form of the equation is non-trivial, but contains e-to-t terms, as we would expect.

An explicit solution can begin by replacing the position variable with a velocity variable and rewriting the equation as a separable differential equation.

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$100\dot{v} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} = 981ms^{-2} - 0.8m^{-1}v^2$$

$$\frac{100}{981ms^{-2} - 0.8m^{-1}v^2}dv = dt$$

$$\int\frac{\frac{100}{-0.8m^{-1}}}{\frac{981}{-0.8m^{-1}}ms^{-2} + v^2}dv = \int dt$$

$$\int\frac{-125m}{v^2 - 1226.25m^2s^{-2}}dv = t + C_1$$

$$\int\frac{-125m}{\left(v + 35.02\frac{m}{s}\right)\left(v - 35.02\frac{m}{s}\right)}dv = t + C_1$$

Figure 2.27 Example: Developing an integral

This can be reduced with a partial fraction expansion.

$$\int\left[\frac{A}{\left(v+35.02\frac{m}{s}\right)}+\frac{B}{\left(v-35.02\frac{m}{s}\right)}\right]dv = t+C_1$$

$$Av-A\left(35.02\frac{m}{s}\right)+Bv+B\left(35.02\frac{m}{s}\right) = -125m$$

$$v(A+B)+35.02\frac{m}{s}(-A+B) = -125m$$

$$A+B = 0 \qquad\qquad A = -B$$

$$35.02\frac{m}{s}(-A+B) = -125m$$

$$(-(-B)+B) = -\frac{125}{35.02}s \qquad\qquad B = -1.785s$$

$$A = 1.785s$$

$$\int\left[\frac{1.785s}{\left(v+35.02\frac{m}{s}\right)}+\frac{-1.785s}{\left(v-35.02\frac{m}{s}\right)}\right]dv = t+C_1$$

The integral can then be solved using an identity from the integral table. In this case the integration constants can be left off because they are redundant with the one on the right hand side.

$$1.785s\ln\left|v+35.02\frac{m}{s}\right|-1.785s\ln\left|v-35.02\frac{m}{s}\right| = t+C_1$$

$$1.785s\ln\left|\frac{v+35.02\frac{m}{s}}{v-35.02\frac{m}{s}}\right| = t+C_1$$

$$\int(a+bx)^{-1}dx = \frac{\ln|a+bx|}{b}+C$$

$$\left|\frac{v+35.02\frac{m}{s}}{v-35.02\frac{m}{s}}\right| = e^{\frac{t}{1.785s}+C_1}$$

$$\left|\frac{v+35.02\frac{m}{s}}{v-35.02\frac{m}{s}}\right| = e^{C_1}e^{\frac{t}{1.785s}}$$

Figure 2.28 Example: Solution of the integral

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{t}{1.785s}}$$

An initial velocity of zero can be assumed to find the value of the integration constant

$$\left|\frac{0 + 35.02\frac{m}{s}}{0 - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{0}{1.785s}} \qquad 1 = C_2$$

This can then be simplified, and the absolute value sign eliminated.

$$\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}} = \pm e^{\frac{t}{1.785s}}$$

$$v + 35.02\frac{m}{s} = \pm v e^{\frac{t}{1.785s}} \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}}$$

$$v\left(1 \mp e^{\frac{t}{1.785s}}\right) = \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}} - 35.02\frac{m}{s}$$

$$v = 35.02\frac{m}{s}\left(\frac{\mp e^{\frac{t}{1.785s}} - 1}{1 \mp e^{\frac{t}{1.785s}}}\right) \qquad 0 = 35.02\frac{m}{s}\left(\frac{\mp 1 - 1}{1 \mp 1}\right) = \left(\frac{1 - 1}{1 + 1}\right) = \frac{0}{2}$$

$$v = 35.02\frac{m}{s}\left(\frac{e^{\frac{t}{1.785s}} - 1}{1 + e^{\frac{t}{1.785s}}}\right)$$

Figure 2.29 Example: Solution of the integral and application of the initial conditions

As evident from the example, non-linear equations are involved and don’t utilize routine methods. Typically the numerical methods discussed in the next chapter are preferred.

Non-Linear Equation Terms

If our models include a device that is non-linear and we want to use a linear technique to solve the equation, we will need to linearize the model before we can proceed. A non-linear system can be approximated with a linear equation using the following method.

1. Pick an operating point or range for the component.
2. Find a constant value that relates a change in the input to a change in the output.
3. Develop a linear equation.
4. Use the linear equation for the analysis.

A linearized differential equation can be approximately solved using known techniques as long as the system doesn’t travel too far from the linearized point. The example in Figure 2.30 shows the linearization of a non-linear equation about a given operating point. This equation will be approximately correct as long as the first derivative doesn't move too far from 100. When this value does, the new velocity can be calculated.

Assume we have the non-linear differential equation below. It can be solved by linearizing the value about the operating point

Given,

$$\dot{y}^2 + 4y = 200 \qquad\qquad y(0) = 10$$

We can make the equation linear by replacing the velocity squared term with the velocity times the actual velocity. As long as the system doesn't vary too much from the given velocity the model should be reasonably accurate.

$$\dot{y} = \pm\sqrt{200 - 4y}$$

$$\dot{y}(0) = \pm\sqrt{200 - 4(10)} = \pm 12.65$$

$$12.65\dot{y} + 4y = 20$$

This system may now be solved as a linear differential equation. If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this.

Homogeneous:

$$12.65\dot{y} + 4y = 0$$

$$12.65A + 4 = 0 \qquad\qquad A = -0.316$$

$$y_h = Ce^{-0.316t}$$

Particular:

$$y_p = A$$

$$12.65(0) + 4A = 200 \qquad\qquad A = 50$$

Initial Conditions

$$y(t) = Ce^{-0.316t} + 50$$

$$10 = Ce^{0} + 50 \qquad\qquad C = -40$$

$$y(t) = -40e^{-0.316t} + 50$$

Figure 2.30 Example: Linearizing a differential equation

If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this. For example we could decide to recalculate the equation value after 0.1s.

$$y(0.1) = -40e^{-0.316(0.1)} + 50 = 11.24$$

$$\frac{d}{dt}y(0.1) = -40(-0.316)e^{-0.316(0.1)} = 12.25$$ Note: a small change

$$12.25y' + 4y = 20$$

Now recalculate the solution to the differential equation.

Homogeneous:

$$12.25\dot{y} + 4y = 0$$

$$12.25A + 4 = 0 \qquad A = -0.327$$

$$y_h = Ce^{-0.327t}$$

Particular:

$$y_p = A$$

$$12.25(0) + 4A = 200 \qquad A = 50$$

Initial Conditions:

$$y(t) = Ce^{-0.327t} + 50$$

$$11.24 = Ce^{0.1} + 50 \qquad C = -35.070575$$

$$y(t) = -35.07e^{-0.316t} + 50$$

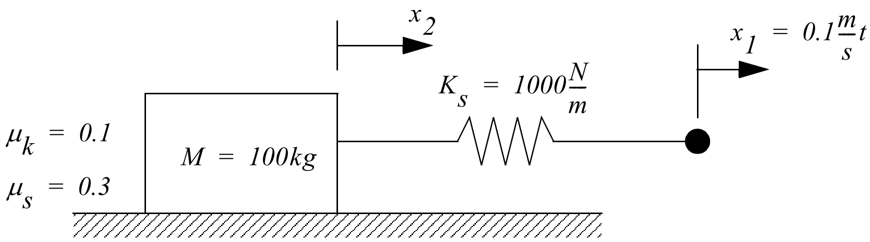
Notice that the values have shifted slightly, and as the analysis progresses the equations will adjust slowly. Higher accuracy can be obtained using smaller steps in time.

Figure 2.31 Example: Linearizing a differential equation

Changing Systems

In practical systems, the forces at work are continually changing. For example a system often experiences a static friction force when motion is starting, but once motion starts it is replaced with a smaller kinetic friction. Another example is tension in a cable. When in tension a cable acts as a spring. But, when in compression the force goes to zero.

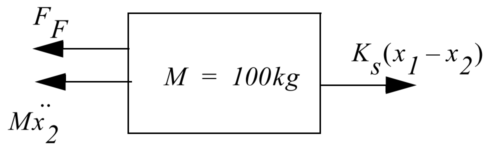
Consider the example in Figure 2.32 where a mass is pulled by an elastic cable. The right hand side of the cable is being pulled at a constant rate, while the block is free to move, only restricted by friction forces and inertia. At the beginning all components are at rest and undeflected. The solution is done by solving for each stage of motion. At the beginning there is no tension on the spring, hence no force. As x1 increase, so does the force. The mass remains stationary until the spring from the overcomes static friction and the mass begins to move. The beginning of this motion requires a new differential equation, and solution. As this type of analysis continues the end of each motion stage is used to switch to a new set of initial conditions, initial time, directions for friction forces, and new differential equation. In simple terms, each phase of motion requires the solution of a differential equation. Needless to say this can be very time consuming.



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

For the cable/spring in tension

$$x_1 - x_2 \geq 0$$



$$\sum F_x = -F_F + K_s(x_1 - x_2) = M\ddot{x}_2$$

$$-F_F + 1000\frac{N}{m}\left(0.1\frac{m}{s}t - x_2\right) = 100kg\ddot{x}_2$$

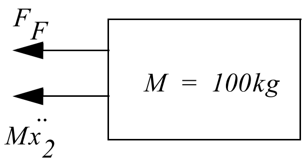
$$100kg\ddot{x}_2 + 1000\frac{N}{m}x_2 = 1000\frac{N}{m}0.1\frac{m}{s}t - F_F$$

$$\ddot{x}_2 + 10\frac{N}{kgm}x_2 = 1\frac{N}{kgs}t - \frac{F_F}{100kg}$$

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

For the cable/spring in compression

$$x_1 - x_2 < 0$$



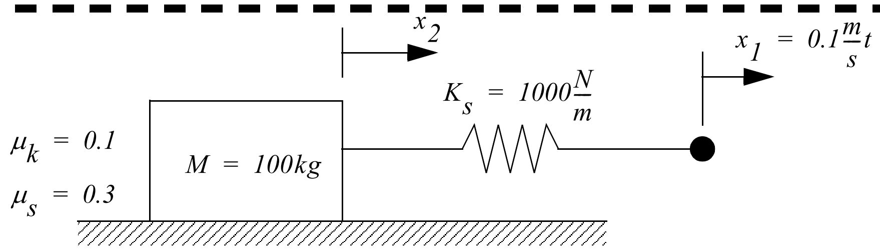
$$\sum F_x = -F_F = M\ddot{x}_2$$

$$-F_F = 100kg\ddot{x}_2$$

$$100kg\ddot{x}_2 = -F_F$$

$$\ddot{x}_2 = -\frac{F_F}{100kg}$$

Figure 2.32 Example: A differential equation for a mass pulled by a springy cable



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

$$N = 100kg9.81\frac{N}{kg} = 981N$$

static friction $\frac{d}{dt}x_2 = 0\frac{m}{s}$

$$|F_F| < (\mu_s N)$$

kinetic friction $\left|\frac{d}{dt}x_2\right| > 0\frac{m}{s}$

$$|F_F| = \mu_k N = 98.1N$$

Figure 2.33 Example: Friction forces for the mass

The analysis of the system begins by assuming the system starts at rest and undeflected. In this case the cable/spring will be undeflected with no force, and the mass will be experiencing static friction. Therefore the block will stay in place until the cable stretches enough to overcome the static friction.

$$x_2 = 0 \qquad \ddot{x}_2 = 0 \qquad F_F = 294.3N$$

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

$$0 + 10s^{-2}0 = 1\frac{m}{s^3}t - \frac{294.3N}{100kg}$$

$$1\frac{m}{s^3}t = \frac{294.3kgm}{100kgs^2}$$

$$t = 2.943s$$

Therefore the system is static from 0 to 2.943s

Figure 2.34 Example: Analysis of the object before motion begins

After motion begins the object will only experience kinetic friction, and continue to accelerate until the cable/spring becomes loose in compression. This stage of motion requires the solution of a differential equation. The value of time will be set equal to zero, t=0s, for this step of the motion. As a result the equation for the particular equation must be adjusted to have an offset of 2.93s. The time variable ‘t’ is given a subscript of 2, indicating that it is the second segment of motion.

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}(t_2 + 2.943s) - \frac{98.1N}{100kg}$$

For the homogeneous,

$$\ddot{x}_2 + 10s^{-2}x_2 = 0$$

$$A^2 + 10s^{-2} = 0 \qquad A = \pm 3.16js^{-1}$$

$$x_h = C_1 \sin(3.16t_2 + C_2)$$

For the particular,

$$x_p = At_2 + B \qquad \dot{x}_p = A \qquad \ddot{x}_p = 0$$

$$0 + 10s^{-2}(At_2 + B) = 1\frac{m}{s^3}(t_2 + 2.943s) - \frac{98.1N}{100kg}$$

$$10s^{-2}A = 1\frac{m}{s^3} \qquad A = 0.1\frac{m}{s}$$

$$10s^{-2}B = 2.93\frac{m}{s^2} - \frac{98.1N}{100kg} \qquad B = 0.1949m$$

Figure 2.35 Example: Analysis of the object after motion begins

For the initial conditions, the mass is still at zero position and velocity.

$$x(0s) = 0m \qquad \frac{d}{dt}x(0s) = 0\frac{m}{s}$$

$$x(t_2) = C_1 sin(3.16t_2 + C_2) + 0.1\frac{m}{s}(t_2 + 2.943s) + (-0.1949)m$$

$$0 = C_1 sin(3.16(0s) + C_2) + 0.1\frac{m}{s}(0s + 2.943s) + (-0.1949)m$$

$$C_1 sin(C_2) = -0.0994s$$

$$\frac{d}{dt}x(t_2) = 3.16C_1 cos(3.16t_2 + C_2) + 0.1\frac{m}{s}$$

$$0 = 3.16C_1 cos(3.16(0) + C_2) + 0.1\frac{m}{s}$$

$$C_1 cos(C_2) = -0.0316$$

$$\frac{C_1 sin(C_2)}{C_1 cos(C_2)} = \frac{-0.0994}{-0.0316} = 3.146 = tan(C_2) \qquad C_2 = 1.263$$

$$C_1 = \frac{-0.0994}{sin(1.263)} = -0.1043$$

$$x(t_2) = -0.1043 sin(3.16t_2 - 1.263) + 0.1t_2 + 0.0994$$

$$\frac{d}{dt}x(t_2) = -0.1043(3.16) cos(3.16t_2 - 1.263) + 0.1$$

Finally we write the equations, and find the point where the spring becomes slack, in compression. This is done by setting the newly found equation for ‘x’ with the original spring displacement. Note: units are now being removed to simplify calculation.

$$0.1(t_2 + 2.943) = -0.1043 sin(3.16t_2 - 1.263) + 0.1t_2 + 0.0994$$

$$sin(3.16t_2 - 1.263) = \frac{0.0994 - 0.2943}{0.1043}$$

$$t_2 = \frac{\text{asin}\left(\frac{0.0994 - 0.2943}{0.1043}\right) + 1.506}{3.16} = \text{undefined}$$

Given that the result is undefined, the spring never becomes slack. Next we check to see if the mass stops moving. In this case the result is defined so the mass stops moving at 3.7422s. The next analysis step would be to decide when it begins to slip again.

$$0 = -0.1043(3.16) cos(3.16t_2 - 1.263) + 0.1$$

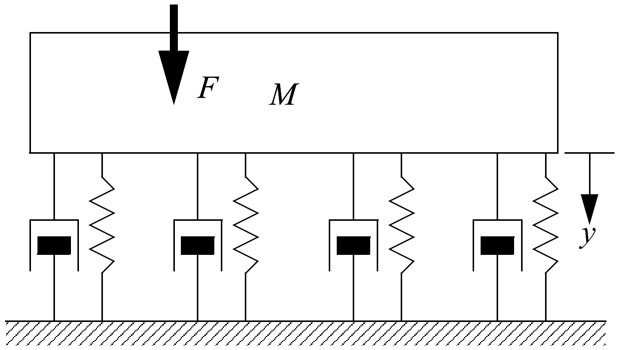
$$t_2 = \frac{\text{acos}\left(\frac{-0.1}{(-0.1043)(3.16)}\right) + 1.263}{3.16} = 0.7992$$

$$t_{sticks} = 0.7992 + 2.943 = 3.7422s$$

Figure 2.36 Example: Analysis of the object after motion begins

2.3 Case Study

A typical vibration control system design is described in Figure 2.37.



The model to the left describes a piece of reciprocating industrial equipment. The mass of the equipment is 10000kg. The equipment operates such that a force of 1000N with a frequency of 2Hz is exerted on the mass. We have been asked to design a vibration isolation mounting system. The criteria we are given is that the mounts should be 30cm high when unloaded, and 25cm when loaded with the mass. In addition, the oscillations while the machine is running cannot be more than 2cm total. In total there will be four mounts mounted around the machine. Each isolator will be composed of a spring and a damper.

Figure 2.37 Example: A vibration control system

There are a number of elements to the design and analysis of this system, but as usual the best place to begin is by developing a free body diagram, and a differential equation. This is done in Figure 2.38.

$$+\downarrow \quad \sum F_y = F - 4K_s y - 4K_d \dot{y} + Mg = M\ddot{y}$$

$$M\dot{y}' + 4K_d \dot{y} + 4K_s y = F + Mg$$

$$\ddot{y} + \frac{4K_d \dot{y}}{M} + \frac{4K_s y}{M} = \frac{F}{M} + g$$

$$\ddot{y} + \frac{4K_d \dot{y}}{10000Kg} + \frac{4K_s y}{10000Kg} = \frac{1000N}{10000Kg} sin(2(2\pi)t) + 9.81ms^{-2}$$

$$\ddot{y} + 0.0004Kg^{-1} K_d \dot{y} + 0.0004Kg^{-1} K_s y = 0.1ms^{-2} sin(4\pi t) + 9.81ms^{-2}$$

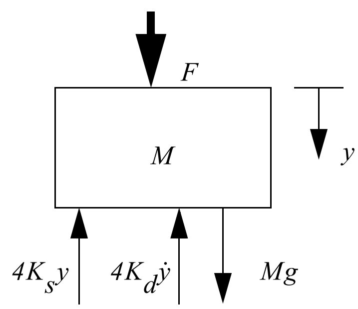


Figure 2.38 Example: FBD and derivation of equation

Using the differential equation, the spring values can be found by assuming the machine is at rest. This is done in Figure 2.39.

When the system is at rest the equation is simplified; the acceleration and velocity terms both become zero. In addition, we will assume that the cyclic force is not applied for the unloaded/loaded case. This simplifies the differential equation by eliminating several terms.

$$0.0004Kg^{-1}K_s y = 9.81ms^{-2}$$

Now we can consider that when unloaded the spring is 0.30m long, and after loading the spring is 0.25m long. This will result in a downward compression of 0.05m, in the positive y direction.

$$0.0004Kg^{-1}K_s(0.05m) = 9.81ms^{-2}$$

$$K_s = \frac{9.81}{0.0004(0.05)}Kgms^{-2}m^{-1}$$

$$\therefore K_s = 491KNm^{-1}$$

Figure 2.39 Example: Calculation of the spring coefficient

At this point we have determined the range of motion of the mass. The remaining unknown is the damping factor. This can be calculated by completing the particular solution of the differential equation and identifying the damped motion term of the equation. This calculation begins in Figure 2.40.

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 0.0004Kg^{-1}(491KNm^{-1})y = 0.1ms^{-2}sin(4\pi t) + 9.81ms^{-2}$$

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 196s^{-2}y = 0.1ms^{-2}sin(4\pi t) + 9.81ms^{-2}$$

The particular solution can now be found by guessing a value, and solving for the coefficients. (Note: The units in the expression are uniform (i.e., the same in each term) and will be omitted for brevity.)

$$y = A sin(4\pi t) + B cos(4\pi t) + C$$

$$y' = 4\pi A cos(4\pi t) - 4\pi B sin(4\pi t)$$

$$y'' = -16\pi^2 A sin(4\pi t) - 16\pi^2 B cos(4\pi t)$$

$$\therefore(-16\pi^2 A sin(4\pi t) - 16\pi^2 B cos(4\pi t)) + 0.0004K_d(4\pi A cos(4\pi t) - 4\pi B sin(4\pi t))$$
$$+ 196(A sin(4\pi t) + B cos(4\pi t) + C) = 0.1 sin(4\pi t) + 9.81$$

$$-16\pi^2 B + 0.0004K_d 4\pi A + 196A = 0$$
$$B = A\left(31.8 \times 10^{-6}K_d + 1.24\right)$$
$$-16\pi^2 A + 0.0004K_d(-4\pi B) + 196A = 0.1$$

$$A(-16\pi^2 + 196) + B\left(-5.0 \times 10^{-3}K_d\right) = 0.1$$
$$A(-16\pi^2 + 196) + A\left(31.8 \times 10^{-6}K_d + 1.24\right)\left(-5.0 \times 10^{-3}K_d\right) = 0.1$$
$$A = \frac{0.1}{-16\pi^2 + 196 + \left(31.8 \times 10^{-6}K_d + 1.24\right)\left(-5.0 \times 10^{-3}K_d\right)}$$
$$A = \frac{0.1}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$
$$B = \frac{3.18 \times 10^{-6}K_d - 0.124}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$
$$C = 9.81ms^{-2}$$

Figure 2.40 Example: Particular solution of the differential equation

The particular solution can be used to find a damping factor that will give an overall oscillation of 0.02m, as shown in Figure 2.41. In this case Mathcad was used to find the solution, although it could have also been found by factoring out the algebra, and finding the roots of the resulting polynomial.

In the previous particular solution the values were split into cosine and sine components. The magnitude of oscillation can be calculated with the Pythagorean formula.

$$magnitude = \sqrt{A^2 + B^2}$$

$$magnitude = \frac{\sqrt{(0.1)^2 + \left((3.18 \text{ Þ } 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \text{ Þ } 10^{-9}) + K_d(-(6.2 \text{ Þ } 10^{-3})) + 38.1}$$

The design requirements call for a maximum oscillation of 0.02m, or a magnitude of 0.01m.

$$0.01 = \frac{\sqrt{(0.1)^2 + \left((3.18 \text{ Þ } 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \text{ Þ } 10^{-9}) + K_d(-(6.2 \text{ Þ } 10^{-3})) + 38.1}$$

A given-find block was used in Mathcad to obtain a damper value of,

$$K_d = 3411N\frac{s}{m}$$

Aside: the Mathcad solution

$$f(k) := \frac{\sqrt{0.01 + \left[\left(3.18 \cdot 10^{-6} \cdot k\right) - 0.124\right]^2}}{\left[k \cdot k \cdot \left(-159 \cdot 10^{-9}\right)\right] + k \cdot \left(-6.2 \cdot 10^{-3}\right) + 38.1}$$

$$k_d := 1$$

given

$$f(k_d) = 0.01$$

$$find(k_d) = 3.411 \times 10^3$$

Figure 2.41 Example: Determining the damper coefficient

The values of the spring and damper coefficients can be used to select actual components. Some companies will design and build their own components. Components can also be acquired by searching catalogs, or requesting custom designs from other companies.

2.4 Summary

- First and second-order differential equations were analyzed explicitly.
- First and second-order responses were examined.
- The topic of analysis was discussed.
- A case study looked at a second-order system.
- Non-linear systems can be analyzed by making them linear.

2.5 Problems With Solutions

Problem 2.1 Solve the following first order differential equation given the initial condition.

$$\dot{x} + 2x = 0 \qquad x(0) = 3$$

Problem 2.2 *Solve the following second order homogeneous differential equation given the initial condition.*

$$\ddot{x} + 2\dot{x} + x = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 2$$

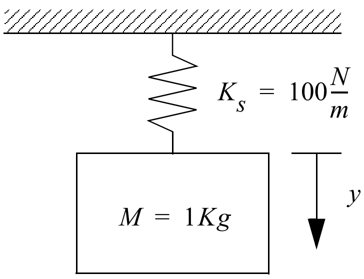
Problem 2.3 Solve the following second order non-homogeneous differential equation given the initial condition.

$$\ddot{x} + 2\dot{x} + x = 1 \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

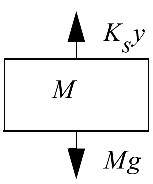
Problem 2.4 Convert the following equation to phase-shift form.

$$5\sin 6t + 7\cos 6t$$

Problem 2.5 The following differential equation was derived for a mass suspended with a spring. At time 0s the system is released and allowed to drop. It then oscillates. Solve the differential equation to find the motion as a function of time.



FBD:



$+\uparrow$

$$\sum F_y = K_s y - Mg = -M\ddot{y}$$

$$\left(100\frac{N}{m}\right)y - (1Kg)\left(9.81\frac{N}{Kg}\right) = (-1Kg)\ddot{y}$$

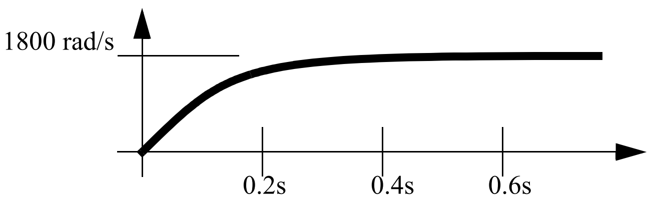
$$\left(1\frac{Nm}{s^2}\right)\ddot{y} + \left(100\frac{N}{m}\right)y = 9.81N$$

$$(1Kg)\ddot{y} + \left(100\frac{Kgm}{ms^2}\right)y = 9.81\frac{Kgm}{s^2}$$

$$\ddot{y} + (100s^{-2})y = 9.81ms^{-2}$$

$$y_0 = 0m \qquad \dot{y}_0 = 0ms^{-1}$$

Problem 2.6 *Find the differential equation equations using the response when a step input of Vs=12V is applied:*

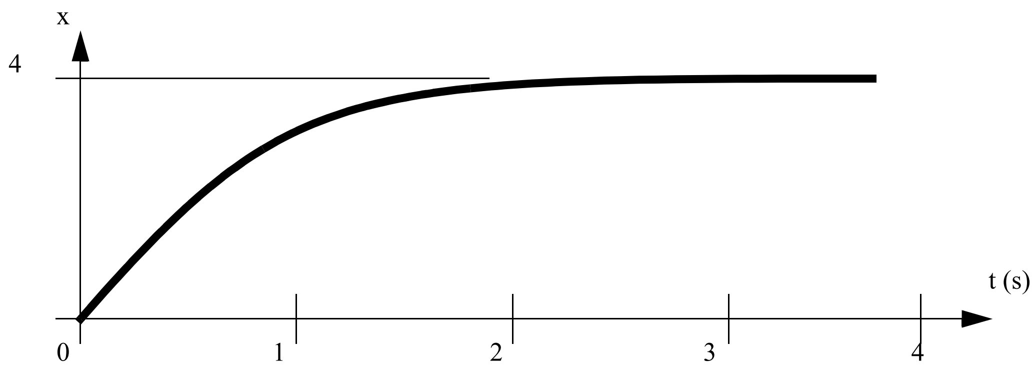


$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

Problem 2.7 Use the general form given below to find the final solution without solving the differential equation. Assume the system starts at y=-20.

$$\dot{y} + 10y = 5 \longrightarrow y(t) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

Problem 2.8 Determine the first order differential equation given the graphical response shown below. Assume the input is a step function.



Problem 2.9 What are the damping coefficient and damped frequencies for the equations.

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

$$\ddot{x}_2 = -\frac{F_F}{100kg}$$

Problem 2.10 Solve the following differential equation with the three given cases. All of the systems have a step input ‘y’ and start undeflected and at rest.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = y$$

initial conditions $\dot{x} = 0$, $x = 0$, $y = 1$

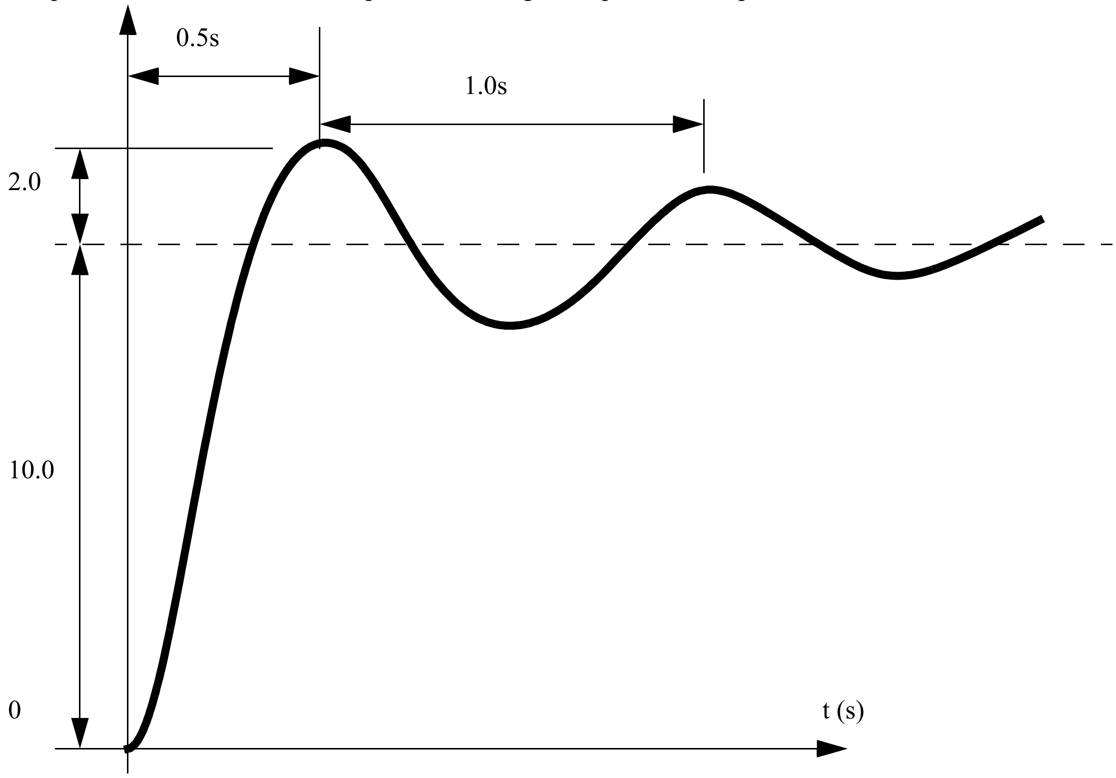
case 1: $\zeta = 0.50$ $\omega_n = 10$

case 2: $\zeta = 1.0$ $\omega_n = 10$

case 3: $\zeta = 2.0$ $\omega_n = 10$

Problem 2.11 Write the homogeneous differential equation for a second order system with the first peak at 1s and 10% over-shoot. The system variable is ‘x’.

Problem 2.12 The second order response below was obtained experimentally. Determine the parameters of the differential equation that resulted in the response assuming the input was a step function.



Problem 2.13 A system is to be approximated with a mass-spring-damper model using the following parameters: mass 28N, viscous damping 6Ns/m, and stiffness 36N/m. Calculate the undamped natural frequency (Hz) of the system, the damping ratio and describe the type of response you would expect if the mass were displaced and released. What additional damping would be required to make the system critically damped?

$$M\ddot{x} + K_d\dot{x} + K_s x = F$$

Problem 2.14 Solve the differential equation below using homogeneous and particular solutions. Assume the system starts undeflected and at rest with no acceleration.

$$\dddot{\theta} + 40\ddot{\theta} + 20\dot{\theta} + 2\theta = 4$$

Problem 2.15 Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2$$

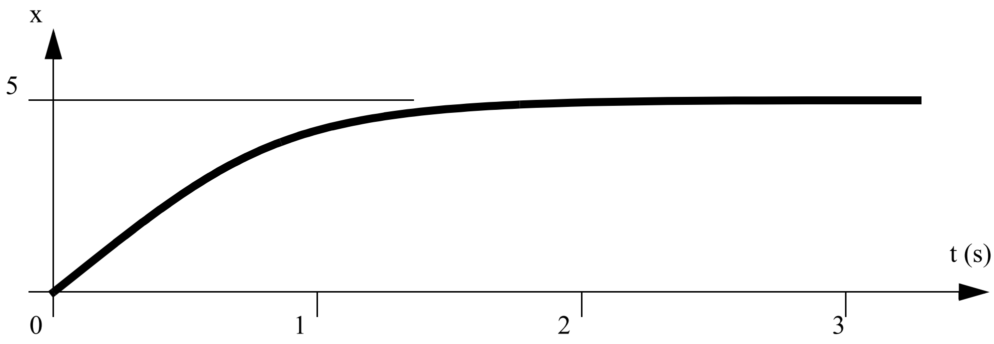
initial conditions

$$V_i = 5$$
$$V_o = 0$$
$$\dot{V}_o = 0$$

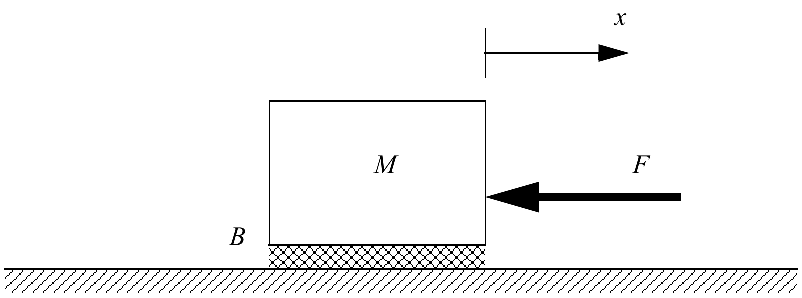
Problem 2.16 A spring mass system supports a mass of 34N. If it has a spring constant of 20.6N/cm, what is the systems natural frequency?

Problem 2.17 Using a standard lumped parameter model the weight is 36N, stiffness is 2.06*10^3 N/m and damping is 100Ns/m. What are the natural frequency (Hz) and damping ratio?

Problem 2.18 Write the differential equation for a first order system with a variable ‘x’. The system has the response shown in the graph below for an input of F=6.

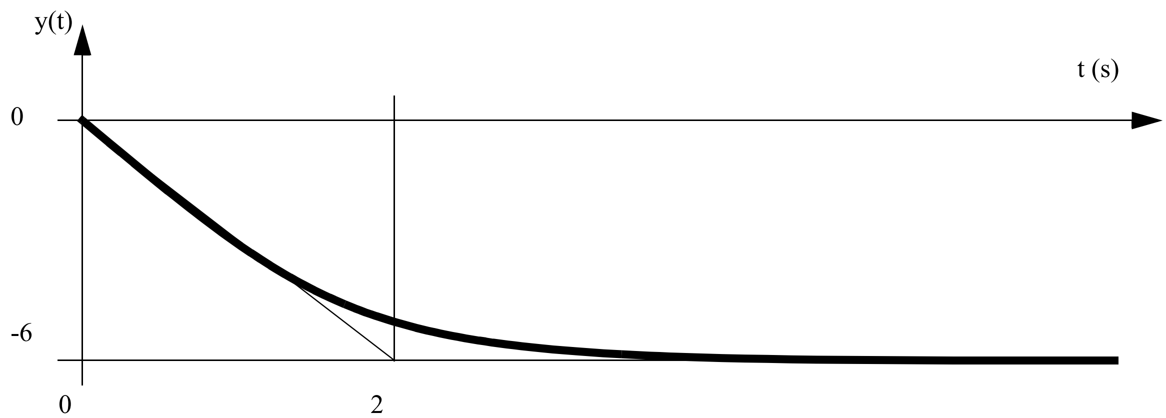


Problem 2.19 The mass, M, illustrated below starts at rest. It can slide across a surface, but the motion is opposed by viscous friction (damping) with the coefficient B. Initially the system starts at rest, when a constant force, F, is applied. Write the differential equation for the mass, and solve the differential equation. Leave the results in variable form.



Problem 2.20 Write a differential equation for a system that has a time constant of 2 s. For an input of 3, the steady state output is 6.

Problem 2.21 A system is tested with a step input of F = 1N. The resulting output 'y' is shown in the graph below. a) Find the differential equation for the system. b) Find the explicit response (i.e., solve the differential equation) for an input of F=sin(t)N.



Problem 2.22 Solve the following differential equation with the three given cases. All of the systems have a sinusoidal input 'y'

and start undeflected and at rest.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = y$$

initial conditions: $\dot{x} = 0$, $x = 0$, $y = \sin(t)$

case 1: $\zeta = 0.5$ $\omega_n = 10$

case 2: $\zeta = 1$ $\omega_n = 10$

case 3: $\zeta = 2$ $\omega_n = 10$

Problem 2.23 Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0, and the system undeflected and at rest.

$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2$$

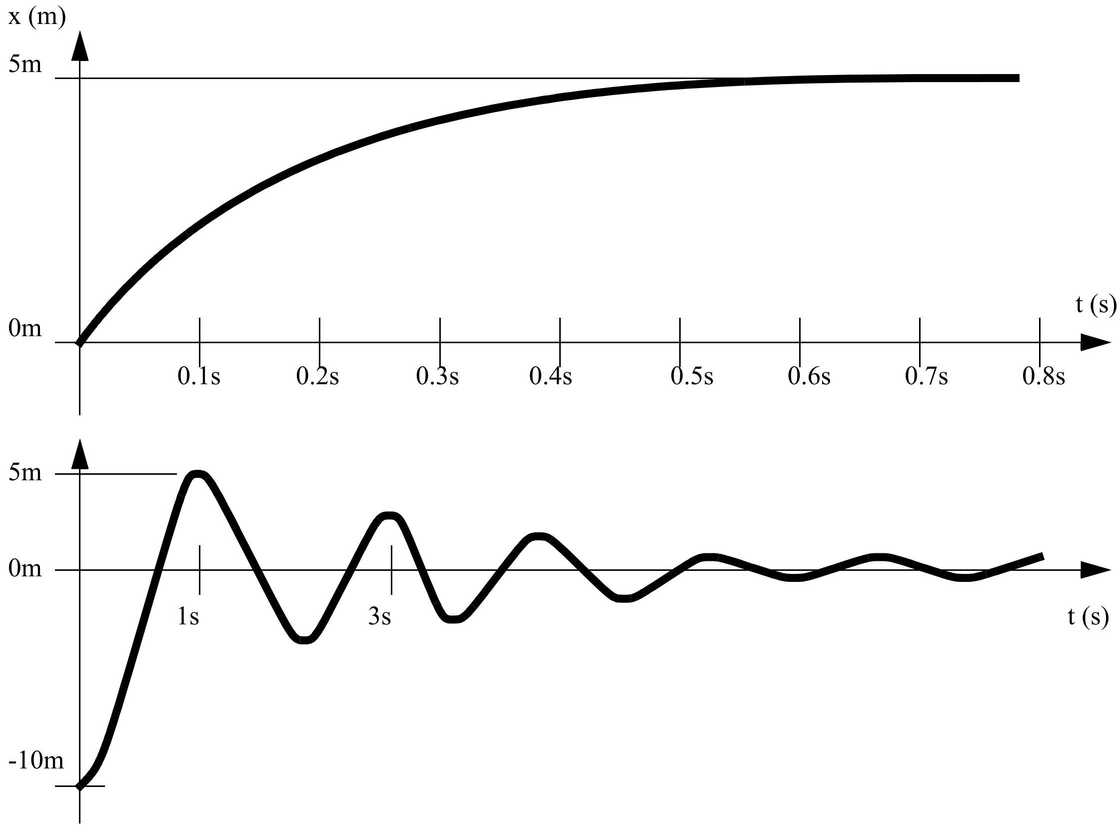
initial conditions: $V_i = 5$, $V_o = 0$, $\dot{V}_o = 1$

Problem 2.24 What is the differential equation for a second-order system that responds to a step input with an overshoot of 20%, with a delay of 0.4 seconds to the first peak?

Problem 2.25 What would the displacement amplitude after 100ms for a second order system having a natural frequency of 13 rads/sec and a damping ratio of 0.20. Assume an initial displacement of 50mm, and a steady state displacement of 0mm. The system is release from rest. (Hint: Find the response as a function of time.)

Problem 2.26 Explain with graphs how to develop first and second-order equations using experimental data.

Problem 2.27 Develop equations (function of time) for the first and second order responses shown below.



Problem 2.28 Find the explicit response of the following differential equation to the given step input. Assume the initial conditions are all zero. (Note: u(t) simply means that the value is zero before t=0.)

$$\dddot{x} + 10\ddot{x} + 100\dot{x} + 1000x = 4F \qquad F(t) = 10u(t)$$

Problem 2.29 A mass-spring-damper system has a mass of 10 kg and a spring coefficient of 1KN/m. Select a damper coefficient so that the system will have an overshoot of 20% for a step input.

2.6 Problem Solutions

Answer 2.1

$$x(t) = 3e^{-2t}$$

Answer 2.2

$$x(t) = e^{-t} + 3te^{-t}$$

Answer 2.3

$$x(t) = -e^{-t} - te^{-t} + 1$$

Answer 2.4

$$8.602\sin(6t + 0.951)$$

Answer 2.5

homogeneous: (we make a guess)

$$y_h = e^{At} \qquad \dot{y}_h = Ae^{At} \qquad \ddot{y}_h = A^2 e^{At}$$

$$A^2 e^{At} + (100s^{-2})e^{At} = 0$$

$$A^2 = -100s^{-2} \qquad A = \pm 10js^{-1}$$

$$y_h = C_1 \cos(10t + C_2)$$

Particular: (we make a guess)

$$y_p = A \qquad \dot{y}_p = 0 \qquad \ddot{y}_p = 0$$

$$(0) + (100s^{-2})A = 9.81ms^{-2}$$

$$(100s^{-2})A = 9.81ms^{-2}$$

$$A = \frac{9.81ms^{-2}}{100s^{-2}} = 0.0981m$$

$$y_p = 0.0981m$$

$$y = y_h + y_p = C_1 \cos(10t + C_2) + 0.0981m$$

$$y' = -10C_1 \sin(10t + C_2)$$

Initial Conditions:

for d/dt y0 = 0m :

$$0 = -10C_1 \sin(10(0) + C_2) \qquad C_2 = 0$$

for y0 = 0m :

$$0 = C_1 \cos(10(0) + (0)) + 0.0981m$$

$$-0.0981m = C_1 \cos(0) \qquad C_1 = -0.0981m$$

$$\boxed{y(t) = (-0.0981m)\cos(10t) + 0.0981m}$$

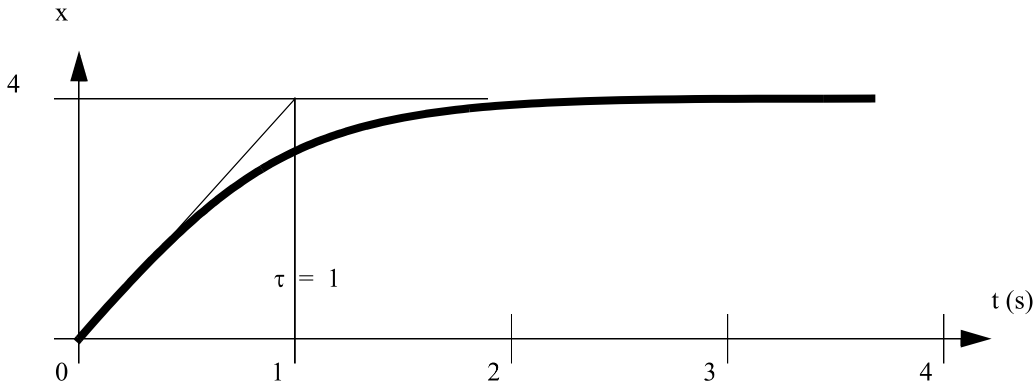
Answer 2.6

$$\left(\frac{d}{dt}\right)\omega + \frac{1}{0.15}\omega = \frac{1800}{12(0.15)}V_s$$

Answer 2.7

$$y(t) = 0.5 - 20.5e^{-10t}$$

Answer 2.8



Given the equation form,

$$\dot{x} + \frac{1}{\tau}x = A$$

The values at steady state will be

$$\dot{x} = 0 \qquad x = 4$$

So the unknown 'A' can be calculated.

$$0 + \frac{1}{1}4 = A \qquad A = 4$$

$$\dot{x} + \frac{1}{1}x = 4$$

$$\dot{x} + x = 4$$

Answer 2.9

$$\ddot{x}_2 + 10s^{-2}x_2 = \ddot{x} + \dot{x}(2\omega_n\zeta) + x(\omega_n^2)$$

$$\omega_n^2 = 10s^{-2} \qquad \omega_n = \sqrt{10}\frac{rad}{s}$$

$$2\omega_n\zeta = 0 \qquad \zeta = 0$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \omega_n = \sqrt{10}\frac{rad}{s}$$

$$\ddot{x}_2 = \ddot{x} + \dot{x}(2\omega_n\zeta) + x(\omega_n^2)$$

$$\omega_n^2 = 0 \qquad \omega_n = 0$$

$$2\omega_n\zeta = 0 \qquad \zeta = 0$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \omega_n = 0$$

Answer 2.10

case 1: $x(t) = -0.0115e^{-5t}\cos(8.66t - 0.524) + 0.010$

case 2: $x(t) = -0.010e^{-10t} - 0.10te^{-10t} + 0.010$

case 3: $x(t) = 775 \cdot 10^{-6}e^{-37.32t} - 0.0108e^{-2.679t} + 0.010$

Answer 2.11

$$\ddot{x} + 4.605\dot{x} + 15.171x = 0$$

Answer 2.12

For the first peak:

$$\frac{b}{\Delta x} = e^{-\sigma t_p}$$

$$\frac{2}{10} = e^{-\sigma 0.5} \qquad \omega_d \approx \frac{\pi}{t_p}$$

$$\ln\left(\frac{2}{10}\right) = -\sigma 0.5$$

$$\sigma = -2\ln\left(\frac{2}{10}\right) = 3.219$$

For the damped frequency:

$$\zeta = \frac{1}{\sqrt{\left(\frac{\pi}{t_p\sigma}\right)^2 + 1}}$$

$$\omega_d = \frac{2\pi}{1s} = 2\pi$$

These values can be used to find the damping factor and natural frequency

$$\sigma = \zeta\omega_n \qquad \omega_n = \frac{3.219}{\zeta}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$2\pi = \frac{3.219}{\zeta}\sqrt{1-\zeta^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 = \frac{1-\zeta^2}{\zeta^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 + 1 = \frac{1}{\zeta^2} \qquad \zeta = \sqrt{\frac{1}{\left(\frac{2\pi}{3.219}\right)^2 + 1}} = 0.4560$$

$$\omega_n = \frac{3.219}{\zeta} = \frac{3.219}{0.4560} = 7.059$$

This leads to the final equation using the steady state value of 10

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F$$

$$\ddot{x} + 2(0.4560)(7.059)\dot{x} + (7.059)^2 x = F$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = F$$

$$(0) + 6.438(0) + 49.83(10) = F \qquad F = 498.3$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = 498.3$$

Answer 2.13

Given,

$$K_d = 6N\frac{s}{m} \qquad K_s = 36\frac{N}{m} \qquad M = \frac{28N}{9.81\frac{N}{kg}} = 2.85kg$$

The typical transfer function for a mass-spring-damper systems is,

$$\ddot{x} + \dot{x}\left(\frac{K_d}{M}\right) + x\left(\frac{K_s}{M}\right) = \frac{F}{M}$$

The second order parameters can be calculated from this.

$$\ddot{x} + \dot{x}(2\zeta\omega_n) + x(\omega_n^2) = y(t)$$

$$\omega_n = \sqrt{\frac{K_s}{M}} = \sqrt{\frac{36\frac{N}{m}}{2.85kg}} = \sqrt{\frac{36\frac{kgm}{ms^2}}{2.85kg}} = \sqrt{12.63s^{-2}} = 3.55\frac{rad}{s} = 0.6Hz$$

$$\zeta = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{6N\frac{s}{m}}{2(3.55)\frac{rad}{s}2.85kg} = 0.296$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.39\frac{rad}{s}$$

If pulled and released the system would have a decaying oscillation about 0.54Hz

A critically damped system would require a damper coefficient of....

$$\zeta = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{K_d}{2(3.55)\frac{rad}{s}2.85kg} = 1.00 \qquad K_d = 20.2\frac{Ns}{m}$$

Answer 2.14

$$\dddot{\theta} + 40\ddot{\theta} + 20\dot{\theta} + 2\theta = 4$$

Homogeneous

$$\theta_h(t) = C_1 e^{-39.495t} + C_2 e^{-0.1379t} + C_3 e^{-0.3672t}$$

Particular

$$\theta_p(t) = 2.0$$

Initial Conditions are zero

$$\theta(t) = C_1 e^{-39.495t} + C_2 e^{-0.1379t} + C_3 e^{-0.3672t} + 2.0$$

$$C_1 + C_2 + C_3 = -2.0$$

$$\dot{\theta}(t) = -39.495 C_1 e^{-39.495t} - 0.1379 C_2 e^{-0.1379t} - 0.3672 C_3 e^{-0.3672t}$$

$$-39.495 C_1 - 0.1379 C_2 - 0.3672 C_3 = 0$$

$$\ddot{\theta}(t) = (39.495)^2 C_1 e^{-39.495t} + (0.1379)^2 C_2 e^{-0.1379t} + (0.3672)^2 C_3 e^{-0.3672t}$$

$$(39.495)^2 C_1 + (0.1379)^2 C_2 + (0.3672)^2 C_3 = 0$$

In Scilab:
A = [1, 1, 1;-39.495] ; [-0.1379, -0.3672] ; [(-39.495)^2, (-0.1379)^2, (-0.3672)^2]
B = [-2 ; 0; 0]
inv(A)*B

$$\theta(t) = -65.8\times10^{-6} e^{-39.495t} + (-3.214) e^{-0.1379t} + (1.214) e^{-0.3672t} + 2.0$$

Answer 2.15

$$V_o(t) = -8.465 e^{-0.6t} \sin(1.960t + 1.274) + 8.095$$

or

$$V_o(t) = -8.465 e^{-0.6t} \cos(1.960t - 0.2971) + 8.095$$

Answer 2.16 24.37 rad/sec

Answer 2.17 fn=3.77Hz, damp.=0.575

Answer 2.18

$$\dot{x} + \frac{1}{0.9} x = 0.926F$$

Answer 2.19

$$x(t) = \frac{-FM}{B^2}e^{-\frac{B}{M}t} - \frac{F}{B}t + \frac{FM}{B^2}$$

Answer 2.20

$$\dot{V}_o + \frac{1}{\tau}V_o = CV_i$$

for steady state,

$$(0) + \frac{1}{2}(6) = C(3) \qquad \therefore C = 1$$

$$\dot{V}_o + 0.5V_o = V_i$$

Answer 2.21

a) $$\dot{y} + 0.5y = -3F$$

b) $$y(t) = 2.40e^{-0.5t} + 2.683\sin(t - 1.107)$$

Answer 2.22

case 1: $$x(t) = -0.00117e^{-5t}\sin(8.66t - 1.061) + 0.0101\sin(t - 0.101)$$

case 2: $$x(t) = (1.96 \cdot 10^{-3})e^{-10t} + (9.9 \cdot 10^{-3})te^{-10t} + (9.9 \cdot 10^{-3})\sin(t - 0.20)$$

case 3: $$x(t) = (3.5 \cdot 10^{-3})e^{-2.679t} - (18 \cdot 10^{-6})e^{-37.32t} + (9.4 \cdot 10^{-3})\sin(t - 0.382)$$

Answer 2.23

$$V_0(t) = -8.331e^{-0.6t}\cos(1.96t - 0.238) + 8.095$$

Answer 2.24

$$\ddot{x} + 8.048\dot{x} + 77.88x = F(t)$$

Answer 2.25

$$y(t) = 0.0510e^{-2.6t}\cos(12.74t - 0.201)$$

$$y(0.1\text{s}) = 0.0188m$$

Answer 2.26 Key points:
First-order: find initial final values, find time constant with 63% or by slope, and use these in standard equation
Second-order: find damped frequency from graph, find time to first peak, and use these in cosine equation

Answer 2.27

$$x(t) = 5\left(1 - e^{\frac{-t}{0.18}}\right) \qquad x(t) = -10.24e^{-0.693t}\cos(\pi t - 0.217)$$

Answer 2.28

$$x(t) = 0.040 - 0.020e^{-10t} - \frac{0.040}{\sqrt{2}}\sin\left(10t + \frac{\pi}{4}\right)$$

Answer 2.29

Given,

$$M = 10kg \qquad K_s = 1000\frac{N}{m} \qquad \frac{b}{\Delta x} = 0.2$$

The differential equation for a mass spring damper system with known terms is,

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \ddot{y} + \left(\frac{K_d}{M}\right)\dot{y} + \left(\frac{K_s}{M}\right)y = \ddot{y} + \left(\frac{K_d}{10}\right)\dot{y} + (100)y$$

$$\omega_n = \sqrt{100} = 10\frac{rad}{s} \qquad K_d = M2\zeta\omega_n = 20,000\zeta$$

The overshoot can be used with the approximation equations to find the damping factor.

$$\frac{b}{\Delta x} = e^{-\sigma t_p} = 0.2 \qquad \sigma t_p = -\ln(0.2)$$

$$\zeta = \frac{1}{\sqrt{\left(\frac{\pi}{t_p\sigma}\right)^2 + 1}} = \frac{1}{\sqrt{(-\ln(0.2))^2 + 1}} = 0.5278$$

The overshoot can be used with the approximation equations to find the damping factor.

$$\frac{b}{\Delta x} = e^{-\sigma t_p} = 0.2 \qquad \sigma t_p = -\ln(0.2)$$

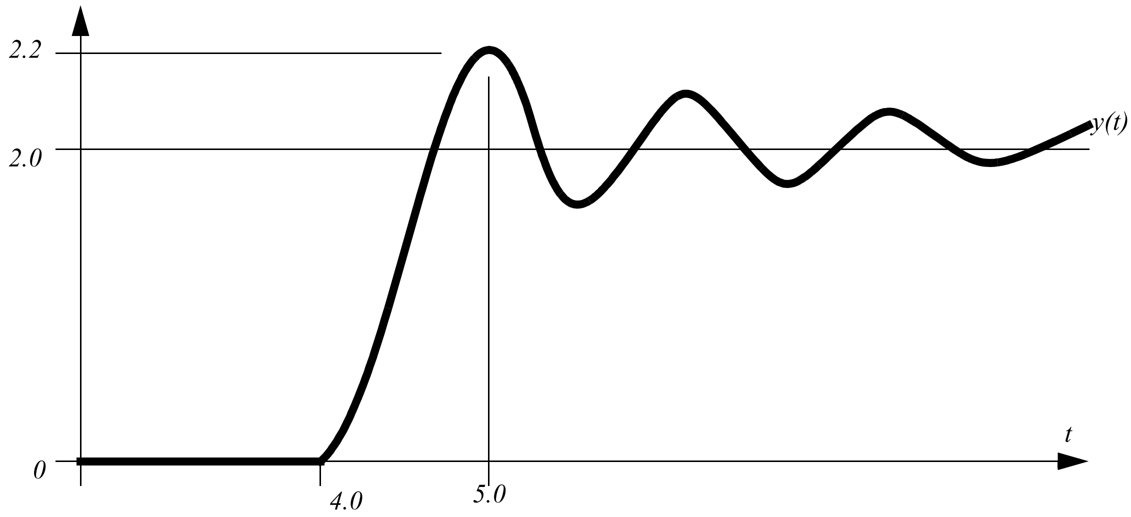
$$K_d = 20,000\zeta = 10556\frac{Ns}{m}$$

2.7 Problems Without Solutions

Problem 2.30 Solve the following differential equation to obtain an explicit function of time. Assume the equation describes a system that starts at rest and undeflected.

$$\ddot{x} + 10\dot{x}^2 = 10$$

Problem 2.31 Write a function of time for the graph. (Note: measure, using a ruler, to get values.) Find the natural frequency and damping factor to develop the differential equation. Using the dashed lines determine the settling time.



$$t < 4 \qquad y(t) = 0$$

$$t \geq 4 \qquad y(t) =$$

2.8 Review of Basic Algebra

• Although well known, it is easy to make mistakes with simple operations. This is more true when the methods have not been used in a long while.

• These operations are generally universal, and are described in sufficient detail for our use.

• Basic properties include,

commutative $\qquad a + b = b + a$

distributive $\qquad a(b + c) = ab + ac$

associative $\qquad a(bc) = (ab)c \qquad a + (b + c) = (a + b) + c$

Figure 2.42 Basic algebra properties

• The quadratic equation appears in almost every engineering discipline, therefore is of great importance.

$$ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 2.43 Quadratic equation

Problems

Problem 2.32 Are the following expressions equivalent?

a) $A(5+B)-C = 5A+B-C$

b) $\frac{A+B}{C+D} = \frac{A}{C}+\frac{B}{D}$

c) $5(5^4) = 5^5$

d) $(x+6)(x-6) = x^2+36$

e) $\sqrt{\frac{(x+1)^6}{(x+1)^2}} = x^2+2x+1$

Problem 2.33 Simplify the following expressions.

a) $x(x+2)^2-3x$

b) $\frac{(x+3)(x+1)x^2}{(x+1)^2 x}$

c) $\frac{64}{16}$

d) $\frac{15}{21}+\frac{3}{28}$

e) $(x^2y^3)^4$

f) $\sqrt{4x^2-8y^4}$

g) $\frac{5}{3}\left(\frac{8}{9}\right)$

h) $\frac{\left(\frac{5}{4}\right)}{5}$

i) $(y+4)^3(y-2)$

j) $\sqrt{x^2y}$

k) $\frac{x+1}{x+2} = 4$

Problem 2.34 Simplify the following expressions.

a) $\frac{A+B}{AB}$

b) $\frac{AB}{A+B}$

c) $\left(\frac{(x^4y^5)}{x^2}\right)^3$

Answer 2.34

ans: a) $\frac{A+B}{AB} = \frac{A}{AB}+\frac{B}{AB} = \frac{1}{B}+\frac{1}{A}$

b) $\frac{AB}{A+B}$ cannot be simplified

c) $\left(\frac{(x^4y^5)}{x^2}\right)^3 = (x^2y^5)^3 = x^6y^{15}$

Problem 2.35 Rearrange the following equation so that only 'y' is on the left hand side.

$$\frac{y+x}{y+z} = x+2$$

Answer 2.35

ans:

$$\frac{y+x}{y+z} = x+2$$

$$y+x = (x+2)(y+z)$$

$$y+x = xy+xz+2y+2z$$

$$y-xy-2y = xz+2z-x$$

$$y(-x-1) = xz+2z-x$$

$$y = \frac{xz+2z-x}{-x-1}$$

Problem 2.36 Solve the following equation to find 'x'.

$$2x^2+8x = -8$$

Answer 2.36

ans:

$$2x^2+8x = -8$$

$$x^2+4x+4 = 0$$

$$(x+2)^2 = 0$$

$$x = -2, -2$$

Problem 2.37 Manipulate the following equation to solve for 'x'.

$$x^2+3x = -2$$

Answer 2.37

$$x^2+3x = -2$$

$$x^2+3x+2 = 0$$

$$x = \frac{-3\pm\sqrt{3^2-4(1)(2)}}{2(1)} = \frac{-3\pm\sqrt{9-8}}{2} = \frac{-3\pm 1}{2} = -1, -2$$

2.9 Trigonometry Review

• The basic trigonometry functions are,

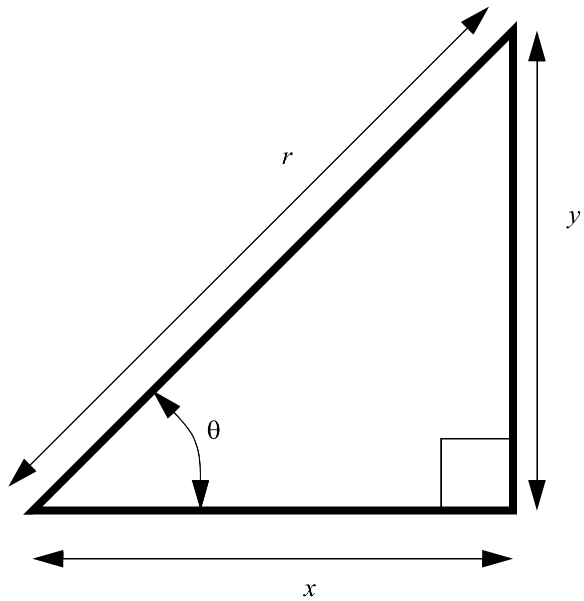
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

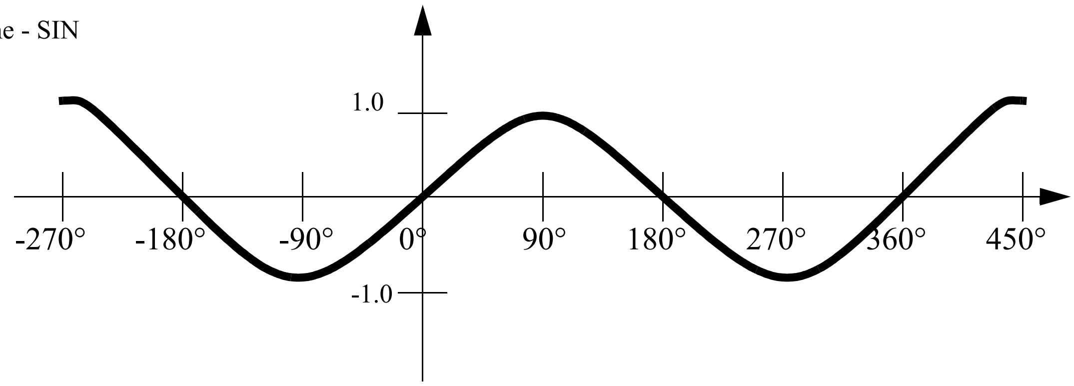
Pythagoreans formula:

$$r^2 = x^2 + y^2$$

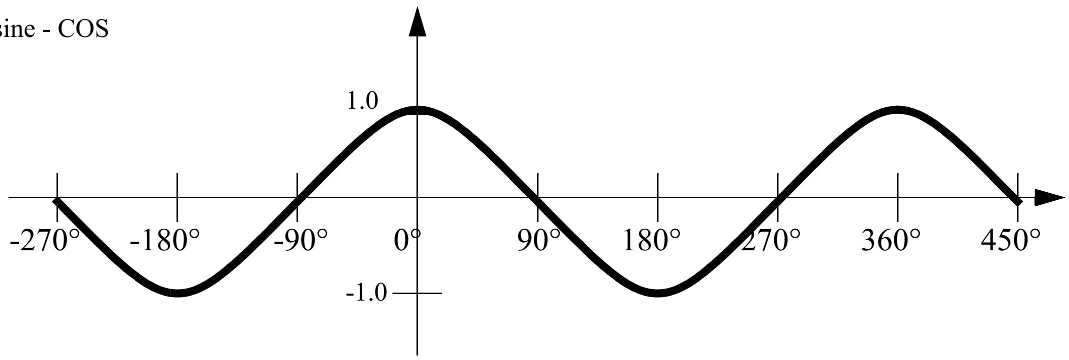


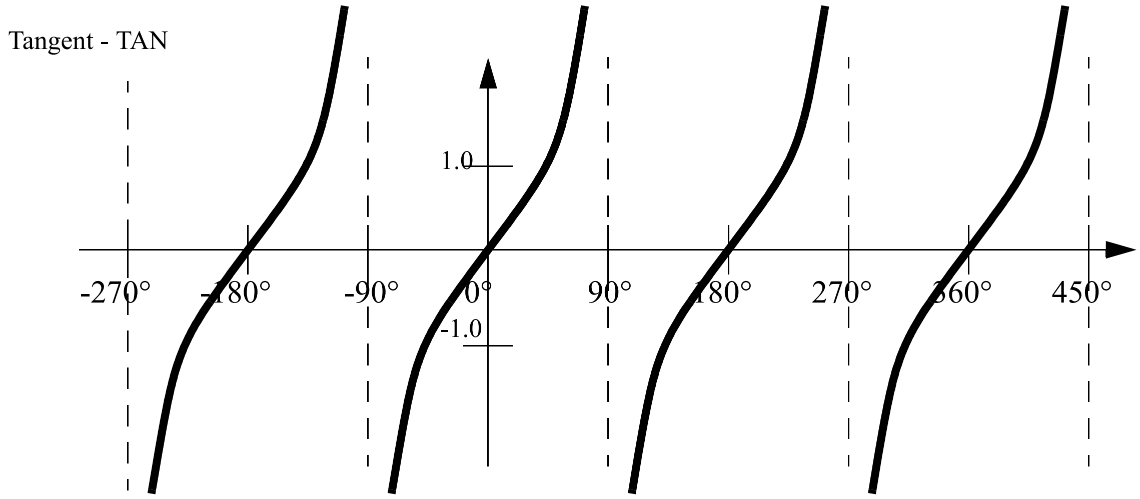
• Graphs of these functions are given below,

Sine - SIN



Cosine - COS





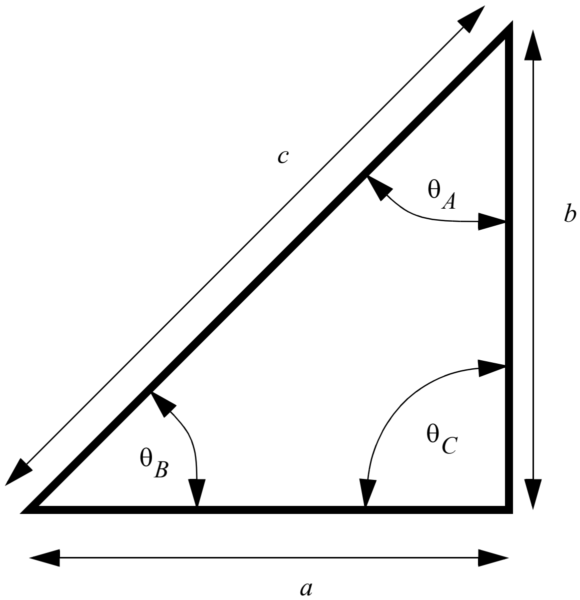
• NOTE: Keep in mind when finding these trig values, that any value that does not lie in the right hand quadrants of Cartesian space, may need additions of ±90° or ±180°.

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$

Sine Law:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}$$



• Now a group of trigonometric relationships will be given. These are often best used when attempting to manipulate equations.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\theta = \cos(\theta - 90°) = \cos(90° - \theta) = \text{etc.}$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2 \quad \text{OR} \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2 \quad \text{OR} \quad \cos(2\theta) = (\cos\theta)^2 - (\sin\theta)^2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$$

$$\cot(\theta_1 \pm \theta_2) = \frac{\cot\theta_1 \cot\theta_2 \mp 1}{\tan\theta_2 \pm \tan\theta_1}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

-ve if in left hand quadrants

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

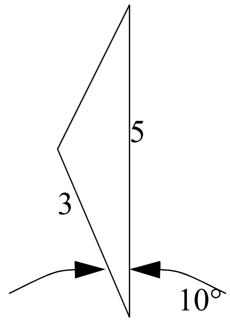
$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

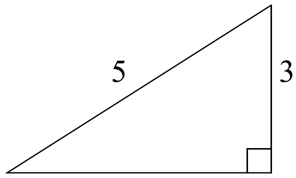
• Numerical values for these functions are given below.

θ (deg)	$\sin\theta$	$\cos\theta$	$\tan\theta$
-90	-1.0	0.0	-infinity
-60	-0.866	0.5	
-45	-0.707	0.707	-1
-30	-0.5	0.866	
0	0	1	0
30	0.5	0.866	
45	0.707	0.707	1
60	0.866	0.5	
90	1.0	0.0	infinity

Problems

Problem 2.38 Find all of the missing side lengths and corner angles on the two triangles below.





Problem 2.39 Simplify the following expression.

$$\cos\theta\cos\theta - \sin\theta\sin\theta =$$

Answer 2.39

$$\cos\theta\cos\theta - \sin\theta\sin\theta = \cos(\theta + \theta) = \cos(2\theta)$$

Problem 2.40 Manipulate the following equation to solve for 'x'.

$$\sin x = \cos x$$

Answer 2.40

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \operatorname{atan} 1$$

$$x = \ldots, -135°, 45°, 225°, \ldots$$

Problem 2.41 Simplify the following expression.

$$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right)$$

Answer 2.41

$$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right) = (\cos 2\theta)^2 + (\sin 2\theta)^2 = 1$$

2.10 Review of Basic Calculus

• NOTE: Calculus is very useful when looking at real systems. Many students are turned off by the topic because they "don't get it". But, the secret to calculus is to remember that there is no single "truth" - it is more a loose collection of tricks and techniques. Each one has to be learned separately, and when needed you must remember it, or know where to look.

Differentiation

• The basic principles of differentiation are,

Both u, v and w are functions of x, but this is not shown for brevity. Also note that C is used as a constant, and all angles are in radians.

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(Cu) = (C)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u + v + \ldots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \ldots$$

$$\frac{d}{dx}(u^n) = (nu^{n-1})\frac{d}{dx}(u)$$

$$\frac{d}{dx}(uv) = (u)\frac{d}{dx}(v) + (v)\frac{d}{dx}(u)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v}{v^2}\right)\frac{d}{dx}(u) - \left(\frac{u}{v^2}\right)\frac{d}{dx}(v)$$

$$\frac{d}{dx}(uvw) = (uv)\frac{d}{dx}(w) + (uw)\frac{d}{dx}(v) + (vw)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y)\frac{d}{dx}(u) = \text{chain rule}$$

$$\frac{d}{dx}(u) = \frac{1}{\frac{d}{du}(x)}$$

$$\frac{d}{dx}(y) = \frac{\frac{d}{du}(y)}{\frac{d}{du}(x)}$$

• Differentiation rules specific to basic trigonometry and logarithm functions

$$\frac{d}{dx}(\sin u) = (\cos u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cos u) = (-\sin u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tan u) = \left(\frac{1}{\cos u}\right)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(e^u) = (e^u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\cot u) = (-\csc u)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sec u) = (\tan u \sec u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\csc u) = (-\csc u \cot u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tanh u) = (\text{sech}\, u)^2\frac{d}{dx}(u)$$

• L'Hospital's rule can be used when evaluating limits that go to infinity.

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{\left(\frac{d}{dt}\right)f(x)}{\left(\frac{d}{dt}\right)g(x)}\right) = \lim_{x\to a}\left(\frac{\left(\frac{d}{dt}\right)^2 f(x)}{\left(\frac{d}{dt}\right)^2 g(x)}\right) = \ldots$$

• Some techniques used for finding derivatives are,

Leibnitz's Rule, (notice the form is similar to the binomial equation) can be used for finding the derivatives of multiplied functions.

$$\left(\frac{d}{dx}\right)^n(uv) = \left(\frac{d}{dx}\right)^0(u)\left(\frac{d}{dx}\right)^n(v) + \binom{n}{1}\left(\frac{d}{dx}\right)^1(u)\left(\frac{d}{dx}\right)^{n-1}(v)$$

$$\binom{n}{2}\left(\frac{d}{dx}\right)^2(u)\left(\frac{d}{dx}\right)^{n-2}(v) + \ldots + \binom{n}{n}\left(\frac{d}{dx}\right)^n(u)\left(\frac{d}{dx}\right)^0(v)$$

Integration

• Some basic properties of integrals include,

In the following expressions, u, v, and w are functions of x. in addition to this, C is a constant. and all angles are radians.

$$\int C dx = ax + C$$

$$\int Cf(x)dx = C\int f(x)dx$$

$$\int (u + v + w + \ldots)dx = \int u dx + \int v dx + \int w dx + \ldots$$

$$\int u dv = uv - \int v du = \text{integration by parts}$$

$$\int f(Cx)dx = \frac{1}{C}\int f(u)du \qquad u = Cx$$

$$\int F(f(x))dx = \int F(u)\frac{d}{du}(x)du = \int \frac{F(u)}{f'(x)}du \qquad u = f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int e^x dx = e^x + C$$

• Some of the trigonometric integrals are,

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int (\sin x)^2 dx = -\frac{\sin x \cos x + x}{2} + C$$

$$\int (\cos x)^2 dx = \frac{\sin x \cos x + x}{2} + C$$

$$\int (\sin x)^3 dx = -\frac{\cos x((\sin x)^2 + 2)}{3} + C$$

$$\int (\cos x)^3 dx = \frac{\sin x((\cos x)^2 + 2)}{3} + C$$

$$\int (\cos x)^4 dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\int \cos x(\sin x)^n dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \tanh x dx = \ln(\cosh x) + C$$

$$\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

• Some other integrals of use that are basically functions of x are,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int (a + bx)^{-1} dx = \frac{\ln|a + bx|}{b} + C$$

$$\int (a + bx^2)^{-1} dx = \frac{1}{2\sqrt{(-b)a}} \ln\left(\frac{\sqrt{a} + 2\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}\right) + C, a > 0, b < 0$$

$$\int x(a + bx^2)^{-1} dx = \frac{\ln(bx^2 + a)}{2b} + C$$

$$\int x^2(a + bx^2)^{-1} dx = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \operatorname{atan}\left(\frac{x\sqrt{ab}}{a}\right) + C$$

$$\int (a^2 - x^2)^{-1} dx = \frac{1}{2a} \ln\left(\frac{a + x}{a - x}\right) + C, a^2 > x^2$$

$$\int (ax + b)^{-1} dx = \frac{1}{a} \ln(ax + b) + C$$

$$\int x(x^2 \pm a^2)^{-\frac{1}{2}} dx = \sqrt{x^2 \pm a^2} + C$$

$$\int (a + bx + cx^2)^{-1} dx = \frac{1}{\sqrt{c}} \ln\left[\sqrt{a + bx + cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C, c > 0$$

$$\int (a + bx + cx^2)^{-1} dx = \frac{1}{\sqrt{-c}} \operatorname{asin}\left[\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right] + C, c < 0$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int x(a+bx)^{\frac{1}{2}}dx = -\frac{2(2a-3bx)(a+bx)^{\frac{3}{2}}}{15b^2}$$

$$\int (1+a^2x^2)^{\frac{1}{2}}dx = \frac{x(1+a^2x^2)^{\frac{1}{2}} + \dfrac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{a}}{2}$$

$$\int x(1+a^2x^2)^{\frac{1}{2}}dx = \frac{a\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}}}{3}$$

$$\int x^2(1+a^2x^2)^{\frac{1}{2}}dx = \frac{ax}{4}\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}} - \frac{8}{8a^2}x(1+a^2x^2)^{\frac{1}{2}} - \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{8a^3}$$

$$\int (1-a^2x^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x(1-a^2x^2)^{\frac{1}{2}} + \frac{\mathrm{asin}(ax)}{a}\right]$$

$$\int x(1-a^2x^2)^{\frac{1}{2}}dx = -\frac{a}{3}\left(\frac{1}{a^2}-x^2\right)^{\frac{3}{2}}$$

$$\int x^2(a^2-x^2)^{\frac{1}{2}}dx = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{8}\left[x(a^2-x^2)^{\frac{1}{2}} + a^2\,\mathrm{asin}\left(\frac{x}{a}\right)\right]$$

$$\int (1+a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\ln\left[x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right]$$

$$\int (1-a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\mathrm{asin}(ax) = -\frac{1}{a}\mathrm{acos}(ax)$$

• Integrals using the natural logarithm base ‘e’,

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

Problems

Problem 2.42 Find the derivative of the function below with respect to time.

$$\frac{3t}{(2+t)^2} + e^{2t}$$

Answer 2.42

$$\left(\frac{d}{dt}\right)\left(\frac{3t}{(2+t)^2} + e^{2t}\right) = \left(\frac{d}{dt}\right)(3t(2+t)^{-2}) + \left(\frac{d}{dt}\right)(e^{2t}) = 3(2+t)^{-2} - 6t(2+t)^{-3} + 2e^{2t}$$

Problem 2.43 Find the following derivatives.

a) $\frac{d}{dt}(\sin t + \cos t)$

b) $\frac{d}{dt}((t+2)^{-2})$

c) $\frac{d}{dt}(5te^{8t})$

d) $\frac{d}{dt}(5\ln t)$

Answer 2.43

a) $\frac{d}{dt}(\sin t + \cos t) = \frac{d}{dt}(\sin t) + \frac{d}{dt}(\cos t) = \cos t - \sin t$

b) $\frac{d}{dt}((t+2)^{-2}) = -2(t+2)^{-3}$

c) $\frac{d}{dt}(5te^{8t}) = 5e^{8t} + 40te^{8t}$

d) $\frac{d}{dt}(5\ln t) = \frac{5}{t}$

Problem 2.44 Find the following integrals

a) $\int 6t^2 dt$

b) $\int 14e^{7t} dt$

c) $\int \sin(0.5t) dt$

d) $\int \frac{5}{x} dx$

Answer 2.44

a) $$\int 6t^2 dt = 6\left(\frac{t^3}{3}\right) = 2t^3 + C$$

b) $$\int 14e^{7t} dt = 14\left(\frac{e^{7t}}{7}\right) + C = 2e^{7t} + C$$

c) $$\int \sin(0.5t)dt = \frac{-\cos(0.5t)}{0.5} + C = -2\cos(0.5t) + C$$

d) $$\int \frac{5}{x} dx = 5\ln(x) + C$$

Problem 2.45 Find the following derivative.

$$\frac{d}{dt}(5te^{4t} + (t+4)^{-3})$$

Problem 2.46 Find the following derivatives.

a) $$\frac{d}{dx}\left(\frac{1}{x+1}\right)$$

b) $$\frac{d}{dt}(e^{-t}\sin(2t-4))$$

Answer 2.46

a) $$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}$$

b) $$\frac{d}{dt}(e^{-t}\sin(2t-4)) = -e^{-t}\sin(2t-4) + 2e^{-t}\cos(2t-4)$$

Problem 2.47 Solve the following integrals.

a) $$\int e^{2t} dt$$

b) $$\int (\sin\theta + \cos 3\theta)d\theta$$

Answer 2.47

a) $$\int e^{2t} dt = 0.5e^{2t} + C$$

b) $$\int (\sin\theta + \cos 3\theta)d\theta = -\cos\theta + \frac{1}{3}\sin 3\theta + C$$

Problem 2.48 Find the response x' + 5x = 0 if x(0) = 4.

$$\dot{x} + 5x = 0 \qquad x(0) = 4$$

Answer 2.48

$$\dot{x} + 5x = 0 \qquad x(0) = 4$$

Homogeneous:

$$x_h = Ce^{At}$$

$$CAe^{At} + 5(Ce^{At}) = 0 \qquad A = -5$$

$$x_h = Ce^{-5t}$$

Initial Conditions:

$$x(t) = Ce^{-5t} + \text{no particular solution}$$

$$x(0) = Ce^{-5(0)} = 4 \qquad C = 4$$

$$\boxed{x(t) = 4e^{-5t}}$$

Problem 2.49 Find the homogeneous solution for x'' + 4x' + 20x = 0

$$\ddot{x} + 4\dot{x} + 20x = 0$$

Answer 2.49

$$\ddot{x} + 4\dot{x} + 20x = 0$$

Homogeneous:

$$x_h = Ce^{At}$$

$$CA^2e^{At} + 4CAe^{At} + 20Ce^{At} = 0$$

$$A^2 + 4A + 20 = 0 \qquad A = -2 \pm 4j$$

$$\boxed{x_h = C_1e^{-2t}\sin(4t + C_2)}$$

3. Numerical Analysis

Topic 3.1 *State variable form for differential equations.*
Topic 3.2 *Numerical integration with software and calculators.*
Topic 3.3 *Numerical integration theory: first-order, Taylor series and Runge-Kutta.*
Topic 3.4 *Using tabular data.*
Topic 3.5 *A design case.*
Objective 3.1 *To be able to solve systems of differential equations using numerical methods.*

For engineering analysis it is always preferable to develop explicit equations that include symbols, but this is not always practical. In cases where the equations are too costly to develop, numerical methods can be used. As their name suggests, numerical methods use numerical calculations (i.e., numbers not symbols) to develop a unique solution to a differential equation. The solution is often in the form of a set of numbers, or a graph. This can then be used to analyze a particular design case. The solution is often returned quickly so that trial and error design techniques may be used. But, without a symbolic equation the system can be harder to understand and manipulate.

This chapter focuses on techniques that can be used for numerically integrating systems of differential equations.

3.1 The General Method

The general process of analyzing systems of differential equations involves first putting the equations into standard form, and then integrating these with one of a number of techniques. State variable equations are the most common standard equation form. In this form all of the equations are reduced to first-order differential equations. These first-order equations are then easily integrated to provide a solution for the system of equations.

State Variable Form

$$\left(\frac{d}{dt}\right)x = Ax + Bu \qquad \text{state variable equation}$$

$$y = Cx + Du \qquad \text{output equation}$$

where,

x = state/output vector (variables such as position)

u = input vector (variables such as input forces)

A = transition matrix relating outputs/states

B = matrix relating inputs to outputs/states

y = non-state value that can be found directly (i.e. no integration)

C = transition matrix relating outputs/states

D = matrix relating inputs to outputs/states

Figure 3.1 The general state variable form

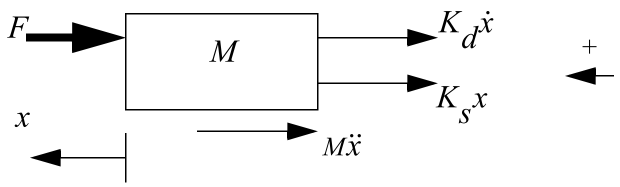
At any time a system can be said to have a state. Consider a car for example, the state of the car is described by its position and velocity. Factors that are useful when identifying state variables are:

- The variables should describe energy storing elements (potential or kinetic).
- The variables must be independent.
- They should fully describe the system elements.

After the state variables of a system have been identified, they can be used to write first-order state variable equations. The general form of state variable equations is shown in Figure 3.1. Notice that the state variable equation is linear, and the value of x is used to calculate the derivative. The output equation is not always required, but it can be used to calculate new output values.

An example of a state variable equation is shown in Figure 3.2. As always, the FBD is used to develop the differential equation. The resulting differential equation is second-order, but this must be reduced to first-order. Using the velocity variable, 'v' the second-order differential equation can be reduced to a first-order equation. An equation is also required to define the velocity as the first derivative of the position, 'x'. In the example the two state equations are manipulated into a matrix form. This form can be useful, and may be required for determining a solution. For example, HP calculators require the matrix form, while TI calculators use the equation forms. Software such as Mathcad can use either form. The main disadvantage of the matrix form is that it will only work for linear differential equations.

Given the FBD shown below, the differential equation for the system is,



$$\sum F_x = -F - K_d\dot{x} - K_s x = M\ddot{x}$$

$$-F - K_d\dot{x} - K_s x = M\ddot{x}$$

The equation is second-order, so two state variables will be needed. One obvious choice for a state variable in this equation is 'x'. The other choice can be the velocity, 'v'. Equation (1) defines the velocity variable. The velocity variable can then be substituted into the differential equation for the system to reduce it to first-order.

$$\dot{x} = v \qquad \text{eqn 3.1}$$

$$M\ddot{x} = -F - K_d\dot{x} - K_s x$$

$$M\dot{v} = -F - K_d v - K_s x$$

$$\dot{v} = x\left(\frac{-K_s}{M}\right) + v\left(\frac{-K_d}{M}\right) + \left(\frac{-F}{M}\right) \qquad \text{eqn 3.2}$$

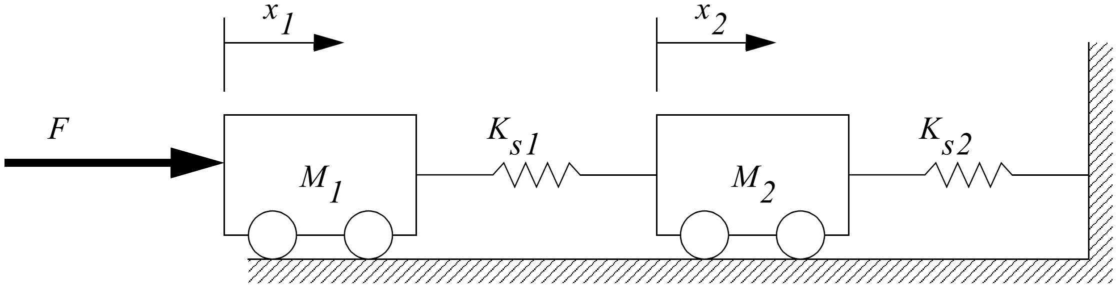
equation 3.1 and equation 3.2 can also be put into a matrix form similar to that given in Figure 3.1.

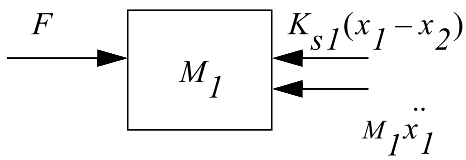
$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-F}{M} \end{bmatrix}$$

Note: To have a set of differential equations that is solvable, there must be the same number of state equations as variables. If there are too few equations, then an additional equation must be developed using an unexploited relationship. If there are too many equations, a redundancy or over constraint must be eliminated.

Figure 3.2 Example: A state variable equation

Consider the two cart problem in Figure 3.3. The carts are separated from each other and the wall by springs, and a force is applied to the left hand side. Free body diagrams are developed for each of the carts, and differential equations developed. For each cart a velocity state variable is created. The equations are then manipulated to convert the second-order differential equations to first-order state equations. The four resulting equations are then put into the state variable matrix form.





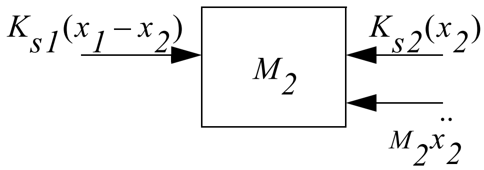
$$\sum F_x = F - K_{s1}(x_1 - x_2) = M_1\ddot{x}_1$$

$$M_1\ddot{x}_1 + K_{s1}x_1 - K_{s1}x_2 = F$$

$$\dot{x}_1 = v_1 \qquad \text{eqn 3.3}$$

$$M_1\dot{v}_1 + K_{s1}x_1 - K_{s1}x_2 = F$$

$$\dot{v}_1 = \frac{F}{M_1} - \frac{K_{s1}}{M_1}x_1 + \frac{K_{s1}}{M_1}x_2 \qquad \text{eqn 3.4}$$



$$\sum F_x = K_{s1}(x_1 - x_2) - K_{s2}(x_2) = M_2\ddot{x}_2$$

$$M_2\ddot{x}_2 + (K_{s1} + K_{s2})x_2 - K_{s1}x_1 = 0$$

$$\dot{x}_2 = v_2 \qquad \text{eqn 3.5}$$

$$M_2\dot{v}_2 + (K_{s1} + K_{s2})x_2 - K_{s1}x_1 = 0$$

$$\dot{v}_2 = \frac{K_{s1}}{M_2}x_1 - \left(\frac{K_{s1} + K_{s2}}{M_2}\right)x_2 \qquad \text{eqn 3.6}$$

The state equations can now be combined in a matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s1}}{M_1} & 0 & \frac{K_{s1}}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s1}}{M_2} & 0 & \frac{-K_{s1} - K_{s2}}{M_2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M_1} \\ 0 \\ 0 \end{bmatrix}$$

Figure 3.3 Example: Two cart state equation

In some cases we will develop differential equations that cannot be directly reduced because they have more than one term at the highest order. For example, if a second-order differential equation has two second derivatives it cannot be converted to a state equation in the normal manner. In this case the two high order derivatives can be replaced with a dummy variable. In mechanical systems this often happens when masses are neglected. Consider the example problem in Figure 3.4, both 'y' and 'u' are first derivatives. To solve this problem, the highest order terms ('y' and 'u') are moved to the left of the equation. A dummy variable, 'q', is then created to replace these two variables with a single variable. This also creates an output equation as shown in Figure 3.1.

Given the equation,

$$3\dot{y} + 2y = 5\dot{u}$$

Step 1: put both the first-order derivatives on the left hand side,

$$3\dot{y} - 5\dot{u} = -2y$$

Step 2: replace the left hand side with a dummy variable,

$$q = 3y - 5u \qquad \dot{q} = -2y$$

Step 3: solve the equation using the dummy variable, then solve for y as an output eqn.

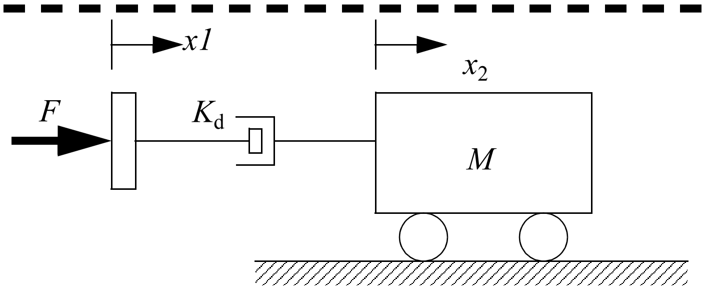
$$y = \frac{q + 5u}{3}$$

Step 4: Substitute the equation for y into the state equation:

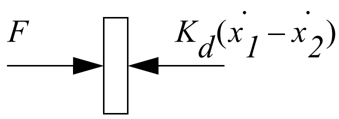
$$\dot{q} = -2y = -2\left(\frac{q + 5u}{3}\right) = q\left(-\frac{2}{3}\right) + u\left(\frac{-10}{3}\right)$$

Figure 3.4 Using dummy variables for multiple high order terms

At other times it is possible to eliminate redundant terms through algebraic manipulation, as shown in Figure 3.5. In this case the force on both sides of the damper is the same, so it is substituted into the equation for the cart. But, the effects on the damper must also be integrated, so a dummy variable is created for the integration. An output equation was created to calculate the value for x_1.



The FBDs and equations are;



$$\xrightarrow{+} \sum F_x = F - K_d(\dot{x}_1 - \dot{x}_2) = 0$$

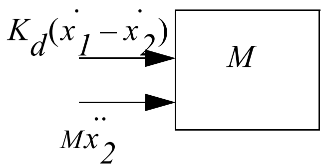
$$K_d(\dot{x}_1 - \dot{x}_2) = F \qquad \text{eqn 3.7}$$

$$q = x_1 - x_2$$

$$K_d(\dot{q}) = F$$

$$\dot{q} = \frac{F}{K_d} \qquad \text{eqn 3.8}$$

$$x_1 = x_2 + q \qquad \text{eqn 3.9}$$



$$\xrightarrow{+} \sum F_x = K_d(\dot{x}_1 - \dot{x}_2) = M\ddot{x}_2$$

$$F = M\ddot{x}_2$$

$$\dot{x}_2 = v_2 \qquad \text{eqn 3.10}$$

$$\dot{v}_2 = \frac{F}{M} \qquad \text{eqn 3.11}$$

The state equations (2, 4, 5) can be put in matrix form. The output equation (2) can also be put in matrix form.

$$\frac{d}{dt}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{F}{K_d} \\ 0 \\ \frac{F}{M} \end{bmatrix} \qquad \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

Figure 3.5 Example: A dummy variable

3.1 Numerical Integration

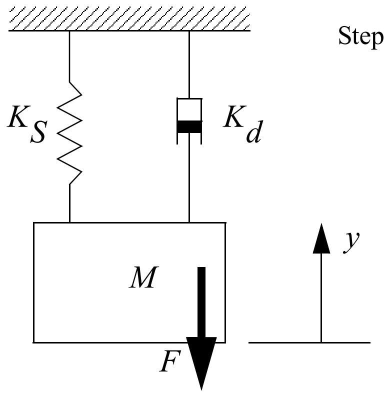
Repetitive calculations can be used to develop an approximate solution to a set of differential equations. Starting from given initial conditions, the equation is solved with small time steps. Smaller time steps result in a higher level of accuracy, while larger time steps give a faster solution.

Numerical Integration

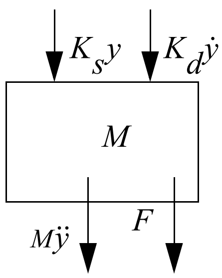
Numerical solutions can be developed with hand calculations, but this is a very time consuming task. In this section we will explore some common tools for solving state variable equations. The analysis process follows the basic steps listed below.

1. Generate the differential equations to model the process.
2. Select the state variables.
3. Rearrange the equations to state variable form.
4. Add additional equations as required.
5. Enter the equations into a computer or calculator and solve.

An example in Figure 3.6 shows the first four steps for a mass-spring-damper combination. The FBD is used to develop the differential equations for the system. The state variables are then selected, in this case the position, y, and velocity, v, of the block. The equations are then rearranged into state equations. The state equations are also put into matrix form, although this is not always necessary. At this point the equations are ready for solution.



Step 1: Develop equations



$$\sum F_y = -F - K_d\left(\frac{d}{dt}\right)y - K_s y = M\left(\frac{d}{dt}\right)^2 y$$

$$F + K_d\left(\frac{d}{dt}\right)y + K_s y + M\left(\frac{d}{dt}\right)^2 y = 0$$

Step 2: We need to identify state variables. In this case the height is clearly a defining variable. We will also need to use the vertical velocity, because the acceleration is a second derivative (we can only have first derivatives). Using the height, y, and velocity, v, as state variables we may now proceed to rewriting the equations. (Note: this is just an algebraic trick, but essential when setting up these matrices.)

Step 3:

$$\left(\frac{d}{dt}\right)y = v$$

$$\left(\frac{d}{dt}\right)v = \frac{-F - K_d v - K_s y}{M}$$

Step 4: We put the equations into a state variable matrix form.

$$\begin{bmatrix} \left(\frac{d}{dt}\right)y \\ \left(\frac{d}{dt}\right)v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_S}{M} & \frac{-K_d}{M} \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{F}{M}$$

Figure 3.6 Example: Dynamic system

Figure 3.7 shows the method for solving state equations on a TI-86 graphing calculator. (Note: this also works on other TI-8x calculators with minor modifications.) In the example a sinusoidal input force, F, is used to make the solution more interesting. The next step is to put the equation in the form expected by the calculator. When solving with the TI calculator the state variables must be replaced with the predefined names Q1, Q2, etc. The steps that follow describe the button sequences required to enter and analyze the equations. The result is a graph that shows the solution of the equation. Points can then be taken from the graph using the cursors. (Note: large solutions can sometimes take a few minutes to solve.)

First, we select some parameter values for the equations of Figure 3.6. The input force will be a decaying sine wave.

$$\left(\frac{d}{dt}\right)y = v_y$$

$$\left(\frac{d}{dt}\right)v_y = \frac{-F - K_d v_y - K_s y}{M} = -4e^{-0.5t}\sin(t) - 2v_y - 5y$$

Next, the calculator requires that the state variables be Q1, Q2, ..., Q9, so we replace y with Q1 and v with Q2.

$$Q1' = Q2$$

$$Q2' = -4e^{-0.5t}\sin(t) - 2Q2 - 5Q1$$

Now, we enter the equations into the calculator and solve. To do this roughly follow the steps below. Look at the calculator manual for additional details.

1. Put the calculator in differential equation mode [2nd][MODE][DifEq][ENTER]
2. Go to graph mode and enter the equations above [GRAPH][F1]
3. Set up the axis for the graph [GRAPH][F2] so that time and the x-axis is from 0 to 10 with a time step of 0.5, and the y height is from +3 to -3.
4. Enter the initial conditions for the system [GRAPH][F3] as Q1=0, Q2=0
5. Set the axis [GRAPH][F4] as x=t and y=Q
6. (TI-86 only) Set up the format [GRAPH][MORE][F1][FldOff][ENTER]
7. Draw the graph [GRAPH][F5]
8. Find points on the graph [GRAPH][MORE][F4]. Move the left/right cursor to move along the trace, use the up/down cursor to move between traces.

Figure 3.7 Example: Solving state equations with a TI-85 calculator

First, we select some parameter values for the equations of Figure 3.6. The input force will be a decaying sine wave.

$$\left(\frac{d}{dt}\right)y = v_y$$

$$\left(\frac{d}{dt}\right)v_y = \frac{-F - K_d v_y - K_s y}{M} = -4e^{-0.5t} sin(t) - 2v_y - 5y$$

Next, the calculator requires that the state variables be Q1, Q2, ..., Q9, so we replace y with Q1 and v with Q2.

$$Q1' = Q2$$

$$Q2' = -4e^{-0.5t} sin(t) - 2Q2 - 5Q1$$

Now, we enter the equations into the calculator and solve. To do this roughly follow the steps below. Look at the calculator manual for additional details.

1. Put the calculator in differential equation mode [2nd][MODE][DifEq][ENTER]
2. Enter the initial conditions for the system [GRAPH][F3] as Q1=0, Q2=0
3. Set up the format [GRAPH][MORE][F1][FldOff][ENTER]
4. Go to graph mode and enter the equations above [GRAPH][F1]
5. Set up the axis for the graph [GRAPH][F2] so that time and the x-axis is from 0 to 10 with a time step of 0.5, and the y height is from +3 to -3.
6. Set the axis [GRAPH][F4] as x=t and y=Q
7. Draw the graph [GRAPH][F5]
8. Find points on the graph [GRAPH][MORE][F4]. Move the left/right cursor to move along the trace, use the up/down cursor to move between traces.

Figure 3.8 Example: Solving state equations with a TI-89 calculator

State equations can also be solved in Mathcad using built-in functions, as shown in Figure 3.9. The first step is to enter the state equations as a function, ‘D(t, Q)’, where ‘t’ is the time and ‘Q’ is the state variable vector. (Note: the equations are in a vector, but it is not the matrix form.) The state variables in the vector ‘Q’ replace the original state variables in the equations. The ‘rkfixed’ function is then used to obtain a solution. The arguments for the function, in sequence are; the state vector, the start time, the end time, the number of steps, and the state equation function. In this case the 10 second time interval is divided into 100 parts each 0.1s in duration. This time is chosen because of the general response time for the system. If the time step is too large the solution may become unstable and go to infinity. A time step that is too small will increase the computation time marginally. When in doubt, run the calculator again using a smaller time step.

$$D(t,Q) := \begin{bmatrix} Q_1 \\ \left(-4\cdot e^{-0.5\cdot t}\cdot \sin(t)\right) - 2\cdot Q_1 - 5\cdot Q_0 \end{bmatrix}$$

$$QI := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

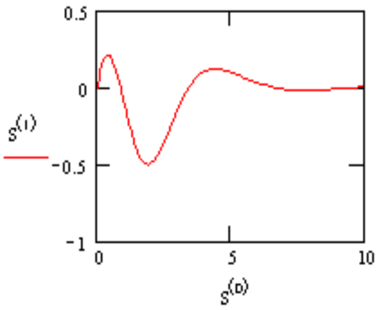
Define state variable equations and initial conditions. Note the origin is at '0' here.

$tMin := 0$ $\quad tMax := 10$ $\quad N := 100$

Define time range and number of steps for the integration.

$S := rkfixed(QI, tMin, tMax, N, D)$

A fixed time step runge-kutta integration



The first state variable S<1> = Q0 plotted against time S<0> = t.

A point in time is,

$i := 40$

$t := \left(S^{\langle 0 \rangle}\right)_i$ $\quad Q_0 := \left(S^{\langle 1 \rangle}\right)_i$ $\quad Q_1 := \left(S^{\langle 2 \rangle}\right)_i$

$t = 4$ $\quad Q_0 = 0.103$ $\quad Q_1 = 0.081$

Figure 3.9 Example: Solving state variable equations with Mathcad

> Note: Notice that for the TI calculators the variables start at Q1, while in Mathcad the arrays start at Q0. Many students encounter problems because they forget this.

Numerical Integration

The simplest form of numerical integration is Euler's first-order method. Given the current value of a function and the first derivative, we can estimate the function value a short time later, as shown in Figure 3.10. (Note: Recall that the state equations allow us to calculate first-order derivatives.) The equation shown is known as Euler's equation. Basically, using a known position and first derivative we can calculate an approximate value a short time, h, later. Notice that the function being integrated curves downward, creating an error between the actual and estimated values at time 't+h'. If the time step, h, were smaller, the error would decrease.

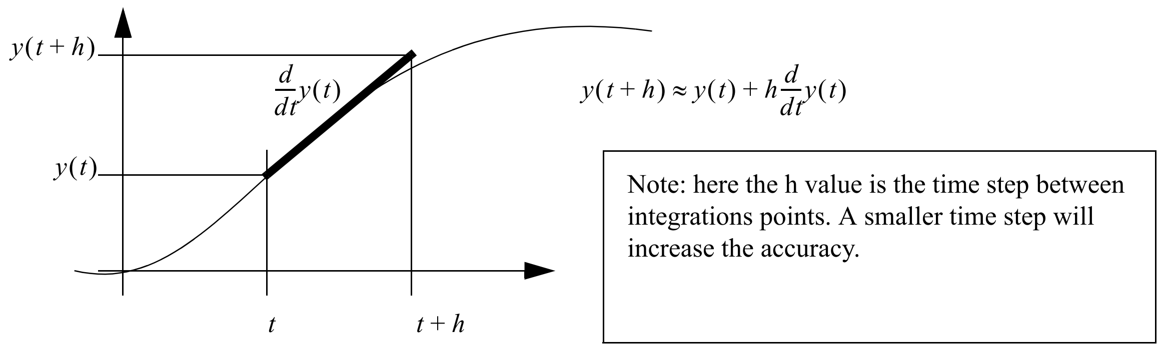


Figure 3.10 First-order numerical integration

The example in Figure 3.11 shows the solution of Newton's equation using Euler's method. In this example we are determining velocity by integrating the acceleration caused by a force. The acceleration is put directly into Euler's equation. This is then used to calculate values iteratively in the table. Notice that the values start before zero so that initial conditions can be used. If the system was second-order we would need two previous values for the calculations.

Given the differential equation,

$$F = M\left(\frac{d}{dt}\right)v$$

we can create difference equations using simple methods.

$$\left(\frac{d}{dt}\right)v = \frac{F}{M}$$ first rearrange equation

$$v(t+h) = v(t) + h\left(\frac{d}{dt}\right)v(t)$$ put this in the Euler equation

$$v(t+h) = v(t) + h\left(\frac{F(t)}{M}\right)$$ finally substitute in known terms

We can now use the equation to estimate the system response. We will assume that the system is initially at rest and that a force of 1N will be applied to the 1kg mass for 4 seconds. After this time the force will rise to 2N. A time step of 2 seconds will be used.

i	t (sec)	F (N)	d/dt v_i	v_i
-1	-2	0	0	0
0	0	1	1	0
1	2	1	1	2
2	4	2	2	4
3	6	2	2	8
4	8	2	2	12
5	10	2	2	16
6	12	2	2	20
7	14	2	etc	etc
8	16	2		

Figure 3.11 Example: First order numerical integration

An example of solving the previous example with a traditional programming language is shown in Figure 3.12. In this example the results will be written to a text file 'out.txt'. The solution iteratively integrates from 0 to 10 seconds with time steps of

0.1s. The force value is varied over the time period with 'if' statements. The integration is done with a separate function.

```
double step(double, double, double);

int main(){
        double          h = 0.1,
                        M = 1.0,
                        F;
        FILE    *fp;
        double  v,
                t;
        if( ( fp = fopen("out.txt", "w")) != NULL){
                v = 0.0;
                for( t = 0.0; t < 10.0; t += h ){
                        if((t >= 0.0) && (t < 4.0)) F = 1.0;
                        if(t >= 4.0) F = 2.0;
                        v = step(v, h, F/M);
                        fprintf(fp, "%f, %f, %f, %f\n", t, v,
   F, M);
                }
        }
        fclose(fp);
}

double step(double v, double h, double slope){
        double          v_new;
        v_new = v + h * slope;
        return v_new;
}
```

Figure 3.12 Example: Solving state variable equations with a C program

```
double step(double, double, double);

public class Integrate extends Object
        public void main() {
                double          h = 0.1,
                                M = 1.0,
                                F;
                FileOut fp = new FileOut("out.txt");
                if(fp.writeStatus != fp.IO_EXCEPTION){
                        double v = 0.0;
                        for( double t = 0.0; t < 10.0; t += h ){
                                if((t >= 0.0) && (t < 4.0)) F = 1.0;
                                if(t >= 4.0) F = 2.0;
                                v = step(v, h, F/M);
                                fp.printf(fp, "%f, %f, %f, %f\n", t, v, F, M);
                        }
                        fp.close();
                }
                fclose(fp);
        }

        public double step(double v, double h, double slope){
                double          v_new;
                v_new = v + h * slope;
                return v_new;
        }
}
```

Figure 3.13 Example: Solving state variable equations with a Java program

The program in Figure 3.14 is for Scilab (a Matlab clone). The state variable function is defined first. This is followed by a definition of the parameters to be used for the numerical integration. Finally the function is solved explicitly (with the exact function).

```
//
// first_order.sce
//
// A first order integration of an accelerating mass
//
// To run this in Scilab use 'File' then 'Exec'.
//
// by: H. Jack Sept., 16, 2002
//


// System component values
mass = 10;
force = 100;

x0 = 8;         // initial conditions
v0 = 12;
X=[x0, v0];

// define the state matrix function
// the values returned are [x, v]
function foo=f(state,t)
        foo = [ state($, 2), force/mass]; // d/dt x = v, d/dt v = F/M
endfunction


// Set the time length and step size for the integration
steps = 100;
t_start = 1;
t_end = 100;
h = (t_end - t_start) / steps;
```

Figure 3.14 Example: First order integration with Scilab

```
//
// Loop for integration
//
for i=1:steps,
        X = [X ; X($,:) + h*f(X, i*h)];
end
printf("The value at the end of first order integration is (x, v) = (%f, %f)\n", ...
        X($,1), ...
        X($,2));


//
// Explicit equation
//
function x=position(x0, v0, a0, t)
        x = (0.5 * a0 * t^2) + (v0 * t) + x0;
endfunction

function v=velocity(v0, a0, t)
        v = (a0 * t) + v0;
endfunction

printf("The value with integration is (x, v) = (%f, %f)\n", ...
                position(x0, v0, force/mass, t_end), ...
                velocity(v0, force/mass, t_end));


//
// The results should be
//       first order integration = (49710, 1002)
//       explicit                = (51208, 1012)
//
// The difference is 1498 for position and 10 for velocity. This is relatively small, but
// shows a clear case of the innacuracy of the numerical solutions.
//
// Note: increasing the number of steps increases the accuracy
//
```

Figure 3.15 Example: First order integration with Scilab (continued)

Taylor Series

First-order integration works well with smooth functions. But, when a highly curved function is encountered we can use a higher order integration equation. The Taylor series equation shown in Figure 3.16 for approximating a function. Notice that the first part of the equation is identical to Euler's equation, but the higher order terms add accuracy.

$$x(t+h) = x(t) + h\left(\frac{d}{dt}\right)x(t) + \frac{1}{2!}h^2\left(\frac{d}{dt}\right)^2 x(t) + \frac{1}{3!}h^3\left(\frac{d}{dt}\right)^3 x(t) + \frac{1}{4!}h^4\left(\frac{d}{dt}\right)^4 x(t) + \ldots$$

Figure 3.16 The Taylor series

An example of the application of the Taylor series is shown in Figure 3.17. Given the differential equation, we must first determine the derivatives and substitute these into Taylor's equation. The resulting equation is then used to iteratively calculate values.

Given,

$$\dot{x} - x = 1 + e^{-20t} + t^3$$

We can write,

$$\left(\frac{d}{dt}\right)x = 1 + e^{-20t} + t^3 + x$$

$$\left(\frac{d}{dt}\right)^2 x = -e^{-20t} + 3t^2$$

$$\left(\frac{d}{dt}\right)^3 x = e^{-20t} + 6t$$

In Taylor series form this becomes,

$$x(t+h) = x(t) + h(1 + e^{-20t} + t^3 + x) + \frac{1}{2!}h^2(-e^{-t} + 3t^2) + \frac{1}{3!}h^3(e^{-t} + 6t)$$

Thus,

$x_0 = 0$

$h = 0.1$

t (s)	x(t)
0	0
0.1	0
0.2	
0.3	
0.4	
0.5	
0.6	
0.7	
0.8	
0.9	

e.g., for t = 0.1s

$$x(0 + 0.1) = 0 + 0.1(2) + \frac{1}{2!}(0.1)^2(-1) + \frac{1}{3!}(0.1)^3(1) =$$

Figure 3.17 Example: Integration using the Taylor series method

Recall that the state variable equations are first-order equations. But, to obtain accuracy the Taylor method also requires higher order derivatives, thus making is unsuitable for use with state variable equations.

Runge-Kutta Integration

First-order integration provides reasonable solutions to differential equations. That accuracy can be improved by using higher order derivatives to compensate for function curvature. The Runge-Kutta technique uses first-order differential equations (such as state equations) to estimate the higher order derivatives, thus providing higher accuracy without requiring other than first-order differential equations.

The equations in Figure 3.18 are for fourth order Runge-Kutta integration. The function ‘f()’ is the state equation or state equation vector. For each time step the values ‘F1’ to ‘F4’ are calculated in sequence and then used in the final equation to find the next value. The ‘F1’ to ‘F4’ values are calculated at different time steps, and values from previous time steps are used to ‘tweak’ the estimates of the later states. The final summation equation has a remote similarity to the first order integration equation. Notice that the two central values in time are more heavily weighted.

$$F_1 = hf(t, x)$$

$$F_2 = hf\left(t + \frac{h}{2}, x + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(t + \frac{h}{2}, x + \frac{F_2}{2}\right)$$

$$F_4 = hf(t + h, x + F_3)$$

Note: in this case the state equation function f(t, x) includes the state variables, x, and time, t. However, in simpler systems the state equations may not include time and it could be replaced with f(x).

$$x(t + h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where,

x = the state variables
f = the differential function or (d/dt) x
t = current point in time
h = the time step to the next integration point

Figure 3.18 Fourth order Runge-Kutta integration

An example of a Runge-Kutta integration calculation is shown in Figure 3.19. The solution begins by putting the state equations in matrix form and defining initial conditions. After this, the four integrating factors are calculated. Finally, these are combined to get the final value after one time step. The number of calculations for a single time step should make obvious the necessity of computers and calculators.

$$\frac{d}{dt}x = v \qquad y = 2 \text{ (assumed input)}$$

$$v_0 = 1$$

$$\frac{d}{dt}v = 3 + 4v + 5y$$

$$x_0 = 3$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix} \qquad h = 0.1$$

For the first time step,

$$F_1 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 0.1\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix}\right) = \begin{bmatrix} 0.1 \\ 1.7 \end{bmatrix}$$

$$F_2 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + \frac{0.1}{2} \\ 1 + \frac{1.7}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.185 \\ 2.04 \end{bmatrix}$$

$$F_3 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + \frac{0.185}{2} \\ 1 + \frac{2.04}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.202 \\ 2.108 \end{bmatrix}$$

$$F_4 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + 0.202 \\ 1 + 2.108 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.3108 \\ 2.5432 \end{bmatrix}$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ v_i \end{bmatrix} + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} 0.1 \\ 1.7 \end{bmatrix} + 2\begin{bmatrix} 0.185 \\ 2.04 \end{bmatrix} + 2\begin{bmatrix} 0.202 \\ 2.108 \end{bmatrix} + \begin{bmatrix} 0.3108 \\ 2.5432 \end{bmatrix}\right) = \begin{bmatrix} 3.1974667 \\ 3.0898667 \end{bmatrix}$$

Figure 3.19 Example: Runge-Kutta integration

The program in Figure 3.20 and Figure 3.22 is used to perform a Runge-Kutta integration of a mass-spring-damper system. The ‘main’ program loops through the time steps and writes the value to a file. The ‘step’ function performs one time step integration for a second order Runge-Kutta integration. It uses the functions ‘add’ and ‘multiply’ to manipulate the state matrix. The ‘derivative’ function updates the state matrix with the new derivative values.

```
/* A program to do Runge Kutta integration of a mass spring damper system */
#include <stdio.h>

void    multiply(double, double[], double[]);
void    add(double[], double[], double[]);
void    step(double, double, double[]);
void    derivative(double, double[], double[]);

#define SIZE            2 /* the length of the state vector */
#define Ks              1000 /* the spring coefficient */
#define Kd              10000 /* the damper coefficient */
#define Mass            10 /* the mass coefficient */
#define Force           100 /* the applied force */

int main(){
        FILE    *fp;

        double  h = 0.001;
        double  t;
        int             j = 0;

        double  X[SIZE];// create state variable list
        X[0] = 0;// set initial condition for x
        X[1] = 0;// set initial condition for v

        if( ( fp = fopen("out.txt", "w")) != NULL){
                fprintf(fp, "   t(s)        x           v \n\n");
                for( t = 0.0; t < 50.0; t += h ){
                        step(t, h, X);
                        if(j == 0) fprintf(fp, "%9.5f %9.5f %9.5f\n", t, X[0], X[1]);
                        j++; if(j >= 10) j = 0;
                }
        }
        fclose(fp);
}
```

Figure 3.20   Example: Runge-Kutta integration C program

```
/* First order integration done here (could be replaced with runge kutta)*/
void step(double t, double h, double X[]){
        double  tmp[SIZE],
                dX[SIZE],
                F1[SIZE],
                F2[SIZE],
                F3[SIZE],
                F4[SIZE];
        /* Calculate F1 */
        derivative(t, X, dX);
        multiply(h, dX, F1);

        /* Calculate F2 */
        multiply(0.5, F1, tmp);
        add(X, tmp, tmp);
        derivative(t+h/2.0, tmp, dX);
        multiply(h, dX, F2);

        /* Calculate F3 */
        multiply(0.5, F2, tmp);
        add(X, tmp, tmp);
        derivative(t+h/2.0, tmp, dX);
        multiply(h, dX, F3);

        /* Calculate F4 */
        add(X, F3, tmp);
        derivative(t+h, tmp, dX);
        multiply(h, dX, F4);

        /* Calculate the weighted sum */
        add(F2, F3, tmp);
        multiply(2.0, tmp, tmp);
        add(F1, tmp, tmp);
        add(F4, tmp, tmp);
        multiply(1.0/6.0, tmp, tmp);
        add(tmp, X, X);
}

/* State Equations Calculated Here */
void derivative(double t, double X[], double dX[]){
        dX[0] = X[1];
        dX[1] = (-Ks/Mass)*X[0] + (-Kd/Mass)*X[1] + (Force/Mass);
}

/* A subroutine to add vectors to simplify other equations */
void add(double X1[], double X2[], double R[]){
        for(int i = 0; i < SIZE; i++) R[i] = X1[i] + X2[i];
}

/* A subroutine to multiply a vector by a scalar to simplify other equations*/
void multiply(double X, double V[], double R[]){
        for(int i = 0; i < SIZE; i++) R[i] = X*V[i];
}
```

Figure 3.21 Example: Runge-Kutta integration C program (cont'd)

A Scilab program is given in Figure 3.22 to perform a Runge Kutta integration.

```
// runge_kutta.sce
// A first order integration of an accelerating mass
// To run this in Scilab use 'File' then 'Exec'.
// by: H. Jack Sept., 15, 2003

// System component values
mass = 10;
force = 100;

x0 = 8;        // initial conditions
v0 = 12;
X=[x0, v0];

// define the state matrix function
// the values returned are [x, v]
function foo=f(state,t)
        foo = [ state($, 2), force/mass]; // d/dt x = v, d/dt v = F/M
endfunction

// Set the time length and step size for the integration
steps = 1000;
t_start = 0;
t_end = 100;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
        t = [t ; t($,:) + h];
        F1 = h * f(X($,:), t($,:));
        F2 = h * f(X($,:) + F1/2.0, t($,:) + h/2.0);
        F3 = h * f(X($,:) + F2/2.0, t($,:) + h/2.0);
        F4 = h * f(X($,:) + F3, t($,:) + h);
        X = [X ; X($,:) + (F1 + 2.0*F2 + 2.0*F3 + F4)/6.0];
end

// print some results to compare
printf("The value at the end of first order integration is (x, v) = (%f, %f)\n", ...
        X($,1), ...
        X($,2));
printf("The position (using an equation) should be %f\n", 0.5*force/mass*(t_end-
   t_start)^2.0 + v0*(t_end - t_start) + x0);

// Graph the values
plot2d(t, X, [-2, -5], leg="position@velocity");
                // leg - the legend titles
                // style - draw lines with marks
                // nax - grid lines for the graph
xtitle('Time (s)');
```

Figure 3.22 Example: Runge-Kutta integration Scilab program

3.2 System Response

In most cases the result of numerical analysis is graphical or tabular. In both cases details such as time constants and damped frequencies can be obtained by the same methods used for experimental analysis. In addition to these methods there is a technique that can determine the steady-state response of the system.

Steady-State Response

The state equations can be used to determine the steady-state response of a system by setting the derivatives to zero, and then solving the equations. Consider the example in Figure 3.23. The solution begins with a state variable matrix. (Note: this can also be done without the matrix also.) The derivatives on the left hand side are set to zero, and the equations are rearranged and solved with Cramer's rule.

Given the state variable form of the system,

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

Set the derivatives to zero for a static steady state case,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

Then solve for x and v,

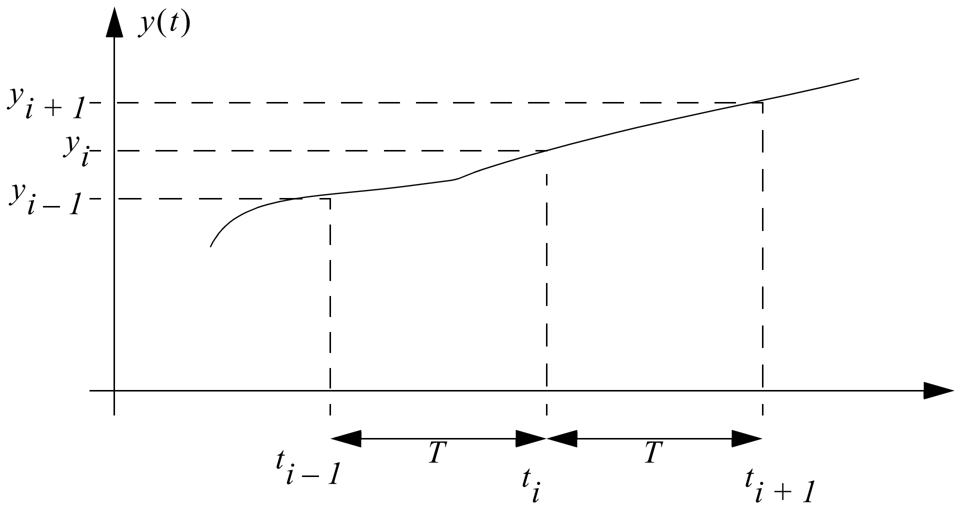
$$\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{F}{M} \end{bmatrix}$$

$$x = \frac{\left|\begin{bmatrix} 0 & 1 \\ -\frac{F}{M} & -\frac{K_d}{M} \end{bmatrix}\right|}{\left|\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\right|} = \frac{\left(\frac{F}{M}\right)}{\left(\frac{K_s}{M}\right)} = \frac{F}{K_S} \qquad v = \frac{\left|\begin{bmatrix} 0 & 0 \\ -\frac{K_s}{M} & -\frac{F}{M} \end{bmatrix}\right|}{\left|\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\right|} = \frac{0}{\left(\frac{K_s}{M}\right)} = 0$$

Figure 3.23 Example: Finding the steady-state response

3.3 Differentiating and Integrating Data

When doing experiments, data is often collected in the form of individual data points (not as complete functions). It is often necessary to integrate or differentiate these values. The basic equations for integrating and differentiating are shown in Figure 3.24. Given data points, y, collected at given times, t, we can integrate and differentiate using the given equations. The integral is basically the average height of the two points multiplied by the width to give an area, or integral. The first derivative is basically the slope between two points. The second derivative is the change in slope values for three points. In a computer based system the time points are often equally spaced in time, so the difference in time can be replaced with a sample period, T. Ideally the time steps would be as small as possible to increase the accuracy of the estimates.



$$\int_{t_{i-1}}^{t_i} y(t_i)dt \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

Figure 3.24 Example: Integration and differentiation using data points

An example of numerical integration using Scilab is given in Figure 3.25 and Figure 3.26.

```
//
// integrate.sce
//
// A simple program to integrate a function
//
// To run this in Scilab use 'File' then 'Exec'.
//
// by: H. Jack Sept., 9, 2002
//


// define the function
function foo=f(x)
        foo = 5 * x + 2 * log(sin(x) / x + 2);
endfunction

// Set the time length and step size
steps = 10;
x_start = 1;
x_end = 10;
x_delta = (x_end - x_start) / steps;


//
// Loop for rectangular integration
//
total = 0; // set the initial sum to zero
for i=0:steps,
        x = x_start + i * x_delta;
        total = total + f(x);
end
total = total * x_delta;
printf("Rectangular integration value %f\n", total);
```

Figure 3.25 Example: Integration with a Scilab program

```
// Loop for trapezoidal integration
//
total = 0; // set the initial sum to zero
for i=0:steps,
        x = x_start + i * x_delta;
        if i == 0 then
                total = total + f(x);
        elseif i == steps then
                total = total + f(x);
        else
                total = total + 2 * f(x);
        end
end
total = total * x_delta / 2;
printf("Trapezoidal integration value %f\n", total);


//
// Loop for Simpson's rule integration
//
total = 0; // set the initial sum to zero
even = 0;
for i=0:steps,
        x = x_start + i * x_delta;
        if i == 0 then
                total = total + f(x);
        elseif i == steps then
                total = total + f(x);
        else
                even = even + 1;
                if even > 1 then
                        total = total + 4 * f(x);
                        even = 0;
                else
                        total = total + 2 * f(x);
                end
        end
end
total = total * x_delta / 3;
printf("Simpsons rule integration value %f\n", total);
```

Figure 3.26   Example: Integration with a Scilab Program (cont'd)

3.4 Advanced Topics

Switching Functions

When analyzing a system, we may need to choose an input that is more complex than inputs such as steps, ramps, sinusoids and parabolas. The easiest way to do this is to use switching functions. Switching functions turn on (have a value of 1) when their arguments are greater than or equal to zero, or off (a value of 0) when the argument is negative. Examples of the use of switching functions are shown in Figure 3.27. By changing the values of the arguments we can change when a function turns on or off.

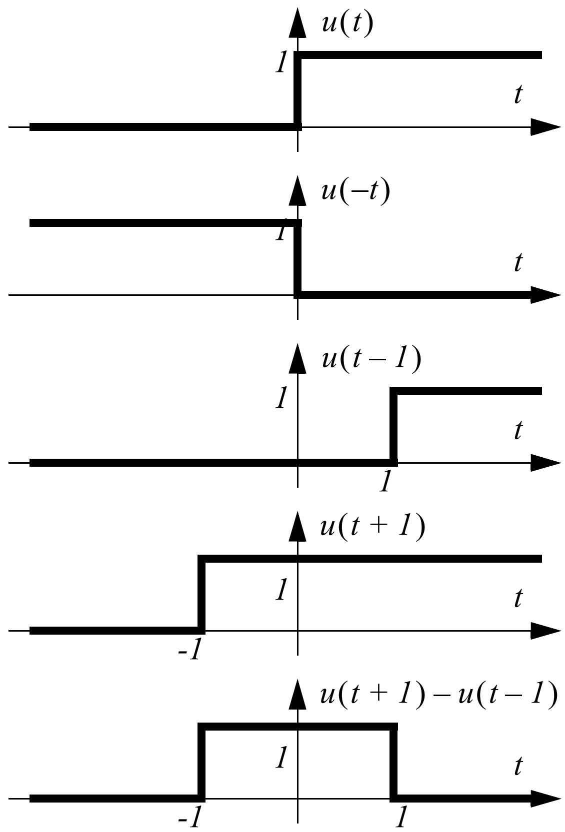
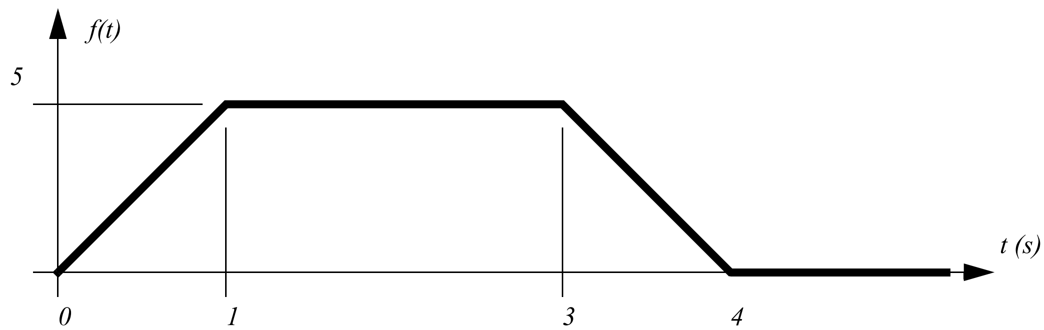


Figure 3.27 Example: Switching functions

Switching functions can be multiplied with other functions to create a complex function by turning parts of the function on or off. An example of a curve created with switching functions is shown in Figure 3.28.



$$f(t) = 5t(u(t) - u(t-1)) + 5(u(t-1) - u(t-3)) + (20 - 5t)(u(t-3) - u(t-4))$$

or

$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

Figure 3.28 Example: Switching functions to create a non-smooth function

The unit step switching function is available in Mathcad and makes creation of complex functions relatively trivial. Step functions are also easy to implement when writing computer programs, as shown in Figure 3.29.

For the function $f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$

```
double u(double t){
        if(t >= 0) return 1.0;
        return 0.0;
}

double function(double t){
        double          f;

        f =     5.0 * t * u(t)
                - 5.0 * (t - 1.0) * u(t - 1.0)
                - 5.0 * (t - 3.0) * u(t - 3.0)
                + 5.0 * (t - 4.0) * u(t - 4.0);

        return f;
}
```

Figure 3.29 Example: A subroutine implementing the example in Figure 3.28

Interpolating Tabular Data

In some cases we are given tables of numbers instead of equations for a system component. These can still be used to do numerical integration by calculating coefficient values as required, in place of an equation.

Tabular data consists of separate data points as seen in Figure 3.30. But, we may need values between the data points. A simple method for finding intermediate values is to interpolate with the 'lever law'. (Note: it is called this because of its' similarity to the equation for a lever.) The table in the example only gives flow rates for a valve at 10 degree intervals, but we want flow rates at 46 and 23 degrees. A simple application of the lever law gives approximate values for the flow rates.

valve angle (deg.)	flow rate (gpm)
0	0
10	0.1
20	0.4
30	1.2
40	2.0
50	2.3
60	2.4
70	2.4
80	2.4
90	2.4

Given a valve angle of 46 degrees, the flow rate is,

$$Q = 2.0 + (2.3 - 2.0)\left(\frac{46 - 40}{50 - 40}\right) = 2.18$$

Given a valve angle of 23 degrees, the flow rate is,

$$Q = 0.4 + (1.2 - 0.4)\left(\frac{23 - 20}{30 - 20}\right) = 0.64$$

Figure 3.30 Example: Using tables of values to interpolate numerical values using the lever law

The subroutine in Figure 3.31 was written to return the numerical value for the data table in Figure 3.30. In the subroutine the tabular data is examined to find the interval that the flow rate value falls inside. Once this is found the valve angle is calculated as the ratio between the two known values.

```
#define         SIZE    10
double          data[SIZE][2] = {{0.0, 0.0},
                               {10.0,  0.1},
                               {20.0,  0.4},
                               {30.0,  1.2},
                               {40.0,  2.0},
                               {50.0,  2.3},
                               {60.0,  2.4},
                               {70.0,  2.4},
                               {80.0,  2.4},
                               {90.0,  2.4}};

double angle(double rate){
        int i;
        for(i = 0; i < SIZE-1; i++){
                if((rate >= data[i][0]) && (rate <= data[i+1][0]){
                        return  (data[i+1][1] - data[i][1])
                                * (rate - data[i][0]) / (data[i+1][0] - data[i][0])
                                + data[i][1];
                }
        }
        printf("ERROR: rate out of range\n");
        exit(1);
}
```

Figure 3.31 Example: A tabular interpolation subroutine

Modeling Functions with Splines

When greater accuracy is required, smooth curves can be fitted to interpolate points. These curves are known as splines. There are multiple methods for creating splines, but the simplest is to use a polynomial fitted to a set of points.

The example in Figure 3.32 shows a spline curve being fitted for three data points. In this case a second order polynomial is used. The three data points are written out as equations, and then put into matrix form, using the coefficients as the unknown values. The matrix is then solved to obtain the coefficient values for the final equation. This equation can then be used to build a mathematical model of the system.

The data points below might have been measured for the horsepower of an internal combustion engine on a dynamometer.

S (RPM)	P (HP)
1000	105
4000	205
6000	110

In this case there are three data points, so we can fit the curve with a second (3-1) order polynomial. The major task is to calculate the coefficients so that the curve passes through all of the given points.

Data values can be substituted into the equation,

$$P(S) = AS^2 + BS + C$$

This can then be put in matrix form to find the coefficients,

$$P(1000) = A1000^2 + B1000 + C = 105$$
$$P(4000) = A4000^2 + B4000 + C = 205$$
$$P(6000) = A6000^2 + B6000 + C = 110$$

$$\begin{bmatrix} 1000^2 & 1000 & 1 \\ 4000^2 & 4000 & 1 \\ 6000^2 & 6000 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1000000 & 1000 & 1 \\ 16000000 & 4000 & 1 \\ 36000000 & 6000 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 6.667x10^{-8} & -1.667x10^{-7} & 1.000x10^{-7} \\ -6.667x10^{-4} & 1.167x10^{-3} & -5.000x10^{-4} \\ 1.600 & -1.000 & 0.400 \end{bmatrix} \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1.617x10^{-5} \\ 0.114 \\ 7.000 \end{bmatrix}$$

$$P(S) = (-1.617x10^{-5})S^2 + 0.114S + 7.000$$

Note: in this example the inverse matrix is used, but other methods for solving systems of equations are equally valid. If the equations were simpler, substitution might have been a better approach.

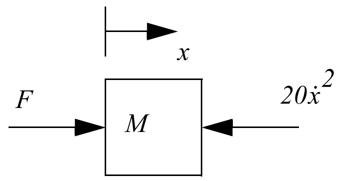
Figure 3.32 Example: Spline fitting

The order of the polynomial should match the number of points. Although, as the number of points increases, the shape of the curve will become less smooth. A common way for dealing with this problem is to fit the spline to a smaller number of points and then verify that it matches the remaining points, or use a least squares method to find the best approximation.

Non-Linear Elements

Despite our deepest wishes for simplicity, most systems contain non-linear components. In the last chapter we looked at dealing with non-linearities by linearizing them locally. Numerical techniques will handle non-linearities easily, but smaller time steps are required for accuracy.

Consider the mass and an applied force shown in Figure 3.33. As the mass moves an aerodynamic resistance force is generated that is proportional to the square of the velocity. This results in a non-linear differential equation. This equation can be numerically integrated using a technique such as Runge-Kutta. Note that the state equation matrix form cannot be used because it requires linear equations.



$$\sum F_x = F - 20\dot{x}^2 = M\ddot{x}$$

$$\dot{x} = v \qquad \text{eqn 3.14}$$

$$\dot{v} = \frac{-20}{M}v^2 + \frac{F}{M} \qquad \text{eqn 3.15}$$

Figure 3.33 Example: Developing state equations for a non-linear system

3.1 Practical Aspects of Computer Mathematics

As engineers we use computers to perform many calculation quickly. There is an obvious trade-off between computer cost and speed. And, of course, more precision is more costly. For example, if we are using a desktop system for the analysis of experimental data we may be able to buy an expensive computer with a high level of mathematical precision. However a microprocessor that is to be put into an automotive part must have a very low cost. Many computers have math functions built into an Arithmetic Logic Unit (ALU) core. On smaller microcontrollers this will often do integer (2s compliment) addition, subtraction, multiplication, and division. More advanced computers will include a math co-processor unit that is dedicated to returning floating point numerical results quickly.

Numbering Systems

Computer based number representations are ultimately reduced to true or false values. The simplest number is a binary bit with an integer range from 0 to 1, however bits are normally grouped into some larger multiple of 8 based upon the size of the date bus (e.g., 8-byte, 16-word, 32- long word, 64). These are then used to represent numerical values over a range. For example a byte can represent an unsigned integer value from 0 to 255, a signed integer value from -127 to 127, or a 2s compliment integer value from -128 to 127. As expected the precision and range of the number increases with the number of bits. And, by allocating a few of the bits for an exponent, the number can be used to represent large real values. Figure 3.34 shows a very simplistic comparison of the number systems.

Number Type	Program Size	Execution Speed	Range
unsigned integer	in hardware	few clock cycles	16 bit (2 bytes) 0 to 65,???
2s compliment	in hardware	few clock cycles	16 bit (2 bytes) -32,768 to 32,767
fixed point	adds 2K	a hundred clock cycles	4 bytes typical
floating point (single)	adds 10K	hundreds of clock cycles	4 bytes
floating point (double)	adds 12K	hundreds of clock cycles	8 bytes

Figure 3.34 Typical Number System Trade-offs

The subject of computer numbering systems is very broad and this book is not intended to teach the fundamentals of these numbering systems. But, to provide some design direction,

- If possible use 2s compliment integer calculations with 2 or 4 bytes. These will run easily on inexpensive hardware, And if used on faster computers there will be a substantial speed bonus. The problems are the limited range and loss of fractions.
- Floating point calculations are the standard on regular (non-embedded) computers. The numbers are 7 place, single precision, 4 byte numbers or 14 place, double precision, 8 byte numbers. Unless space is at a premium, use double precision. To implement these subroutines in small computers can consume large amounts of available memory and processor time.
- Fixed point numbers do not allow all of the flexibility of the floating point values, but preserve the fractional results. These are often used to get the speed or memory benefits of floating point calculations on lower end computers.

Figure 3.35 shows two subroutines that are effectively identical, except for the choice of number system. Both will loop 999 times, incrementing the index value by adding one. The print statement within the loop also requires are division and modulo operations. All three of the mathematical operations are slower for double precision. Moreover the 'printf' statement will require more time to print the results for the floating point numbers. Overall the integer loop will probably run an order of magnitude faster.

```
void loop_fast(){
        int     i;
        for(i = 0; i < 1000; i++){
                printf("Count %d.%d\n", i/10, i%100);
        }
}


void loop_slow(){
        double  r;
        for(r = 0.0; r < 1000.0; r++){
                printf("Count %f.%f\n", r/10.0, r%100.0);
        }
}
```

Figure 3.35 Example: Program speed based upon number system

Speed

In the design of controls systems the execution time for a program may be critical to system stability. Or, when running large numerical calculations, small changes can save days or weeks of computer time. Each operation requires a finite number of computer CPU cycles with the number varying based upon the instruction. For example a sign change is very fast, addition and subtraction can be slower, multiplication and division slower still, and a trigonometric operation is among the slowest. If the operations are done in hardware they will be much faster. If done in software the speed will vary depending upon the compiler. It is often possible to reduce the computation time by reducing the number of slower operations. In Figure 3.36 there is a simple manipulation that eliminates one addition, or a more elaborate method that eliminates one trigonometric operation. In practice this would probably reduce the calculation time by at least one quarter to a half.

$$A = \cos(5t+6) + \sin(2.5t+3)$$ (2 mult, 3 add, 2 trig)

noticing that one argument is twice that of another allows the elimination of an add,

$$x = 2.5t + 3$$

$$A = \cos(2x) + \sin(x)$$

(2 mult, 2 add, 2 trig)

$$\cos(2x) = (\cos(x))^2 - (\sin(x))^2$$

$$1 = (\cos(x))^2 + (\sin(x))^2$$

$$\therefore A = \cos(2x) + \sin(x)$$

$$\therefore A = (\cos(x))^2 - (\sin(x))^2 + \sin(x)$$

this can be continued to eliminate a trig operation,

$$\therefore A = 1 - (\sin(x))^2 - (\sin(x))^2 + \sin(x)$$

$$\therefore A = 1 - (2\sin(x))^2 + \sin(x)$$

$$y = \sin(2.5t + 3)$$

$$A = 1 - y(y) + y$$ (2 mult, 3 add, 1 trig)

The C code would look like,

```
#include <math.h>

double A(double t){
        double y;
        y = sin(2.5 * t + 3);
        return 1.0 - y * y + y;
}
```

Figure 3.36 Rearranging Expressions to Increase Execution Speed

Accuracy

Computer calculations are generally repeatable - meaning that repeating a calculation will give exactly the same result. Although this does not mean that the result is correct. For example a small error (one in a million) repeated a million times becomes significant. Considering the iterative nature of numerical calculations this scenario is likely to occur (note: not just possible). In these cases it is important to review the results with the following rules.

- Understand that the magnitude of errors increases with the number of calculations.
- Find a way to measure errors in calculations.
- Where possible correct for errors in calculations.

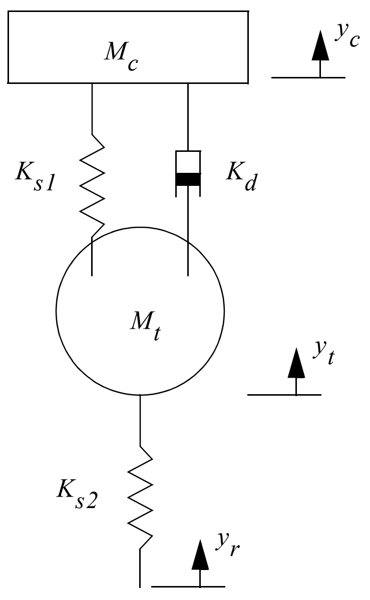
A common problem with floating point numbers is determining when the values are equal. Consider the values 2.00000001 and 1.99999998, for all practical purposes they are equal. But, from the standpoint of a computer they differ by 0.00000003 and are not equal. This can be overcome using a subroutine like that shown in Figure 3.37.

```
#define ERROR 0.00001

int equal(double A, double B){
        if( fabs(A - B) < ERROR ){
                return 0;
        } else {
                return 1;
        }
}
```

Figure 3.37 Example: Error Allowances for Equivalences

3.2 Case Study

Consider the simplified model of a car suspension shown in Figure 3.38. The model distributes the vehicle weight over four tires with identical suspensions, so the mass of the vehicle is divided by four. In this model the height of the road will change and drive the tire up, or allow it to drop down. The tire acts as a stiff spring, with little deflection. The upper spring and damper are the vibration isolation units. The damper has been designed to stiffen as the damper is compressed. The given table shows how the damper coefficient varies with the amount of compression.



$$K_{s1} = 50000\frac{N}{m}$$

$$K_{s2} = 200000\frac{N}{m}$$

$$M_c = 400Kg$$

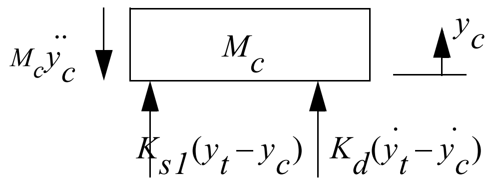
$$M_t = 20Kg$$

$$L = y_c - y_t$$

L (cm)	Kd (Ns/m)
30	1000
0	1600
-20	2000
-30	2250

Figure 3.38 Example: A model of a car suspension system

For our purposes we will focus only on the translation of the tire, and ignore its rotational motion. The differential equations describing the system are developed in Figure 3.39.



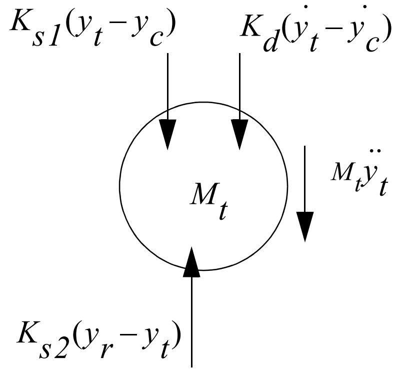
$$\sum F = K_{s1}(y_t - y_c) + K_d(\dot{y}_t - \dot{y}_c) - M_c g = M_c \dot{y}_c$$

$$M_c \ddot{y}_c = (K_{s1})y_t + (-K_{s1})y_c + (K_d)\dot{y}_t + (-K_d)\dot{y}_c - M_c g$$

$$\frac{d}{dt}y_c = v_c \qquad \text{eqn 3.16}$$

$$\frac{d}{dt}y_t = v_t \qquad \text{eqn 3.17}$$

$$\dot{v}_c = \left(\frac{K_{s1}}{M_c}\right)y_t + \left(\frac{-K_{s1}}{M_c}\right)y_c + \left(\frac{K_d}{M_c}\right)v_t + \left(\frac{-K_d}{M_c}\right)v_c + (-g) \qquad \text{eqn 3.18}$$



$$\sum F = K_{s2}(y_r - y_t) - K_{s1}(y_t - y_c) - K_d(\dot{y}_t - \dot{y}_c) - M_t g = M_t \ddot{y}_t$$

$$M_t \ddot{y}_t = (K_{s2})y_r + (-K_{s2} - K_{s1})y_t + (K_{s1})y_c + (-K_d)\dot{y}_t + (K_d)\dot{y}_c - M_t g$$

$$\dot{v}_t = \left(\frac{K_{s2}}{M_t}\right)y_r + \left(\frac{-K_{s2} - K_{s1}}{M_t}\right)y_t + \left(\frac{K_{s1}}{M_t}\right)y_c + \left(\frac{-K_d}{M_t}\right)v_t + \left(\frac{K_d}{M_t}\right)v_c + (-g) \qquad \text{eqn 3.19}$$

Figure 3.39 Example: Differential and state equations for the car suspension system

The damping force must be converted from a tabular form to equation form. This is done in Figure 3.40.

There are four data points, so a third order polynomial is required.

L (cm)	Kd (Ns/m)
30	1000
0	1600
-20	2000
-30	2250

$$K_d(L) = AL^3 + BL^2 + CL + D$$

The four data points can now be written in equation form, and then put into matrix form.

$$1000 = A(0.3)^3 + B(0.3)^2 + C(0.3) + D$$
$$1600 = A(0)^3 + B(0)^2 + C(0) + D$$
$$2000 = A(-0.2)^3 + B(-0.2)^2 + C(-0.2) + D$$
$$2250 = A(-0.3)^3 + B(-0.3)^2 + C(-0.3) + D$$

$$\begin{bmatrix} 0.027 & 0.09 & 0.3 & 1 \\ 0 & 0 & 0 & 1 \\ -0.008 & 0.04 & -0.2 & 1 \\ -0.027 & 0.09 & -0.3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1000 \\ 1600 \\ 2000 \\ 2250 \end{bmatrix}$$

The matrix can be solved to find the coefficients, and the final equation written.

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -2778 \\ 277.8 \\ -1833 \\ 1600 \end{bmatrix}$$

$$K_d(L) = (-2778)L^3 + (277.8)L^2 + (-1833)L + 1600$$

Figure 3.40 Example: Fitting a spline to the damping values

The system is to be tested for overall deflection when exposed to obstacles on the road. For the initial conditions we need to find the resting heights for the tire and car body. This can be done by setting the accelerations and velocities to zero, and finding the resulting heights.

The initial accelerations and velocities are set to zero, assuming the car has settled to a steady state height. This then yields equations that can be used to calculate the initial deflections. Assume the road height is also zero to begin with.

$$0 = \left(\frac{K_{s1}}{M_c}\right)y_t + \left(\frac{-K_{s1}}{M_c}\right)y_c + \left(\frac{K_d}{M_c}\right)0 + \left(\frac{-K_d}{M_c}\right)0 + (-g)$$

$$y_c = y_t - g\frac{M_c}{K_{s1}}$$

$$0 = \left(\frac{K_{s2}}{M_t}\right)0 + \left(\frac{-K_{s2} - K_{s1}}{M_t}\right)y_t + \left(\frac{K_{s1}}{M_t}\right)y_c + \left(\frac{-K_d}{M_t}\right)0 + \left(\frac{K_d}{M_t}\right)0 + (-g)$$

$$gM_t = (-K_{s2} - K_{s1})y_t + (K_{s1})y_c$$

$$gM_t = (-K_{s2} - K_{s1})y_t + (K_{s1})\left(y_t - g\frac{M_c}{K_{s1}}\right) \qquad y_t = -g\frac{M_c + M_t}{K_{s2}}$$

$$y_c = -g\frac{M_c + M_t}{K_{s2}} - g\frac{M_c}{K_{s1}} \qquad y_c = -g\left(\frac{M_c + M_t}{K_{s2}} + \frac{M_c}{K_{s1}}\right)$$

Figure 3.41 Example: Calculation of initial deflections

The resulting calculations can then be written in a computer program for analysis, as shown in Figure 3.42.

```
#include <stdio.h>
#include <math.h>

#define SIZE    4       /* define state variables */
#define y_c     0
#define y_t     1
#define v_c     2
#define v_t     3

#define N_step  10000   // number of steps
#define h_step  0.001   // define step size

#define Ks1     50000.0 /* define component values */
#define Ks2     200000.0
#define Mc      400.0
#define Mt      20.0
#define grav    9.81

void integration_step(double h, double state[], double derivative[]){
        int i;
        for(i = 0; i < SIZE; i++) state[i] += h * derivative[i];
}

double damper(double L){
        return (-2778*L*L*L + 277.8*L*L - 1833*L + 1600);
}

double y_r(double t){
        /* return 0.0; /* a zero input to test the initial conditions */
        /* return 0.2 * sin(t);/* a sinusoidal oscillation */
         return 0.2; */ /* a step function */
        /* return 0.2 * t; */ /* a ramp function */
}
```

Figure 3.42   Example: Program for numerical analysis of suspension system

```
void d_dt(double t, double state[], double derivative[]){
        double  Kd;
        Kd = damper(state[y_c] - state[y_t]);
        derivative[y_c] = state[v_c];
        derivative[y_t] = state[v_t];
        derivative[v_c] = (Ks1/Mc)*state[y_t] - (Ks1/Mc)*state[y_c]
                + (Kd/Mc)*state[v_t] - (Kd/Mc)*state[v_c] - grav;
        derivative[v_t] = (Ks2/Mt)*y_r(t) - ((Ks2+Ks1)/Mt)*state[y_t]
                + (Ks1/Mt)*state[y_c] - (Kd/Mt)*state[v_t]
                + (Kd/Mt)*state[v_c] - grav;
}

main(){
        double  state[SIZE];
        double  derivative[SIZE];
        FILE            *fp_out;
        double  t;
        int             i;

        state[y_c] = - grav *  ( (Mc/Ks1) + (Mt + Mc)/Ks2 ); /* initial values */
        state[y_t] = - grav * (Mt + Mc) / Ks2;
        state[v_c] = 0.0;
        state[v_t] = 0.0;

        if((fp_out = fopen("out.txt", "w")) != NULL){ /* open the file */
                fprintf(fp_out, " t   Yc     Yt     Vc     Vt  \n");
                for(t = 0.0, i = 0; i < N_step; i++, t += h_step){
                    if((i % 100) == 0) fprintf(fp_out, "%f    %f   %f    %f   %f \n",
                        t, state[y_c], state[y_t], state[v_c], state[v_t]);
                    d_dt(t, state, derivative);
                    integration_step(h_step, state, derivative);
                }
        } else {
                printf("ERROR: Could not open file \n");
        }
        fclose(fp_out);
}
```

Figure 3.43 Example: Program for numerical analysis of suspension system (continued)

This program was then used to test various design cases by selecting input types for changes in the road height, and then calculating how the tire and vehicle heights would change as a result. Some of these results are seen in Figure 3.44. These results were obtained by running the program, and then graphing the results in a spreadsheet program. The input of zero for the road height was used to test the program. As shown the height of the vehicle changes, indicating that the initial height calculations are correct, and the model is stable. The step function shows some oscillations that settle out to a stable final value. The oscillation is relatively slow, and is fully transmitted to the automobile. The ramp function shows that the car follows the rise of the slope with small transient effects at the start.

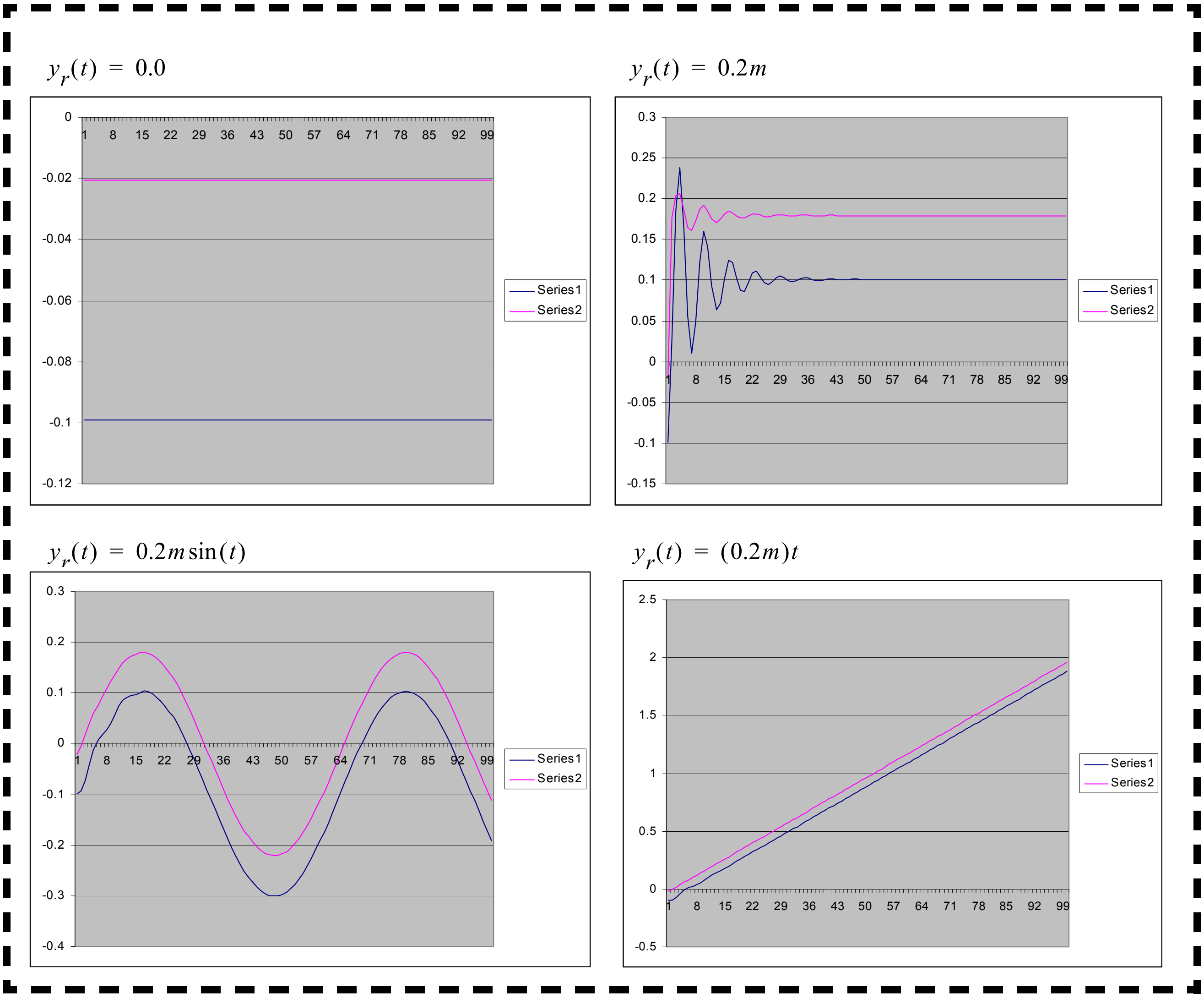


Figure 3.44 Example: Graphs of simulation results

3.1 Summary

- State variable equations are differential equations reduced to first order differential equations.
- First order equations can be integrated numerically.
- Higher order integration, such as Runge-Kutta increase the accuracy.
- Switching functions allow functions terms to be turned on and off to provide more complex function.
- Tabular data can be used to get numerical values.

3.2 Problems With Solutions

Problem 3.1 Put the equation into state variable and matrix form.

$$\sum F = F = M\left(\frac{d}{dt}\right)^2 x$$

Problem 3.2 Convert the following differential equations to state variable form.

$$\ddot{x} + 2\dot{x} + 3x + 5y = 3$$

$$\ddot{y} + \dot{y} + 6y + 9x = \sin(t)$$

Problem 3.3 a) Put the differential equations given below in state variable form. b) Put the state equations in matrix form

$$\ddot{y}_1 + 2\dot{y}_1 + 3y_1 + 4\dot{y}_2 + 5y_2 + 6\dot{y}_3 + t + F = 0$$

$$\dot{y}_1 + 7y_1 + 8\dot{y}_2 + 9y_2 + 10\ddot{y}_3 + 11y_3 + 5\cos(5t) = 0$$

$$\dot{y}_1 + 12\ddot{y}_2 + 13\dot{y}_3 = 0$$

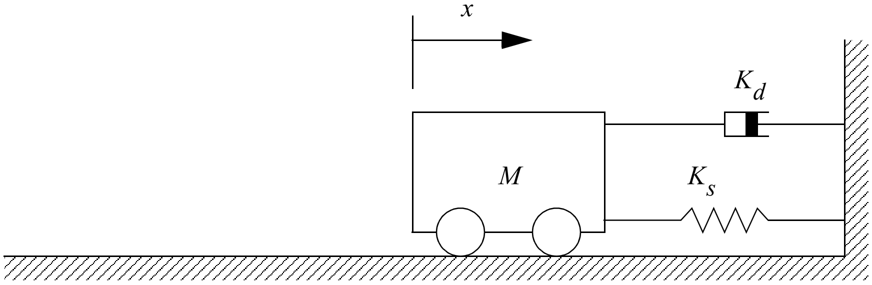
y_1, y_2, y_3 = outputs
F = input

Problem 3.4 Convert the equations to state equations.

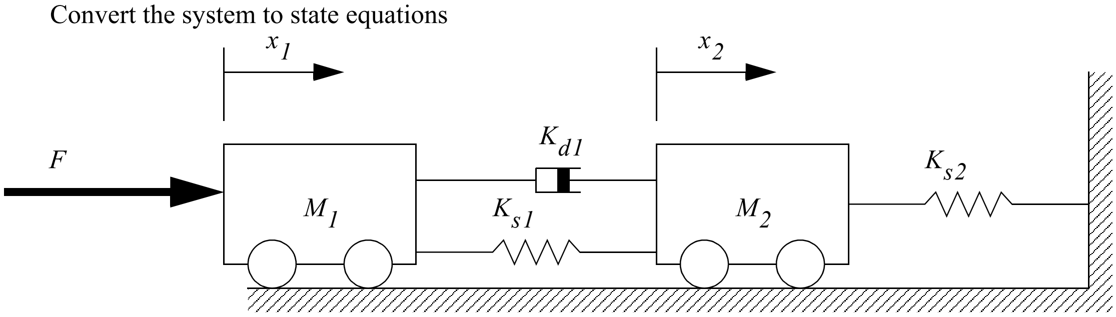
$$K_{S1}(-y_1) + K_{S2}(y_2 - y_1) + M_1(-9.81) + K_d(\dot{y}_2 - \dot{y}_1) = M_1\ddot{y}_1$$

$$-K_{S2}(y_2 - y_1) + M_2(-9.81) - K_d(\dot{y}_2 - \dot{y}_1) = M_2\ddot{y}_2$$

Problem 3.5 Develop the state equations in matrix form



Problem 3.6 Convert the system to state equations



Problem 3.7 Do a first order numerical integration of the derivative below from 0 to 10 seconds in one second steps. Assume the system starts undeflected and at rest. Write the equation for each time step.

$$\frac{d}{dt}x(t) = 5(t-4)^2$$

Problem 3.8 Use first-order integration to solve the differential equation from 0 to 10 seconds with time steps of 1 second. Assume that the system starts at rest and undeflected.

$$\dot{x} + 0.1x = 5$$

Problem 3.9 Solve the differential equation using first order numerical integration, for 0s, 0.5s, 1.0s, and 1.5s. Show the calculations.

$$F = M\left(\frac{d}{dt}\right)^2 x$$

use,

$x(0) = 1$
$v(0) = 2$
$h = 0.5\ s$
$F = 10$
$M = 1$

Problem 3.10 Given the differential equation, integrate the values numerically for the first ten seconds with 1 second steps. Assume the initial value of x is 1. You may use first order or Runge-Kutta integration.

$$\dot{x} + 0.25x = 3$$

Problem 3.11 Numerically integrate one time step of the differential equation using a) first order integration and b) Runge-Kutta integration.c) Write a computer program in a language of your choice (C/C++, Java, Scilab script, etc.) that will numerically integrate the differential equation.

$$\ddot{\theta} + 3\dot{\theta} + 9\theta = 10$$

Problem 3.12 Write a program to implement the following equation to calculate the value of x.

$$x = \sum_{i=0}^{i<100} 5iu(i-20)$$

where,

$$u(t) = 0 \qquad \text{when} \qquad t \le 0$$

$$u(t) = 1 \qquad \text{when} \qquad t > 0$$

Problem 3.13 Develop the state equations and numerically integrate the following differential equation for 20s for the given input function. Show the results in a table.

$$\dddot{Q} + 2\ddot{Q} + 10\dot{Q} + 20Q = 5\theta$$

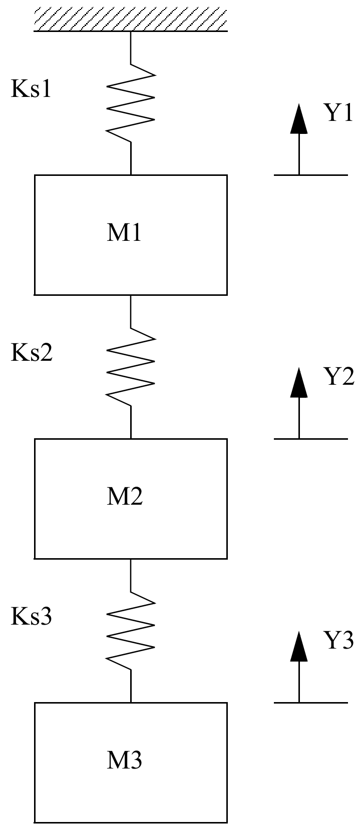
where,

$$\theta = 0 \qquad t < 0$$

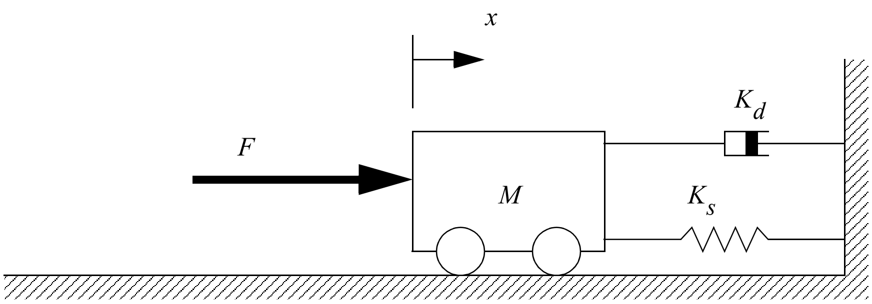
$$\theta = 0.1t \qquad 0 \le t < 10s$$

$$\theta = 1 \qquad t \ge 10s$$

Problem 3.14 Write the state equations for the system shown.

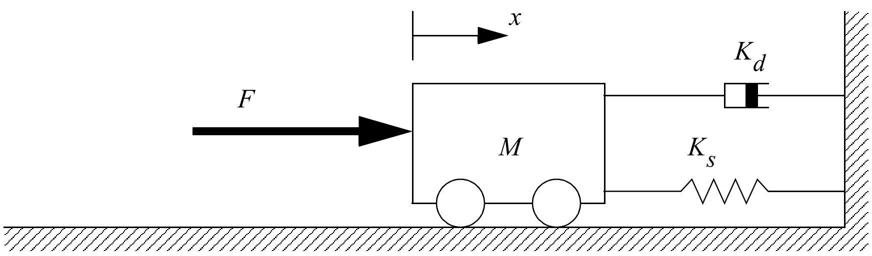


Problem 3.15 Develop state equations for the mass-spring-damper system below.



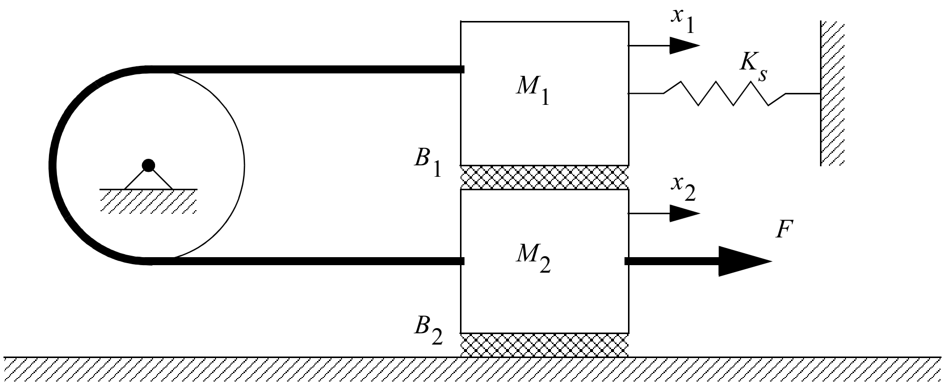
Problem 3.16 a) For the mass-spring-damper system below solve the differential equation as a function of time. Assume the system starts at rest and undeflected. b) Also, solve the problem using your Scilab (and state equations) to verify your solution. Sketch the results.

$K_s = 10$
$K_d = 10$
$M = 10$
$F = 10$



Problem 3.17 The system below is comprised of two masses. There is viscous damping between the masses and between the bottom mass and the floor. The masses are also connected with a cable that is run over a massless and frictionless

pulley. Write the differential equations for the system, and put them in state variable form.



Problem 3.18 Find the response of the following differential equation to the given step input. Assume the initial conditions are all zero. Solve the problem using a numerical method and show the values from 0 to 10 seconds in 1 second intervals.

$$\dddot{x} + 10\ddot{x} + 100\dot{x} + 1000x = 4F \qquad F(t) = 10u(t)$$

Problem 3.19 Given the following differential equation and initial conditions, draw a sketch of the first 5 seconds of the output response. The input is a step function that turns on at t=0. Use at least two different methods, and compare the results.

$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2$$

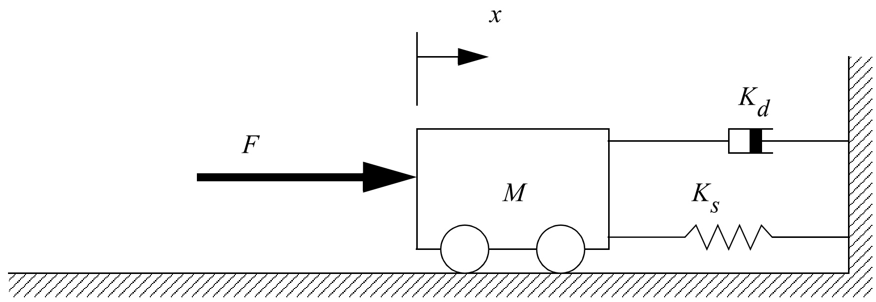
initial conditions

$$V_i(t \geq 0) = 5V$$
$$V_o(0) = 0V$$
$$\dot{V}_o(0) = 1Vs^{-1}$$

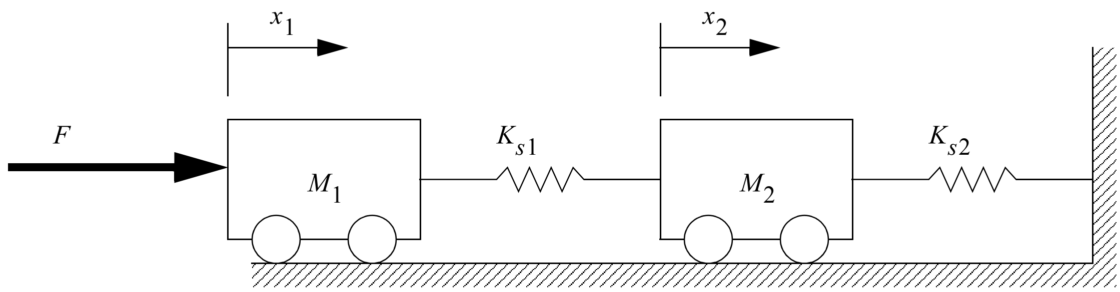
Problem 3.20 The mechanical system below is a mass-spring-damper system. A force ‘F’ of 100N is applied to the 10Kg cart at time t=0s. The motion is resisted by the spring and damper. The spring coefficient is 1000N/m, and the damper coefficient is to be determined. Follow the steps below to develop a solution to the problem. Assume the system always starts undeflected and at rest.

a) Develop the differential equation for the system.
b) Solve the differential equation using damper coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
c) Develop the state equations for the system.
d) Solve the system with a first order numerical analysis using Scilab for damper coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
e) Solve the system with a Runge-Kutta numerical analysis using Scilab for damper coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
f) Write a computer program (in C, Java or Fortran) to do the Runge Kutta numerical integration in step e). Draw a graph of the results.
g) Compare all of the solutions found in the previous steps.
h) Select a damper coefficient to give an overall system damping factor of 1. Verify the results by numerically

integrating.

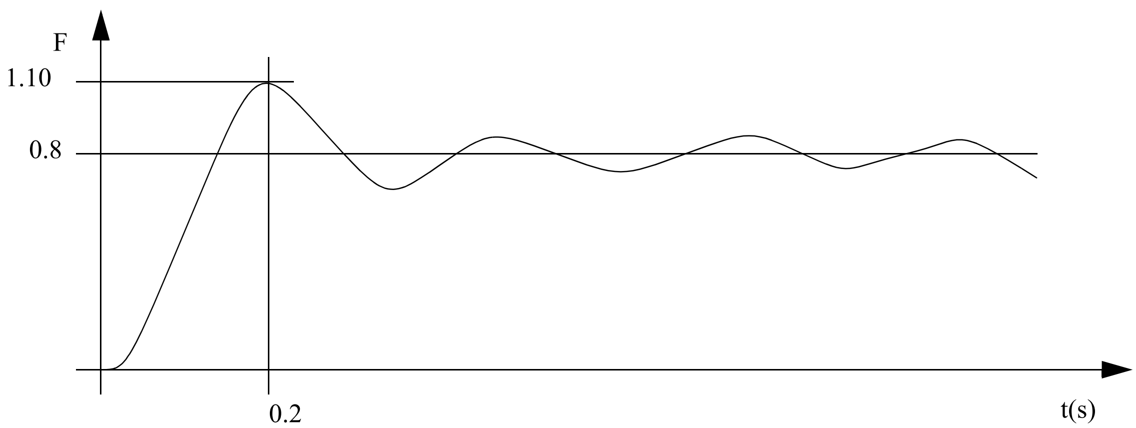


Problem 3.21 For the mechanism illustrated in the figure below the values are $K_{s1}=K_{s2}=100N/m$, $M_1=M_2=1kg$, F=1N. Assume that the system starts at rest, and the springs are undeformed initially.



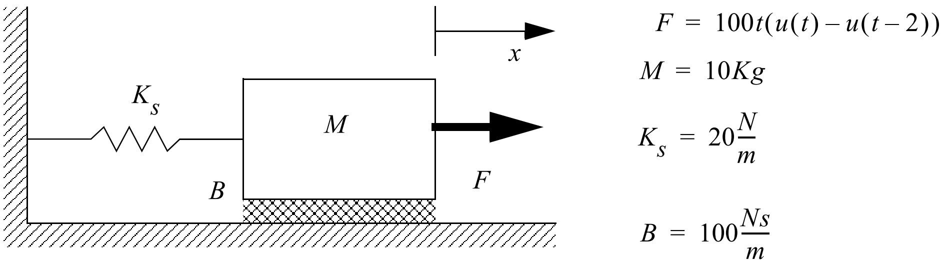
a. Derive the differential equations for the system.
b. Put the equations in state variable form.
c. Put the equations in state variable matrices.
d. Use Scilab to find values for x_1 and x_2 over the first 10 seconds. Provide the results in a table in 1 second intervals.
e. Use Scilab to plot the values for the first 10 seconds.
f. Use a Scilab program and the Runge-Kutta method to produce a graph of the first 10 seconds.
g. Repeat step g. using the first-order approximation method.
h. Use a C program to produce a graph of points for the first 10 seconds.

Problem 3.22 The second order response below was obtained for a unit step input (x) to a system. Develop the state equation and find the response to a ramp input using a numerical method.

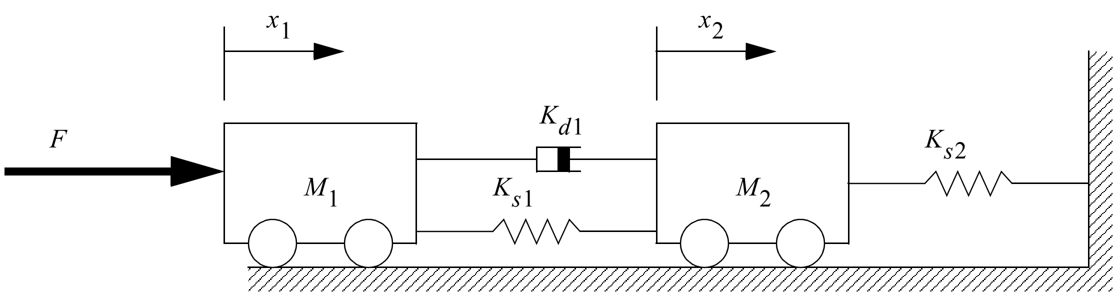


Problem 3.23 The system shown has the applied force that is 0N at 0s, increases to 200N at 2s, and then goes to zero. a) Find the response using a computer program to do numerical integration. b) Find the response by integrating the dif-

ferential equation.

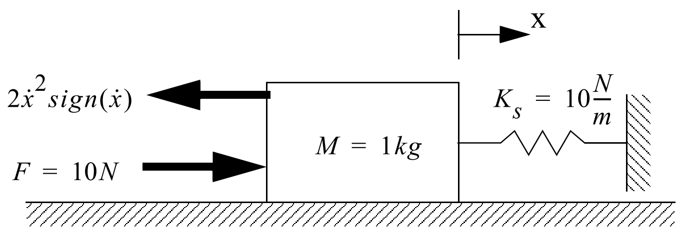


Problem 3.24 For the mechanism shown in the figure below the values are K_{s1}=K_{s2}=100N/m, K_{d1}=10Nm/s, M_1=M_2=1kg, F=1N. Assume that the system starts at rest, and the springs are undeformed initially.

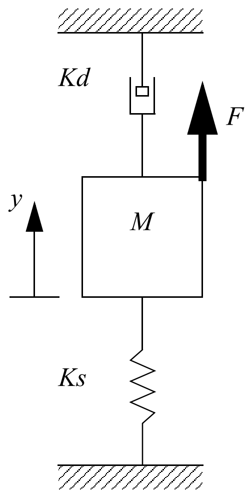


a. Derive the differential equations for the system.
b. Put the equations in state variable form.
c. Put the equations in state variable matrices.
d. Use Scilab and first order integration to find values for x_1 and x_2 over the first 10 seconds. Provide the results in a table in 1 second intervals.
e. Use Scilab to plot the values for the first 10 seconds using the values obtained in part d.
f. Use a Scilab program and the Runge-Kutta method to produce a graph of the first 10 seconds.
g. Use a C program to produce a list of points for the first 10 seconds.
h. Compare the results found in steps d, f and g in a table.

Problem 3.25 Use a numerical method to find the position of the mass below over the first 2s. Record these values in a table and sketch the curve. The mass starts at rest at x=0. The block experiences aerodynamic drag that opposes motion. Assume the surface is frictionless.



Problem 3.26 a) Use numerical integration to find the system response from 0 to 10s. .



$$K_d = 10Nsm^{-1}$$

$$K_s = 100Nm^{-1}$$

$$M = 1kg$$

$$y_0 = 1m \qquad \dot{y}_0 = 0m$$

$$F = 10\sin(10t)$$

Problem 3.27 b) Modify the program to use the variables below and then use trial and error to select a new Kd value that makes it critically damped..

$$K_d = ?Nsm^{-1} \qquad K_s = 100Nm^{-1} \qquad M = 1kg$$

$$y_0 = 1m \qquad \dot{y}_0 = 0m \qquad F = 10$$

Problem 3.28 c) Develop an explicit solution using the given coefficients and input..

$$K_d = 100Nsm^{-1} \qquad K_s = 10Nm^{-1} \qquad M = 1kg$$

$$y_0 = 1m \qquad \dot{y}_0 = 0m \qquad F = 10\sin(10t)$$

3.3 Problem Solutions

Answer 3.1

$$\dot{x} = v$$
$$\dot{v} = \frac{F}{M}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

Answer 3.2

$$\dot{x} = v$$
$$\dot{y} = u$$
$$\dot{v} = -2v - 3x - 5y + 3$$
$$\dot{u} = -u - 6y - 9x + \sin t$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -5 & -2 & 0 \\ -9 & -6 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ v \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \\ \sin t \end{bmatrix}$$

Answer 3.3

$$\dot{y}_1 = v_1$$
$$\dot{y}_2 = v_2$$
$$\dot{y}_3 = v_3$$
$$\dot{v}_1 = -2v_1 - 3y_1 - 4v_2 - 5y_2 - 6v_3 - t - F$$
$$\dot{v}_2 = \frac{-v_1}{12} - \frac{13}{12}v_3$$
$$\dot{v}_3 = \frac{-v_1}{10} - \frac{7}{10}y_1 - \frac{8}{10}v_2 - \frac{9}{10}y_2 - \frac{11}{10}y_3 - \frac{5}{10}\cos(5t)$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -5 & 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & -\frac{1}{12} & 0 & -\frac{13}{12} \\ -\frac{7}{10} & -\frac{9}{10} & -\frac{11}{10} & -\frac{1}{10} & -\frac{8}{10} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -t-F \\ 0 \\ -\frac{5}{10}\cos(5t) \end{bmatrix}$$

Answer 3.4

$$\dot{y}_1 = v_1$$
$$\dot{y}_2 = v_2$$
$$M_1\dot{v}_1 = -K_{S1}y_1 + K_{S2}y_2 - K_{S2}y_1 + (-9.81M_1) + K_d v_2 - K_d v_1$$
$$\dot{v}_1 = y_1\left(\frac{-K_{S1} - K_{S2}}{M_1}\right) + v_1\left(\frac{-K_d}{M_1}\right) + y_2\left(\frac{K_{S2}}{M_1}\right) + v_2\left(\frac{K_d}{M_1}\right) + (-9.81)$$
$$M_2\dot{v}_2 = (-K_{S2}y_2 + K_{S2}y_1) + (-9.81M_2) - K_d v_2 + K_d v_1$$
$$\dot{v}_2 = y_1\left(\frac{K_{S2}}{M_2}\right) + v_1\left(\frac{K_d}{M_2}\right) + y_2\left(\frac{-K_{S2}}{M_2}\right) + v_2\left(\frac{-K_d}{M_2}\right) + (-9.81)$$

Answer 3.5

$$\dot{x} = v$$
$$\dot{v} = v\left(\frac{-K_d}{M}\right) + x\left(\frac{-K_s}{M}\right)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Answer 3.6

$$\dot{x}_1 = v_1$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_1 = v_1\left(\frac{-K_{d1}}{M_1}\right) + x_1\left(\frac{-K_{s1}}{M_1}\right) + v_2\left(\frac{K_{d1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + \frac{F}{M_1}$$
$$\dot{v}_2 = v_2\left(\frac{-K_{d1}}{M_2}\right) + x_2\left(\frac{-K_{s1} - K_{s2}}{M_2}\right) + v_1\left(\frac{K_{d1}}{M_2}\right) + x_1\left(\frac{K_{s1}}{M_2}\right)$$

Answer 3.7

$$h = 1$$

$$x(t+h) = x(t) + h5(t-4)^2$$

$$x(1) = x(0) + 5(0-4)^2 = 80$$

$$x(2) = x(1) + 5(1-4)^2 = 125$$

t	x
0	0
1	80
2	125
3	145
4	150
5	150
6	155
7	175
8	220
9	300
10	425

Answer 3.8

$$x(t+h) = x(t) + h(-0.1x(t) + 5)$$

$$x(10) =$$

Answer 3.9

$$F = M\left(\frac{d}{dt}\right)^2 x$$

$$\dot{x} = v$$

$$\dot{v} = \frac{F}{M} = 10$$

t	$\dot{v}_{i+1} = 10$	$v_{i+1} = v_i + 0.5\dot{v}_i$	$x_{i+1} = x_i + 0.5v_i$
0	10	2	1
0.5	10	7	2
1.0	10	12	5.5
1.5	10	17	11.5

Answer 3.10

$$x' = 3 - 0.25x$$

$$x(t+h) = x(t) + h\left(\frac{d}{dt}x(t)\right)$$

$$x(t+h) = x(t) + 1(3 - 0.25x(t))$$

$$x(t+h) = 0.75x(t) + 3$$

t	x	x’
0	1	2.75
1	3.75	2.06
2	5.81	1.55
3	7.36	1.16
4	8.52	0.870
5	9.39	0.652
6	10.0	0.489
7	10.5	0.367
8	10.9	0.275
9	11.2	0.206
10	11.4	

Answer 3.11 a)

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -3\omega - 9\theta + 10$$

$$\frac{d}{dt}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -9 & -3\end{bmatrix}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + \begin{bmatrix}0 \\ 10\end{bmatrix}$$

This is the solution done using variables.

$$\begin{bmatrix}\theta_{i+1} \\ \omega_{i+1}\end{bmatrix} \approx \begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + h\frac{d}{dt}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} = \begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + h\begin{bmatrix}0 & 1 \\ -9 & -3\end{bmatrix}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + h\begin{bmatrix}0 \\ 10\end{bmatrix}$$

$$\begin{bmatrix}\theta_{i+1} \\ \omega_{i+1}\end{bmatrix} = \begin{bmatrix}\theta_i + h\omega_i \\ \omega_i - 9h\theta_i - 3h\omega_i + 10h\end{bmatrix}$$

b)

$$F_1 \approx h\frac{d}{dt}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} = h\begin{bmatrix}0 & 1 \\ -9 & -3\end{bmatrix}\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + h\begin{bmatrix}0 \\ 10\end{bmatrix} = \begin{bmatrix}h\omega_i \\ -9h\theta_i - 3h\omega_i + 10h\end{bmatrix}$$

$$F_2 \approx h\frac{d}{dt}\left[\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + \frac{F_1}{2}\right] = h\begin{bmatrix}0 & 1 \\ -9 & -3\end{bmatrix}\begin{bmatrix}\theta_i + \dfrac{h\omega_i}{2} \\ \omega_i + \dfrac{-9h\theta_i - 3h\omega_i + 10h}{2}\end{bmatrix} + h\begin{bmatrix}0 \\ 10\end{bmatrix}$$

$$= \begin{bmatrix}h\theta_i + \dfrac{h^2\omega_i}{2} \\ -9h\theta_i - 9\dfrac{h^2\omega_i}{2} - 3h\omega_i + \dfrac{27h^2\theta_i + 9h^2\omega_i + 30h^2}{2} + 10h\end{bmatrix}$$

$$F_3 \approx h\frac{d}{dt}\left[\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + \frac{F_2}{2}\right]$$

$$F_4 \approx h\frac{d}{dt}\left[\begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + F_3\right]$$

$$\begin{bmatrix}\theta_{i+1} \\ \omega_{i+1}\end{bmatrix} \approx \begin{bmatrix}\theta_i \\ \omega_i\end{bmatrix} + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

c)

```
#include <stdio.h>

int main(){
        int     steps = 100,
                i;
        double  theta,
                omega,
                step_t,
                theta_last,
                omega_last;
        theta = 0.0;
        omega = 0.0;
        step_t = 1.0;
        for(i = 0; i < steps; i++){
                theta_last = theta;
                omega_last = omega;
                theta = theta_last + step_t * omega_last;
                omega = omega_last + step_t*(-3 * omega_last - 9 *
   theta_last + 10);
                printf("%f  %f  %f \n", i+step_t, theta, omega);
        }
}
```

Answer 3.12

```
// AP numerical integration

function ret=u(t)
  ret = 0;
  if(t > 0) then
    ret = 1;
  end
endfunction

x = 0;
for i = 0:99
  x = x + 5 * i * u(i - 20);
end

printf("Final value = %f\n", x);

// Final value = 23700.000000
```

Answer 3.13

$$\dot{Q} = R$$

$$\dot{R} = S$$

$$\dot{S} = -2S - 10R - 20Q + 5\theta$$

```
The value at t=0.000000 s is (Q, R, S) = (0.000000, 0.000000, 0.000000)
The value at t=0.500000 s is (Q, R, S) = (0.000987, 0.007226, 0.036225)
The value at t=1.000000 s is (Q, R, S) = (0.010194, 0.030040, 0.040156)
The value at t=1.500000 s is (Q, R, S) = (0.027865, 0.035242, -0.022205)
The value at t=2.000000 s is (Q, R, S) = (0.041190, 0.016960, -0.034372)
The value at t=2.500000 s is (Q, R, S) = (0.047603, 0.013858, 0.025059)
The value at t=3.000000 s is (Q, R, S) = (0.059079, 0.032938, 0.034771)
The value at t=3.500000 s is (Q, R, S) = (0.077599, 0.035896, -0.025631)
The value at t=4.000000 s is (Q, R, S) = (0.090961, 0.016746, -0.034314)
The value at t=4.500000 s is (Q, R, S) = (0.097376, 0.014202, 0.026508)
The value at t=5.000000 s is (Q, R, S) = (0.109131, 0.033513, 0.033950)
The value at t=5.500000 s is (Q, R, S) = (0.127759, 0.035673, -0.027339)
The value at t=6.000000 s is (Q, R, S) = (0.140880, 0.016222, -0.033552)
The value at t=6.500000 s is (Q, R, S) = (0.147208, 0.014455, 0.028172)
The value at t=7.000000 s is (Q, R, S) = (0.159211, 0.034040, 0.033136)
The value at t=7.500000 s is (Q, R, S) = (0.177925, 0.035411, -0.029000)
The value at t=8.000000 s is (Q, R, S) = (0.190795, 0.015699, -0.032701)
The value at t=8.500000 s is (Q, R, S) = (0.197043, 0.014730, 0.029823)
The value at t=9.000000 s is (Q, R, S) = (0.209300, 0.034561, 0.032247)
The value at t=9.500000 s is (Q, R, S) = (0.228089, 0.035123, -0.030641)
The value at t=10.000000 s is (Q, R, S) = (0.240702, 0.015182, -0.031774)
```

```
Q = 0;         // initial conditions
R = 0;
S = 0;
X=[Q, R, S];

// the state matrix function
function foo=f(state,t)
  if (t <= 0) then
    theta = 0;
  elseif (t < 10) then
    theta = 0.1 * t;
  else theta = 1;
  end,
        foo = [state($, 2), state($, 3), -2 * state($, 3) - 10 * state($, 2) - 20 * state($, 1)
   + 5 * theta];
endfunction

// Set the time length and step size for the integration
steps = 10000;
t_start = 0.0;
t_end = 10.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
        t = [t ; t($,:) + h];
                X = [X ; X($,:) + h * f(X($,:), t($,:))];
end

// print some results to compare
print_steps = 20;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  time = j * h + t_start;
  printf("The value at t=%f s is (Q, R, S) = (%f, %f, %f)\n", time, X(j+1,1), X(j+1,2),
   X(j+1,3));
end
```

Answer 3.14

$$\dot{y}_1 = v_1$$

$$\dot{y}_2 = v_2$$

$$\dot{y}_3 = v_3$$

$$\ddot{y}_1 M_1 + K_{s1} y_1 + K_{s2}(y_1 - y_2) + 9.81 M_1 = 0$$

$$\ddot{y}_1 = y_1\left(\frac{-K_{s1} - K_{s2}}{M_1}\right) + v_1(0) + y_2\left(\frac{K_{s2}}{M_1}\right) + v_2(0) + y_3(0) + v_3(0) - 9.81$$

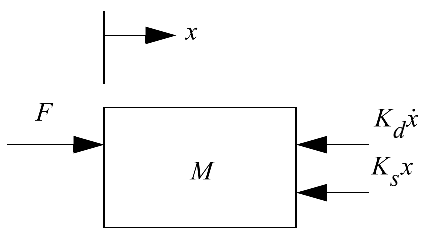
$$\ddot{y}_2 M_2 + K_{s2}(y_2 - y_1) + K_{s3}(y_2 - y_3) + 9.81 M_2 = 0$$

$$\ddot{y}_2 = y_1\left(\frac{K_{s2}}{M_2}\right) + v_1(0) + y_2\left(\frac{-K_{s2} - K_{s3}}{M_2}\right) + v_2(0) + y_3\left(\frac{K_{s3}}{M_2}\right) + v_3(0) - 9.81$$

$$\ddot{y}_3 M_3 + K_{s3}(y_3 - y_2) + 9.81 M_3 = 0$$

$$\ddot{y}_3 = y_1(0) + v_1(0) + y_2\left(\frac{K_{s3}}{M_3}\right) + v_2(0) + y_3\left(\frac{-K_{s3}}{M_3}\right) + v_3(0) - 9.81$$

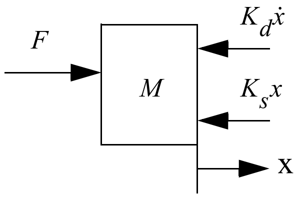
Answer 3.15



$$\dot{x} = v$$

$$\dot{v} = v\left(\frac{-K_d}{M}\right) + x\left(\frac{-K_s}{M}\right) + \left(\frac{F}{M}\right)$$

Answer 3.16



$$\sum F = F - K_d\dot{x} - K_s x = M\ddot{x}$$

$$M\ddot{x} + K_d\dot{x} + K_s x = F$$

$$10\ddot{x} + 10\dot{x} + 10x = 10$$

$$\ddot{x} + \dot{x} + x = 1$$

Homogeneous

$$\ddot{x} + \dot{x} + x = 0$$

$$A^2 + A + 1 = 0$$

$$A = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = -0.5 \pm j\frac{\sqrt{3}}{2}$$

$$x_h = C_1 e^{-0.5t}\cos\left(\frac{\sqrt{3}}{2}t + C_2\right)$$

Particular:

$$x_p = B$$

$$0 + 0 + B = 1$$

$$x_p = 1$$

Initial Conditions:

$$x = x_h + x_p = C_1 e^{-0.5t}\cos\left(\frac{\sqrt{3}}{2}t + C_2\right) + 1$$

$$x' = -\frac{\sqrt{3}}{2}C_1 e^{-0.5t}\sin\left(\frac{\sqrt{3}}{2}t + C_2\right) - 0.5C_1 e^{-0.5t}\cos\left(\frac{\sqrt{3}}{2}t + C_2\right)$$

$$x'(0) = -\frac{\sqrt{3}}{2}C_1\sin(C_2) - 0.5C_1\cos(C_2) = 0$$

$$-\frac{\sqrt{3}}{2}\sin(C_2) = 0.5\cos(C_2)$$

$$\tan(C_2) = \frac{-1}{\sqrt{3}} \qquad C_2 = \mathrm{atan}\left(\frac{-1}{\sqrt{3}}\right) = -0.5236$$

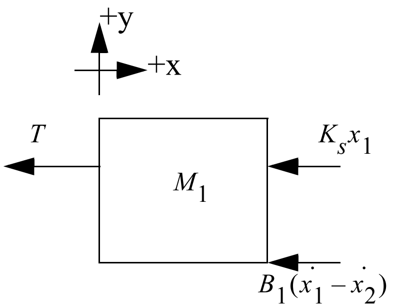
$$x(0) = C_1\cos(-0.5236) + 1 = 0$$

$$C_1 = \frac{-1}{\cos(-0.5236)} = -1.155$$

$$\boxed{x = -1.155e^{-0.5t}\sin\left(\frac{\sqrt{3}}{2}t + 1.047\right) + 1}$$

First peak is at x = 1.219 m, t = 3.63 s on graph

Answer 3.17

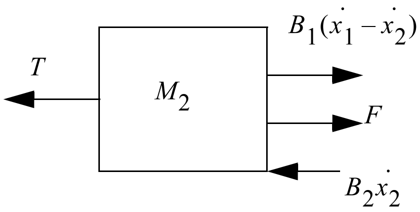


$$\sum F = -T - K_s x_1 - B_1(\dot{x}_1 - \dot{x}_2) = M_1 \ddot{x}_1$$

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = x_1\left(\frac{-K_s}{M_1}\right) + v_1\left(\frac{-B_1}{M_1}\right) + v_2\left(\frac{B_1}{M_1}\right) + \left(\frac{-T}{M_1}\right)$$



$$\sum F = -T + B_1(\dot{x}_1 - \dot{x}_2) + F - B_2\dot{x}_2 = M_2\ddot{x}_2$$

$$\dot{v}_2 = v_1\left(\frac{B_1}{M_2}\right) + v_2\left(\frac{-B_1 - B_2}{M_2}\right) + \left(\frac{-T + F}{M_2}\right)$$

Answer 3.18

t (s)	x(t)
0	0.0000
1	0.0681
2	0.0127
3	0.0575
4	0.0383
5	0.0247
6	0.0678
7	0.0837
8	0.0652
9	0.0296
10	0.0316

Answer 3.19

a)

$$V_o(t) = -8.331 e^{-0.6t} \cos(1.960t - 0.238) + 8.095$$

b)

$$\dot{V}_o = Y_o$$

$$\dot{Y}_o = -1.2 Y_o - 4.2 V_o + 34$$

```
// by: H. Jack Sept., 23, 2003

v0 = 0;        // initial conditions
y0 = 1;
X=[v0, y0];

// define the state matrix function
// the values returned are [x, v]
function foo=f(state,t)
       foo = [ state($, 2), -1.2*state($,2) - 4.2*state($,1) + 34];
endfunction


// Set the time length and step size for the integration
steps = 1000;
t_start = 0;
t_end = 10;
h = (t_end - t_start) / steps;
t = [t_start];


//
// Loop for integration
//
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       F2 = h * f(X($,:) + F1/2.0, t($,:) + h/2.0);
       F3 = h * f(X($,:) + F2/2.0, t($,:) + h/2.0);
       F4 = h * f(X($,:) + F3, t($,:) + h);
       X = [X ; X($,:) + (F1 + 2.0*F2 + 2.0*F3 + F4)/6.0];
end

//
// Graph the values
//
plot2d(t, X, [-2, -5], leg="position@velocity");
xtitle('Time (s)');

//
// Generate points from the given function
//
XX = [v0];
for i=1:steps,
       tt = i * h;
       XX = [XX ; -8.331 * exp(-0.6 * tt) * cos( 1.960 * tt - 0.238) + 8.095];
end
plot2d(t, XX, [-4], leg="explicit");
```

c) The two curves produced by the Scilab program overlap, so the results agree.

Answer 3.20

a) $$\ddot{x} + \left(\frac{K_d}{M}\right)\dot{x} + \left(\frac{K_s}{M}\right)x = \frac{F}{M}$$

b) $$K_d = 100\frac{N}{m} \qquad x_1(t) = -0.115e^{-5t}\cos(5\sqrt{3}t - 0.524) + 0.10$$

$$K_d = 10000\frac{N}{m} \qquad x_1(t) = -0.1e^{-0.1t} + 10^{-5}e^{-999.9t} + 0.10$$

c) $$\frac{d}{dt}\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

g) For 100N/m: all solutions are underdamped and overshoot at;
b) 0.1163 at t = 0.363s
d) 0.1166 at t = 0.361s
e) 0.1163 at t = 0.363s
f) 0.1163 at t = 0.360s

For 10000N/m: all solutions are overdamped. The time to reach the time constant (at 0.06321) is,
b) 0.06321 at t = 10.001s
d) 0.06321 at t = 10.000s
e) 0.06321 at t = 10.000s
f) 0.06321 at t = 10.0s

h) $$K_d = 200\frac{Ns}{m}$$ (verify with numerical integration also)

Answer 3.21

b)

$$\dot{x}_1 = v_1$$

$$\dot{x}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + \left(\frac{F}{M_1}\right)$$

$$\dot{x}_2 = v_2$$

$$\dot{x}_1 = x_1\left(\frac{K_{s1}}{M_2}\right) + x_2\left(\frac{-K_{s1} - K_{s2}}{M_2}\right)$$

c) $$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s1}}{M_1} & 0 & \frac{K_{s1}}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s1}}{M_2} & 0 & \frac{-K_{s1} - K_{s2}}{M_2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M_1} \\ 0 \\ 0 \end{bmatrix}$$

e), f), g)

```
// System component values
Ks1 = 100;
Ks2 = 100;
M1 = 1;
M2 = 1;
F = 1;

x0 = 0;         // initial conditions
v0 = 0;
x1 = 0;
v1 = 0;
X=[x0, v0, x1, v1];

// define the state matrix function the values returned are [x, v]
function foo=f(state,t)
       foo = [ state($,2), -Ks1/M1*state($,1)+Ks1/M1*state($,3)+F/M1, state($,4),
   Ks1/M2*state($,1)-(Ks1+Ks2)/M2*state($,3)];
endfunction

// Set the time length and step size for the integration
steps = 10000;
t_start = 0;
t_end = 10;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       F2 = h * f(X($,:) + F1/2.0, t($,:) + h/2.0);
       F3 = h * f(X($,:) + F2/2.0, t($,:) + h/2.0);
       F4 = h * f(X($,:) + F3, t($,:) + h);
       X = [X ; X($,:) + (F1 + 2.0*F2 + 2.0*F3 + F4)/6.0];
end
// Graph the values for part e)
plot2d(t, X, [-2, -5, -7, -9], leg="position1@velocity1@position2@velocity2");
xtitle('Time (s)');

// printf the values for part f)
printf("\n\nPart e output\n\n");
for time_count=0:20,
       i = (time_count/2) / h + 1;
       printf("Point at t=%f  x1=%f, v1=%f, x2=%f, v2=%f \n", time_count/2, X(i, 1),
   X(i, 2), X(i, 3), X(i, 4));
end

// First order integration for part h)
X=[x0, v0, x1, v1];
t = [t_start];
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       X = [X ; X($,:) + F1 ];
end
printf("\n\nPart g output \n\n");
for time_count=0:20,
       i = (time_count/2) / h + 1;
       printf("Point at t=%f  x1=%f, v1=%f, x2=%f, v2=%f \n", time_count/2, X(i, 1),
   X(i, 2), X(i, 3), X(i, 4));
end
```

h)

The following subroutine is used in place of the subroutine in the program shown in Figure 3.20 and Figure 3.21.

```
//
// State Equations Calculated Here
//
void derivative(double t, double X[], double dX[]){
        dX[0] = X[1];
        dX[1] = -Ks1 / M1 * X[0] + Ks1 / M1 * X[2] + Force / M1;
        dX[2] = X[3];
        dX[3] = Ks1 / M2 * X[0] - (Ks1 + Ks2) / M2 * X[2];
}
```

Answer 3.22

From the graph,

$$b = 1.10 - 0.8 = 0.3 \qquad \Delta x = 0.8 \qquad t_p = 0.2$$

$$\frac{b}{\Delta x} = e^{\sigma t_p} \qquad \therefore \sigma = \frac{1}{t_p}\ln\left(\frac{b}{\Delta x}\right) = \frac{1}{0.2}\ln\left(\frac{0.3}{0.8}\right) = -4.904$$

$$\zeta = \frac{1}{\sqrt{\left(\frac{\pi}{t_p\sigma}\right)^2 + 1}} = \frac{1}{\sqrt{\left(\frac{\pi}{0.2(-4.904)}\right)^2 + 1}} = 0.298$$

$$\omega_n = \frac{-\sigma}{\zeta} = \frac{-(-4.904)}{0.298} = 16.46$$

$$\ddot{F} + 2(0.298)16.46\dot{F} + (16.46)^2 F = Cx$$

For steady state,

$$(0) + 2(0.298)16.46(0) + (16.46)^2(0.8) = C(1) \qquad \therefore C = 216.7$$

$$\ddot{F} + 9.81016\dot{F} + 270.9F = 216.7x$$

$$\dot{F} = G \qquad \text{eqn 3.1}$$

$$\dot{G} = -9.81016G - 270.9F + 216.7x$$

For a unit ramp input x = t

$$\dot{G} = -9.81016G - 270.9F + 216.7t \qquad \text{eqn 3.2}$$

Answer 3.23 a)

$$\sum F = -K_s x + F - B\dot{x} - M\ddot{x} = 0 \qquad F = 100t(u(t) - u(t-2))$$

$$M = 10Kg$$

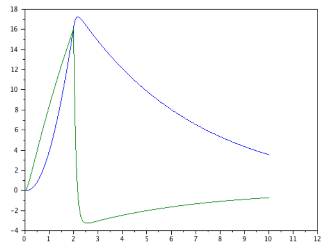
$$-K_s x + F - B\dot{x} - M\ddot{x} = 0$$

$$K_s = 20\frac{N}{m}$$

$$\dot{x} = v$$

$$B = 100\frac{Ns}{m}$$

$$\dot{v} = -\frac{K_s}{M}x + \frac{F}{M} - \frac{B}{M}v$$



```
B = 100; Ks = 20; M = 10;
x_0 = 0; v_0 = 0;
X = [x_0, v_0];
t_start = 0; t_end = 10; n_steps = 1000;
h = (t_end - t_start) / n_steps;
t = [t_start];
function val = u(time)
    val = 0;
    if time >= 0 then
        val = 1;
    end
endfunction
function X_der = state_eqn(X, t)
    x = X($, 1); v = X($, 2); t_now = t($);
    x_der = v;
    v_der = - x * Ks / M - v * B / M + 100 * t_now * (u(t_now)-u(t_now-2));
    X_der = [x_der, v_der];
endfunction
for i=0:n_steps
    X = [X; X($,:) + h * state_eqn(X, t)];
    t = [t; t($) + h];
end
plot(t, X);
```

b)

$$-K_s x + F - B\dot{x} - M\ddot{x} = 0 \qquad F = 100t(u(t) - u(t-2))$$

$$\ddot{x}(10) + \dot{x}\left(\frac{100}{10}\right) + x\left(\frac{20}{10}\right) = F$$

Homogeneous:

$$\ddot{x}(10) + \dot{x}(10) + x(2) = 0$$

$$x_h(t) = C_1 e^{-0.7236t} + C_2 e^{-0.2764t}$$

To start, for the first 2 seconds when the force has a slope of 100. Find the particular solution.

$$\ddot{x}(10) + \dot{x}(10) + x(2) = 100t$$

$$y_p = At + B \qquad \dot{y}_p = A$$

$$0(10) + A(10) + (At + B)(2) = 100t$$

$$2A = 100 \qquad A = 50$$

$$10A + 2B = 0 \qquad B = -5A = -250$$

$$y_p = 50t - 250$$

$$x(t) = C_1 e^{-0.7236t} + C_2 e^{-0.2764t} + 50t - 250$$

$$x(0) = 0 \qquad \dot{x}(0) = 0$$

$$0 = C_1 + C_2 + 0 - 250 \qquad C_1 + C_2 = 250$$

$$\dot{x}(t) = -0.7236 C_1 e^{-0.7236t} + -0.2764 C_2 e^{-0.2764t} + 50$$

$$0 = -0.7236 C_1 + -0.2764 C_2 + 50$$

$$0.7236 C_1 + 0.2764 C_2 = 50$$

$$C_1 = \frac{50 - 0.2764(250)}{0.7236 - 0.2764} = -42.710197$$

$$C_2 = 250 - (-42.710197) = 292.7102$$

$$x(t) = (-42.710197)e^{-0.7236t} + 292.7102e^{-0.2764t} + 50t - 250$$

The value and derivative must be found at t=2s to find the response for the second segment of the motion.

$$x(0) = (-42.710197)e^{-0.7236(2)} + 292.7102e^{-0.2764(2)} + 50(2) - 250$$

$$x(0) = -42.710197e^{-1.4472} + 292.7102e^{-0.5528} - 150 = 8.3612$$

$$\dot{x}(0) = ((-42.710197)(-0.7236))0.23523 + 292.7102(-0.2764)0.575336 + 50 = 10.722$$

After 2s the input force is zero, resulting in the given differential equation. Since the equation is homogeneous it is possible to proceed directly to finding the constants. Note that for this equation t = 0s is equivalent to t=2s in real time.

$$\ddot{x}(10) + \dot{x}(10) + x(2) = 0$$

$$x(t) = C_1 e^{-0.7236t} + C_2 e^{-0.2764t}$$

$$8.3612 = C_1 e^{-0.7236(2)} + C_2 e^{-0.2764(2)} = C_1 0.23523 + C_2 0.57534$$

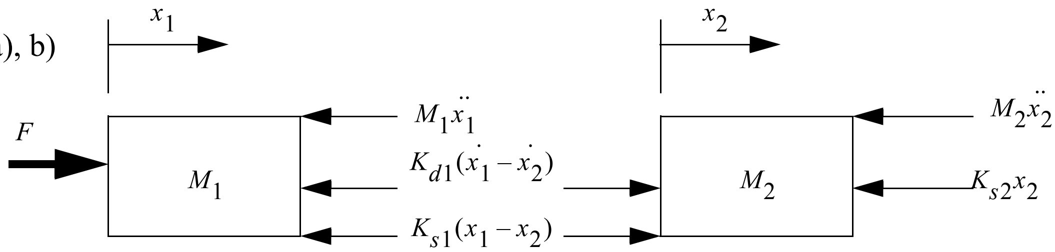
$$10.722 = -0.7236 C_1 e^{-0.7236(2)} + -0.2764 C_2 e^{-0.2764(2)} = -0.1702 C_1 + -0.1590 C_2$$

Solved with Scilab ‘[0.23523, 0.57534 ; -0.1702 , -0.1590]' * [8.3612 ; 10.722]’

$$x(t) = 0.1419e^{-0.7236t} + 3.106e^{-0.2764t}$$

Answer 3.24

a), b)



$$\dot{x}_1 = v_1 \qquad \text{eqn 3.1}$$

$$\dot{x}_2 = v_2 \qquad \text{eqn 3.2}$$

$$\sum F_{M1} \qquad F - M_1\ddot{x}_1 - K_{d1}(\dot{x}_1 - \dot{x}_2) - K_{s1}(x_1 - x_2) = 0$$

$$\dot{v}_1 = \frac{F}{M_1} - \frac{K_{d1}}{M_1}(v_1 - v_2) - \frac{K_{s1}}{M_1}(x_1 - x_2) \qquad \text{eqn 3.3}$$

$$\sum F_{M2} \qquad K_{d1}(\dot{x}_1 - \dot{x}_2) + K_{s1}(x_1 - x_2) - M_2\ddot{x}_2 - K_{s2}x_2 = 0$$

$$\dot{v}_2 = \frac{K_{d1}}{M_2}(v_1 - v_2) + \frac{K_{s1}}{M_2}(x_1 - x_2) - \frac{K_{s2}}{M_2}x_2 \qquad \text{eqn 3.4}$$

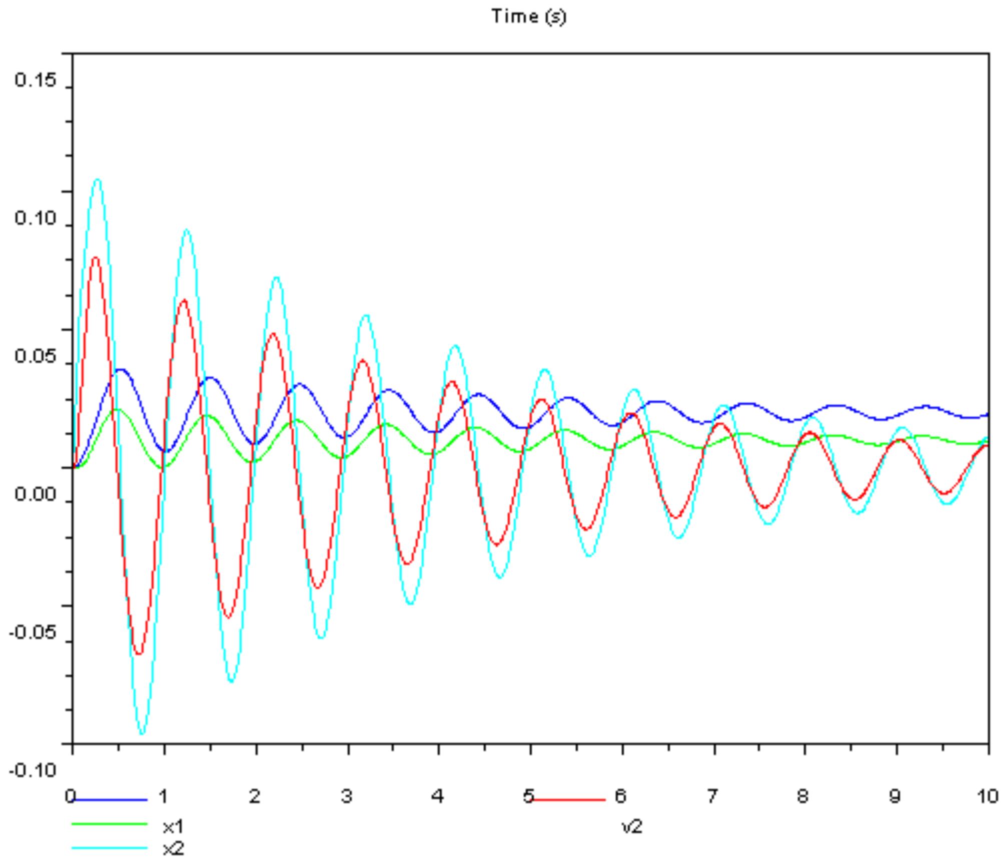
c)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_{s1}}{M_1} & \frac{K_{s1}}{M_1} & -\frac{K_{d1}}{M_1} & \frac{K_{d1}}{M_1} \\ \frac{K_{s1}}{M_2} & \frac{-K_{s1} - K_{s2}}{M_2} & \frac{K_{d1}}{M_2} & -\frac{K_{d1}}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{F}{M_1} \\ 0 \end{bmatrix}$$

d)

```
// A numerical integration for homework problem

Ks1 = 100; // System component values
Ks2 = 100;
Kd1 = 10;
M1 = 1;
M2 = 1;
F = 1;

x1 = 0;        // initial conditions
v1 = 0;
x2 = 0;
v2 = 0;
X=[x1, x2, v1, v2];
A = [0, 0, 1, 0 ; 0, 0, 0, 1 ; -Ks1/M1, Ks1/M1, -Kd1/M1, Kd1/M1 ; Ks1/M2, (-Ks1-Ks2)/
   M2, Kd1/M2, -Kd1/M2]; // define the A matrix
B = [0 ; 0 ; 1/M1 ; 0];

// the state matrix function
function foo=f(state,t)
       tmp = A * state' + B * [F];
       foo = tmp';
endfunction

// Set the time length and step size for the integration
steps = 1000;
t_start = 0.0;
t_end = 10.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
       t = [t ; t($,:) + h];
       X = [X ; X($,:) + h * f(X($,:), t($,:))];
end

// print some results to compare
print_steps = 10;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  time = j * h + t_start;
  printf("The value at t=%f s is (x, v) = (%f, %f)\n", time, X(j+1,1), X(j+1,3));
end

// Graph the values
plot2d(t, X, [2, 3, 4, 5], leg="x1@x2@v1@v2");
xtitle('Time (s)');

// SCILAB OUTPUT
//The value at t=0.000000 s is (x, v) = (0.000000, 0.000000)
//The value at t=1.000000 s is (x, v) = (0.005900, -0.003279)
//The value at t=2.000000 s is (x, v) = (0.008716, 0.008097)
//The value at t=3.000000 s is (x, v) = (0.011166, 0.014847)
//The value at t=4.000000 s is (x, v) = (0.013239, 0.018261)
//The value at t=5.000000 s is (x, v) = (0.014953, 0.019357)
//The value at t=6.000000 s is (x, v) = (0.016336, 0.018918)
//The value at t=7.000000 s is (x, v) = (0.017430, 0.017528)
//The value at t=8.000000 s is (x, v) = (0.018275, 0.015618)
//The value at t=9.000000 s is (x, v) = (0.018913, 0.013489)
//The value at t=10.000000 s is (x, v) = (0.019381, 0.011345)
```

e)



```
// A numerical integration for homework problem
Ks1 = 100; // System component values
Ks2 = 100;
Kd1 = 10;
M1 = 1;
M2 = 1;
F = 1;

x1 = 0;        // initial conditions
v1 = 0;
x2 = 0;
v2 = 0;
X=[x1, x2, v1, v2];
A = [0, 0, 1, 0 ; 0, 0, 0, 1 ; -Ks1/M1, Ks1/M1, -Kd1/M1, Kd1/M1 ; Ks1/M2, (-Ks1-Ks2)/
   M2, Kd1/M2, -Kd1/M2]; // define the A matrix
B = [0 ; 0 ; 1/M1 ; 0];

// the state matrix function
function foo=f(state,t)
       tmp = A * state' + B * [F];
       foo = tmp';
endfunction

// Set the time length and step size for the integration
steps = 1000;
t_start = 0.0;
t_end = 10.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       F2 = h * f(X($,:) + F1/2.0, t($,:) + h/2.0);
       F3 = h * f(X($,:) + F2/2.0, t($,:) + h/2.0);
       F4 = h * f(X($,:) + F3, t($,:) + h);
       X = [X ; X($,:) + (F1 + 2.0*F2 + 2.0*F3 + F4)/6.0];
end

// print some results to compare
print_steps = 10;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  time = j * h + t_start;
  printf("The value at t=%f s is (x, v) = (%f, %f)\n", time, X(j+1,1), X(j+1,3));
end

// Graph the values
plot2d(t, X, [2, 3, 4, 5], leg="x1@x2@v1@v2");
xtitle('Time (s)');
```

g)
```
// A program to do Runge Kutta integration of a mass spring damper system
#include <stdio.h>

void    multiply(double, double[], double[]);
void    add(double[], double[], double[]);
void    step(double, double, double[]);
void    derivative(double, double[], double[]);

#define SIZE        4           // the length of the state vector
#define Ks1         100         // the spring coefficients
#define Ks2         100
#define Kd1         10
#define M1          1           // the masses
#define M2          1
#define Force       1           // the applied force

int main(){
        FILE    *fp;

        double  h = 0.0001;
        double  t;
        int             j = 0;

        double  X[SIZE];// create state variable list
        X[0] = 0;// set initial condition to zero
        X[1] = 0;
        X[2] = 0;
        X[3] = 0;

        if( ( fp = fopen("out.txt", "w")) != NULL){
                fprintf(fp, "   t(s)       x          v \n\n");
                for( t = 0.0; t < 10.0; t += h ){
                        step(t, h, X);
                        if(j == 0) fprintf(fp, "%9.5f %9.5f %9.5f %9.5f %9.5f\n", t, X[0],
   X[1], X[2], X[3]);
                        j++; if(j >= 2000) j = 0;
                }
        }
        fclose(fp);

        return 0;
}

// First order integration done here (could be replaced with runge kutta)
void step(double t, double h, double X[]){
        double  tmp[SIZE],
                        dX[SIZE],
                        F1[SIZE],
                        F2[SIZE],
                        F3[SIZE],
                        F4[SIZE];
        // Calculate F1
        derivative(t, X, dX);
        multiply(h, dX, F1);

        // Calculate F2
        multiply(0.5, F1, tmp);
        add(X, tmp, tmp);
        derivative(t+h/2.0, tmp, dX);
        multiply(h, dX, F2);
```

g) cont

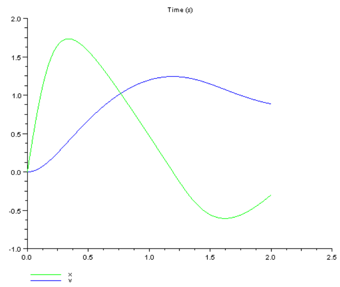
```
        // Calculate F3
        multiply(0.5, F2, tmp);
        add(X, tmp, tmp);
        derivative(t+h/2.0, tmp, dX);
        multiply(h, dX, F3);

        // Calculate F4
        add(X, F3, tmp);
        derivative(t+h, tmp, dX);
        multiply(h, dX, F4);

        // Calculate the weighted sum
        add(F2, F3, tmp);
        multiply(2.0, tmp, tmp);
        add(F1, tmp, tmp);
        add(F4, tmp, tmp);
        multiply(1.0/6.0, tmp, tmp);
        add(tmp, X, X);
}

// State Equations Calculated Here
void derivative(double t, double X[], double dX[]){
        dX[0] = X[2];
        dX[1] = X[3];
        dX[2] = - Ks1/M1*X[0] + Ks1/M1*X[1] -Kd1/M1*X[2] + Kd1/M1*X[3]+ Force/
   M1;
        dX[3] = + Ks1/M2*X[0] - (Ks1+Ks2)/M2*X[1] + Kd1/M2*X[2] - Kd1/M2*X[3];
}

// A subroutine to add vectors to simplify other equations
void add(double X1[], double X2[], double R[]){
        int i;
        for(i = 0; i < SIZE; i++) R[i] = X1[i] + X2[i];
}

// A subroutine to multiply a vector by a scalar to simplify other equations
void multiply(double X, double V[], double R[]){
        int i;
        for(i = 0; i < SIZE; i++) R[i] = X*V[i];
}
```

h)

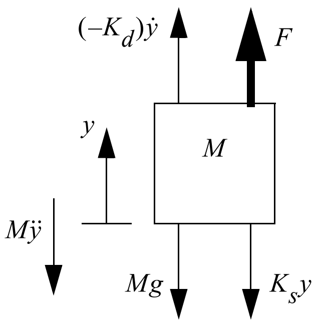
Method	x(10s)
First order Scilab	0.019381
RK Scilab	0.019881
RK C	

Answer 3.25

```
// A numerical integration program

Ks = 10; // Spring coefficient of 10 N/m
F = 10; // An applied force of 10N
M = 1; // A mass of 1Kg

x = 0; // set initial conditions to zero
v = 0;

X = [x, v];

// the state matrix function
function foo=f(state,t)
        foo = [state($, 2), (-2.0 * state($, 2) * abs(state($, 2)) - Ks * state($, 1) +
   10.0) / M];
endfunction

// Set the time length and step size for the integration
steps = 1000;
t_start = 0.0;
t_end = 2.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
        t = [t ; t($,:) + h];
        X = [X ; X($,:) + h * f(X($,:), t($,:))];
end

// print some results to compare
print_steps = 10;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  // time = j * h + t_start;
  time = t(j + 1);
  printf("The value at t=%f s is (x, v) = (%f, %f)\n", time, X(j+1,1), X(j+1,2));
end

// Graph the values
plot2d(t, X, [2, 3], leg="x@v");
xtitle('Time (s)');

// SCILAB OUTPUT
//The value at t=0.000000 s is (x, v) = (0.000000, 0.000000)
//The value at t=0.200000 s is (x, v) = (0.171656, 1.506669)
//The value at t=0.400000 s is (x, v) = (0.507901, 1.704388)
//The value at t=0.600000 s is (x, v) = (0.820884, 1.384830)
//The value at t=0.800000 s is (x, v) = (1.054940, 0.942182)
//The value at t=1.000000 s is (x, v) = (1.196671, 0.467783)
//The value at t=1.200000 s is (x, v) = (1.242429, -0.015709)
//The value at t=1.400000 s is (x, v) = (1.194513, -0.438314)
//The value at t=1.600000 s is (x, v) = (1.085479, -0.605115)
//The value at t=1.800000 s is (x, v) = (0.969314, -0.523109)
//The value at t=2.000000 s is (x, v) = (0.885327, -0.298854)
```



Answer 3.26 .



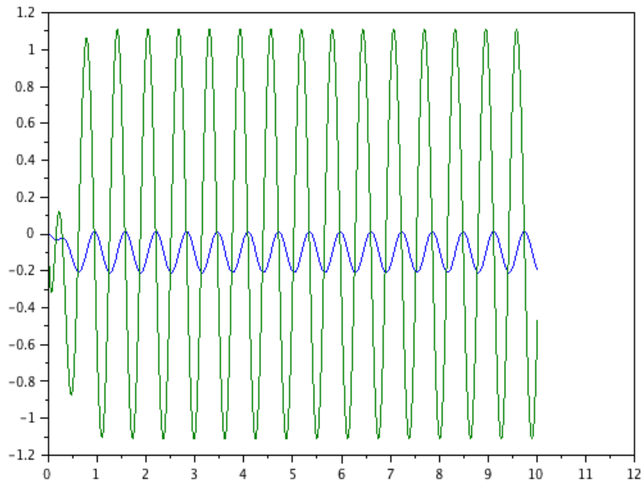
$$\sum F_M = (-K_d)\dot{y} + F - M\ddot{y} - Mg - K_s y = 0$$

$$\ddot{y} + \frac{K_d}{M}\dot{y} + \frac{K_s}{M}y = \frac{F}{M} - g$$

$$\dot{y} = v$$

$$\dot{v} = v\left(\frac{-K_d}{M}\right) + y\left(\frac{-K_s}{M}\right) + \left(\frac{F}{M} - g\right) = v(-10) + x(-100) + 10\sin(10t) - 9.81$$

```
// Exam Problem Solution
Kd = 10;
Ks = 100;
M = 1;
g = 9.81;

y0 = 0;
// y0 = 1.0;
v0 = 0;
X = [y0, v0];
t_start = 0.0;
t_end = 10.0;
t = [t_start];
n_steps = 1000;
h = (t_end - t_start) / n_steps;

function force = F(t)
    force = 10 * sin(10 * t);
    // force = 10;
endfunction

function [derX] = deriv(X, t)
    y = X($, 1);
    v = X($, 2);
    derX = [v , v*(-Kd/M) + y*(-Ks/M) + (F(t)/M - g)];
endfunction

// Euler integration loop
for i = 0 : n_steps
    X = [ X ; X($, :) + h*deriv(X($, :), t($)) ];
    t = [ t ; t($) + h];
end

plot(t, X);
```



Answer 3.27

The program in part a) was used with modified parameters. The table was used to keep track of trial and error values. The t_end was reduced to 2s, and then 1s to show more detail.

The final result is that the damper value must be between 17.45 and 17.46 for critical damping.

Kd	Estimated damping coefficient
10	< 1 (3 major cycles)
100, t_end=2	> 1 (no oscillation)
50	> 1 (no oscillation)
30	> 1 (some curvature)
20	> 1 (getting close)
15	< 1 (some overshoot)
17	< 1 (small overshoot)
18	< 1 (no overshoot)
17.5, t_end = 1	> 1 (no overshoot)
17.2	< 1 (overshoot)
17.3	< 1 (overshoot)
17.4	< 1 (overshoot)
17.45	< 1 (overshoot)
17.48	> 1 (no overshoot)
17.47	> 1 (no overshoot)
17.46	> 1 (no overshoot)

Aside: Finding the roots of the homogeneous finds that a value of 20 is needed for critical damping. As provided the numerical solution suggests an incorrect value of 17.45. This occurs because the initial conditions and gravity mask the underlying phenomenon.

$$R = \frac{-\left(\frac{K_d}{M}\right) \pm \sqrt{\left(\frac{K_d}{M}\right)^2 - 4(1)\left(\frac{K_s}{M}\right)}}{2(1)} \qquad \ddot{y} + \frac{K_d}{M}\dot{y} + \frac{K_s}{M}y = \frac{F}{M} - g$$

$$0 = \sqrt{\left(\frac{K_d}{M}\right)^2 - 4(1)\left(\frac{K_s}{M}\right)}$$

$$K_d = \sqrt{4(1)(K_s)M} = \sqrt{4(1)(100)1} = 20$$

Answer 3.28

UNVERIFIED

$$\ddot{y} + \frac{K_d}{M}\dot{y} + \frac{K_s}{M}y = \frac{F}{M} - g \qquad \ddot{y} + 100\dot{y} + 10y = 10\sin(10t) - 9.81$$

Homogeneous:

$$R = -99.8999, -0.10010$$

$$y_h = C_1 e^{(-99.8999)t} + C_2 e^{(-0.10010)t}$$

Particular:

$$y_p = A\sin(10t) + B\cos(10t) + D$$

$$\dot{y}_p = (10A)\cos(10t) + (-10B)\sin(10t)$$

$$\ddot{y}_p = (-100A)\sin(10t) + (-100B)\cos(10t)$$

$$(-100A)\sin(10t) + (-100B)\cos(10t) + 100((10A)\cos(10t) + (-10B)\sin(10t)) + 10(A\sin(10t) + B\cos(10t) + D) = 10\sin(10t) - 9.81$$

$$(-100A - 1000B + 10A)\sin(10t) + (-100B + 1000A + 10B)\cos(10t) + (10D) = 10\sin(10t) - 9.81$$

$$D = -9.81$$

$$-100B + 1000A + 10B = 0 \qquad A = \frac{9}{100}B$$

$$-100A - 1000B + 10A = 10$$

$$-1000B - 90\left(\frac{9}{100}B\right) = 10 \qquad B = \frac{10}{-1000 - 90\left(\frac{9}{100}\right)} = -0.0099196508$$

$$A = \frac{9}{100}(-0.0099196508) = -0.00089276857$$

$$y_p = (-0.00089276857)\sin(10t) + (-0.0099196508)\cos(10t) - 9.81$$

Initial Conditions:

$$y = C_1 e^{(-99.8999)t} + C_2 e^{(-0.10010)t} + (-0.00089276857)\sin(10t) + (-0.0099196508)\cos(10t) - 9.81$$

$$1 = C_1 + C_2 + (-0.0099196508) - 9.81 \qquad C_1 + C_2 = 10.81992$$

$$\dot{y} = (-99.8999)C_1 e^{(-99.8999)t} + (-0.10010)C_2 e^{(-0.10010)t} +$$
$$(-0.00089276857)10\cos(10t) + (-0.0099196508)(-10)\sin(10t)$$

$$0 = (-99.8999)C_1 + (-0.10010)C_2 + (-0.00089276857)10$$

$$(-99.8999)C_1 + (-0.10010)C_2 = 0.0089276857$$

Scilab:

```
A = [[1, 1] ; [-99.8999, -0.10010]]; B = [10.81992 ; 0.0089276857];
inv(A) * B
```

$$C_1 = -0.01094 \qquad C_2 = 10.831$$

$$y = (-0.01094)e^{(-99.8999)t} + (10.831)e^{(-0.10010)t} + (0.009960)\sin(10t + 0.08976) - 9.81$$

3.1 Problems Without Solutions

Problem 3.29 Integrate the acceleration function

$$F = M\left(\frac{d}{dt}\right)^2 x$$

use,

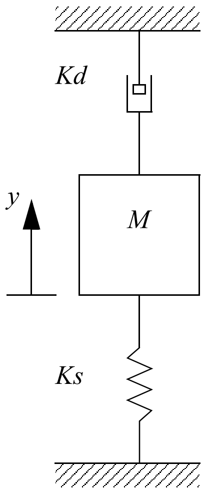
x(0) = 1
v(0) = 2
h = 0.5 s
F = 10
M = 1

Problem 3.30 Given the differential equation:

$$\dddot{x} + \ddot{x} + 100\dot{x} + 100x = F$$

a) What is the homogeneous solution to the equation?
b) What is the particular solution if the input is F = 1 sin(t)?
c) Solve the differential equation to find the response to F = 1 sin(t) if the system starts undeflected, at rest.
d) Write state equations for the differential equation.
e) Use a numerical method to solve the differential equation using the input F = 1 sin(t).
f) Write the time of the first peak for the explicit and numerical solutions. (Hint: They should match.)
g) Rewrite the equation as a transfer function.

Problem 3.31 Use numerical integration to find the system response. Leave the solution in variable form, and then change values to see how the system response changes.



$$K_d = 10Nsm^{-1}$$

$$K_s = 10Nm^{-1}$$

$$M = 1kg$$

$$y_0 = 1m \qquad \dot{y}_0 = 0m$$

Problem 3.32 Convert the third order differential equation to state equation form. With a numerical method of your choice, find the state of the system 1 second later. Show all calculations.

$$\dddot{x} + 4\ddot{x} + 2\dot{x} + 5x = 10$$

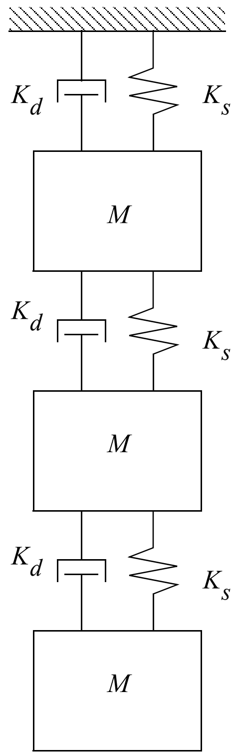
Problem 3.33 The differential equation below describes a first order system that starts with an initial value of k=20. Find the state at two milliseconds using a) explicit integration, b) first order numerical integration and c) Runge-Kutta integration. For the numerical methods use a time step of h=0.001s. The final results must be put in a table for easy comparison.

$$\dot{k} + 10k = 5 \qquad k(0) = 20$$

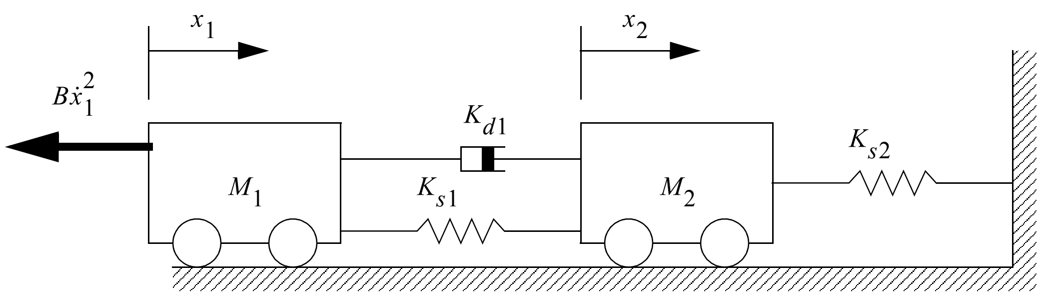
Problem 3.34 Explicitly solve the following differential equation. Verify the result numerically.

$$\dot{v} + 20v^2 = 200$$

Problem 3.35 Given the mass spring damper system below a) develop the state equations, and b) estimate the steady state response. Consider gravity.



Problem 3.36 Write the differential equations AND state equations for the systems below.



Problem 3.37 Solve the following nonlinear equation using numerical integration in scilab. Note that the velocity term is squared for aerodynamic drag. When done save a graph of x from 0 to 10 seconds (use a PDF or image format).

$$\ddot{x} + 20(\dot{x})^2 + 10x = 10$$

Problem 3.38 Solve the following differential equation assuming a unit step input.

$$\dot{y} + 2y = \dot{x} + 3x$$

Problem 3.39 Solve the following differential equation assuming a unit step input.

$$\ddot{y} - \frac{\dot{y}}{6} - \frac{y}{3} = \ddot{x} + 2x$$

3.2 Matrix Math Review

• Matrices allow simple equations that drive a large number of repetitive calculations - as a result they are found in many computer applications.

• A matrix has the form seen below,

n columns

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

m rows

If n = m then the matrix is said to be square. Many applications require square matrices. We may also represent a matrix as a 1-by-3 for a vector.

• Matrix operations are available for many of the basic algebraic expressions, examples are given below. There are also many restrictions - many of these are indicated.

$$A = 2 \qquad B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \qquad C = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix} \qquad D = \begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix} \qquad E = \begin{bmatrix} 24 & 25 & 26 \end{bmatrix}$$

Addition/subtraction

$$A + B = \begin{bmatrix} 3+2 & 4+2 & 5+2 \\ 6+2 & 7+2 & 8+2 \\ 9+2 & 10+2 & 11+2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3+12 & 4+13 & 5+14 \\ 6+15 & 7+16 & 8+17 \\ 9+18 & 10+19 & 11+20 \end{bmatrix}$$

$B + D =$ not valid

$$B - A = \begin{bmatrix} 3-2 & 4-2 & 5-2 \\ 6-2 & 7-2 & 8-2 \\ 9-2 & 10-2 & 11-2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3-12 & 4-13 & 5-14 \\ 6-15 & 7-16 & 8-17 \\ 9-18 & 10-19 & 11-20 \end{bmatrix}$$

$B - D =$ not valid

Multiplication/division

$$A \cdot B = \begin{bmatrix} 3(2) & 4(2) & 5(2) \\ 6(2) & 7(2) & 8(2) \\ 9(2) & 10(2) & 11(2) \end{bmatrix} \qquad \frac{B}{A} = \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\ \frac{6}{2} & \frac{7}{2} & \frac{8}{2} \\ \frac{9}{2} & \frac{10}{2} & \frac{11}{2} \end{bmatrix}$$

$$B \cdot D = \begin{bmatrix} (3 \cdot 21 + 4 \cdot 22 + 5 \cdot 23) \\ (6 \cdot 21 + 7 \cdot 22 + 8 \cdot 23) \\ (9 \cdot 21 + 10 \cdot 22 + 11 \cdot 23) \end{bmatrix} \qquad D \cdot E = 21 \cdot 24 + 22 \cdot 25 + 23 \cdot 26$$

Note: To multiply matrices, the first matrix must have the same number of columns as the second matrix has rows.

$$B \cdot C = \begin{bmatrix} (3 \cdot 12 + 4 \cdot 15 + 5 \cdot 18) & (3 \cdot 13 + 4 \cdot 16 + 5 \cdot 19) & (3 \cdot 14 + 4 \cdot 17 + 5 \cdot 20) \\ (6 \cdot 12 + 7 \cdot 15 + 8 \cdot 18) & (6 \cdot 13 + 7 \cdot 16 + 8 \cdot 19) & (6 \cdot 14 + 7 \cdot 17 + 8 \cdot 20) \\ (9 \cdot 12 + 10 \cdot 15 + 11 \cdot 18) & (9 \cdot 13 + 10 \cdot 16 + 11 \cdot 19) & (9 \cdot 14 + 10 \cdot 17 + 11 \cdot 20) \end{bmatrix}$$

$$\frac{B}{C}, \frac{B}{D}, \frac{D}{B}, \text{etc} = \text{not allowed (see inverse)}$$

Determinant

$$|B| = 3 \cdot \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} - 4 \cdot \begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} + 5 \cdot \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = 3 \cdot (-3) - 4 \cdot (-6) + 5 \cdot (-3) = 0$$

$$\begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} = (7 \cdot 11) - (8 \cdot 10) = -3$$

$$\begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} = (6 \cdot 11) - (8 \cdot 9) = -6$$

$$\begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = (6 \cdot 10) - (7 \cdot 9) = -3$$

$$|D|, |E| = \text{not valid (matrices not square)}$$

Transpose

$$B^T = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix} \qquad D^T = \begin{bmatrix} 21 & 22 & 23 \end{bmatrix} \qquad E^T = \begin{bmatrix} 24 \\ 25 \\ 26 \end{bmatrix}$$

Adjoint

$$\|B\| = \begin{bmatrix} \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} & -\begin{vmatrix} 6 & 8 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 9 & 11 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 9 & 10 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \end{bmatrix}^T$$

The matrix of determinant to the left is made up by getting rid of the row and column of the element, and then finding the determinant of what is left. Note the sign changes on alternating elements.

$$\|D\| = \text{invalid (must be square)}$$

Inverse

$$D = B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve this equation for x,y,z we need to move B to the left hand side. To do this we use the inverse.

$$B^{-1}D = B^{-1} \cdot B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1}D = I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1} = \frac{\|B\|}{|B|}$$

In this case B is singular, so the inverse is undetermined, and the matrix is indeterminate.

$$D^{-1} = \text{invalid (must be square)}$$

Identity Matrix:

This is a square matrix that is the matrix equivalent to ‘1’.

$$B \cdot I = I \cdot B = B$$

$$D \cdot I = I \cdot D = D$$

$$B^{-1} \cdot B = I$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc=I}$$

• The eigenvalue of a matrix is found using,

$$|A - \lambda I| = 0$$

Solving Linear Equations with Matrices

- We can solve systems of equations using the inverse matrix,

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the matrix, then rearrange, and solve.

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

- We can solve systems of equations using Cramer's rule (with determinants),

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with B.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

3.3 Practice Problems

Problem 3.40 Perform the matrix operations below.

Multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} =$$

Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Inverse

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} =$$

Answer 3.40

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 36$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -0.833 & 0.167 & 0.167 \\ 0.167 & -0.333 & 0.167 \\ 0.5 & 0.167 & -0.167 \end{bmatrix}$$

Problem 3.41 Solve the following equations using any technique,

$$5x - 2y + 4z = -1$$

$$6x + 7y + 5z = -2$$

$$2x - 3y + 6z = -3$$

Answer 3.41

x= 0.273
y= -0.072
z= -0.627

Problem 3.42 Solve the following set of equations using a) Cramer's rule and b) an inverse matrix.

$$2x + 3y = 4$$

$$5x + 1y = 0$$

Answer 3.42

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

a)

$$x = \frac{\left|\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{4}{-13} \qquad y = \frac{\left|\begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{-20}{-13} = \frac{20}{13}$$

b)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -(5) \\ -(3) & 2 \end{bmatrix}^T}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}}{-13}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \left(-\frac{1}{13}\right)\begin{bmatrix} 4 \\ -20 \end{bmatrix} = \begin{bmatrix} -\frac{4}{13} \\ \frac{20}{13} \end{bmatrix}$$

Problem 3.43 Perform the following matrix calculation. Show all work.

$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T$$

Answer 3.43

$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \left[\begin{bmatrix} AX+BY+CZ \\ DX+EY+FZ \\ GX+YH+IZ \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \begin{bmatrix} AX+BY+CZ+L \\ DX+EY+FZ+M \\ GX+YH+IZ+N \end{bmatrix}^T$$

$$= \begin{bmatrix} AX+BY+CZ+L & DX+EY+FZ+M & GX+YH+IZ+N \end{bmatrix}$$

Problem 3.44 Perform the matrix calculations given below.

a) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$$

b) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X & Y & Z \end{bmatrix} =$$

Problem 3.45 Perform the following matrix calculations.

a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix}$ b) $\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|$ c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

Answer 3.45

a) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} (ad+bg+cm) & (ae+bh+cn) & (af+bk+cp) \end{bmatrix}$$

b) $$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad-bc$$

c) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{adj}}{\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|} = \frac{\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}}{ad-bc} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-c}{ad-bc} \\ \dfrac{-b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Problem 3.46 Find the value of ‘x’ for the following system of equations.

$x+2y+3z = 5$ $x+4y+8z = 0$ $4x+2y+z = 1$

Answer 3.46

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \frac{\begin{bmatrix} 5 & 2 & 3 \\ 0 & 4 & 8 \\ 1 & 2 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix}} = \frac{5(4-16)+2(8-0)+3(0-4)}{1(4-16)+2(32-1)+3(2-16)} = \frac{-60+16-12}{-12+62-42} = \frac{-56}{8} = -7$$

Problem 3.47 Perform the matrix calculations given below.

a) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$$

b) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X & Y & Z \end{bmatrix} =$$

c) $$det \begin{bmatrix} A & B \\ C & D \end{bmatrix} =$$

d) $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^A =$$

e) $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

Problem 3.48 Solve the following set of equations with the specified methods.

$$3x + 5y = 7$$

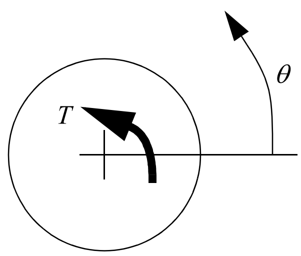
$$4x - 6y = 2$$

a) Inverse matrix
b) Cramer's rule
c) Gauss-Jordan row reduction
d) Substitution

4. Rotation

Topic 4.1	*Basic laws of motion.*
Topic 4.2	*Rotational inertia, combined rotation and translation.*
Topic 4.3	*Springs, damping, and friction.*
Topic 4.4	*Design cases.*
Objective 4.1	*To be able to develop and analyze differential equations for rotational systems.*

The equations of motion for a rotating mass are shown in Figure 4.1. Given the angular position, the angular velocity can be found by differentiating once, the angular acceleration can be found by differentiating again. The angular acceleration can be integrated to find the angular velocity, the angular velocity can be integrated to find the angular position. The angular acceleration is proportional to an applied torque, but inversely proportional to the mass moment of inertia.



equations of motion

$$\omega = \left(\frac{d}{dt}\right)\theta \quad \text{eqn 4.1}$$

$$\alpha = \left(\frac{d}{dt}\right)\omega = \left(\frac{d}{dt}\right)^2\theta \quad \text{eqn 4.2}$$

OR

$$\theta(t) = \int \omega(t)dt = \iint \alpha(t)dtdt \quad \text{eqn 4.3}$$

$$\omega(t) = \int \alpha(t)dt \quad \text{eqn 4.4}$$

$$\alpha(t) = \frac{T(t)}{J_M} \quad \text{eqn 4.5}$$

where,

θ, ω, α = *position, velocity and acceleration*

J_M = *second mass moment of inertia of the body*

T = *torque applied to body*

Figure 4.1 Basic properties of rotation

Note: A 'torque' and 'moment' are equivalent in terms of calculations. The main difference is that 'torque' normally refers to a rotating moment.

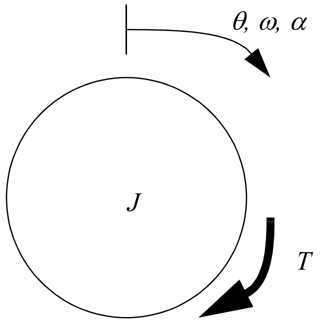
4.1 Modeling

Free Body Diagrams (FBDs) are required when analyzing rotational systems, as they were for translating systems. The force components normally considered in a rotational system include,

- Inertia - opposes acceleration and deceleration
- Springs - resist deflection
- Dampers - oppose velocity

Inertia

When unbalanced torques are applied to a mass it will begin to accelerate, in rotation. The sum of applied torques is equal to the inertia forces shown in Figure 4.2.



$$\sum T = J_M \alpha \quad \text{eqn 4.6}$$

$$J_M = I_{xx} + I_{yy} \quad \text{eqn 4.7}$$

$$I_{xx} = \int y^2 dM \quad \text{eqn 4.8}$$

$$I_{yy} = \int x^2 dM \quad \text{eqn 4.9}$$

Note: The 'mass' moment of inertia will be used when dealing with acceleration of a mass. Later we will use the 'area' moment of inertia for torsional springs.

Figure 4.2 Summing moments and angular inertia

The mass moment of inertia determines the resistance to acceleration. This can be calculated using integration, or found in tables. When dealing with rotational acceleration it is important to use the mass moment of inertia, not the area moment of inertia.

4.1 Eccentric Moments

The center of rotation for free body rotation will be the centroid. Moment of inertia values are typically calculated about the centroid. If the object is constrained to rotate about some point, other than the centroid, the moment of inertia value must be recalculated. The parallel axis theorem provides the method to shift a moment of inertia from a centroid to an arbitrary center of rotation, as shown in Figure 4.3.

$$J_M = \tilde{J}_M + Mr^2$$

where,

J_M = *mass moment about the new point*

$\tilde{J}_M$ = *mass moment about the center of mass*

M = *mass of the object*

r = *distance from the centroid to the new point*

Figure 4.3 Parallel axis theorem for shifting a mass moment of inertia

$$J_A = \tilde{J}_A + Ar^2$$

where,

J_A = *area moment about the new point*

$\tilde{J}_A$ = *area moment about the centroid*

A = *mass of the object*

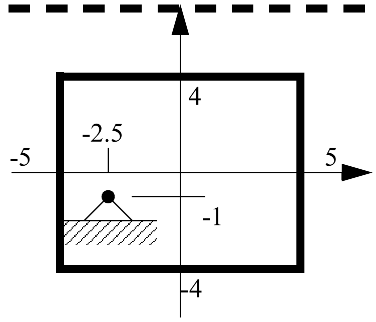
r = *distance from the centroid to the new point*

Figure 4.4 Parallel axis theorem for shifting a area moment of inertia

Aside: If forces do not pass through the center of an object, it will rotate. If the object is made of a homogeneous material, the area and volume centroids can be used as the center. If the object is made of different materials then the center of mass should be used for the center. If the gravity varies over the length of the (very long) object then the center of gravity should be used.

An example of calculating a mass moment of inertia is shown in Figure 4.5. In this problem the density of the material is calculated for use in the integrals. The integrals are then developed using slices for the integration element dM. The integrals for the moments about the x and y axes, are then added to give the polar moment of inertia. This is then shifted from the centroid to the new axis using the parallel axis theorem.

The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia.



First find the density and calculate the moments of inertia about the centroid.

$$\rho = \frac{10Kg}{2(5m)2(4m)} = 0.125Kgm^{-2}$$

$$I_{xx} = \int_{-4}^{4} y^2 dM = \int_{-4}^{4} y^2 \rho 2(5m) dy = 1.25Kgm^{-1} \frac{y^3}{3}\bigg|_{-4}^{4}$$

$$\therefore = 1.25Kgm^{-1}\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 53.33Kgm^2$$

$$I_{yy} = \int_{-5}^{5} x^2 dM = \int_{-5}^{5} x^2 \rho 2(4m) dx = 1Kgm^{-1} \frac{x^3}{3}\bigg|_{-5}^{5}$$

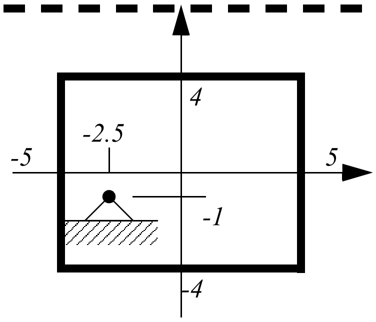
$$\therefore = 1Kgm^{-1}\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 83.33Kgm^2$$

$$J_M = I_{xx} + I_{yy} = 53.33Kgm^2 + 83.33Kgm^2 = 136.67Kgm^2$$

The centroid can now be shifted to the center of rotation using the parallel axis theorem.

$$J_M = \tilde{J}_M + Mr^2 = 136.67Kgm^2 + (10Kg)((-2.5m)^2 + (-1m)^2) = 209.2Kgm^2$$

Figure 4.5 Example: Mass moment of inertia



First, the area moment of inertia is calculated about the centroid by integration. All dimensions are in m.

$$I_{xx} = \int_{-4m}^{4m} y^2 dA = \int_{-4m}^{4m} y^2 2(5m)dy = 10m\frac{y^3}{3}\Bigg|_{-4m}^{4m} = 10m\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 426.7m^4$$

$$I_{yy} = \int_{-5m}^{5m} x^2 dA = \int_{-5m}^{5m} x^2 2(4m)dx = 8m\frac{x^3}{3}\Bigg|_{-5m}^{5m} = 8m\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 666.7m^4$$

$$\tilde{J}_A = I_{xx} + I_{yy} = (426.7 + 666.7)m^4 = 1093.4m^4$$

Next, shift the area moment of inertia from the centroid to the other point of rotation.

$$J_A = \tilde{J}_A + Ar^2$$

$$= 1093.4m^4 + ((4m - (-4m))(5m - (-5m)))((-1m)^2 + (-2.5m)^2)$$

$$= 1673m^4$$

Note: The basic definitions for the area moment of inertia are shown to the right.

$$I_{xx} = \int y^2 dA \quad \text{eqn 4.10}$$

$$I_{yy} = \int x^2 dA \quad \text{eqn 4.11}$$

$$J_A = I_{xx} + I_{yy} \quad \text{eqn 4.12}$$

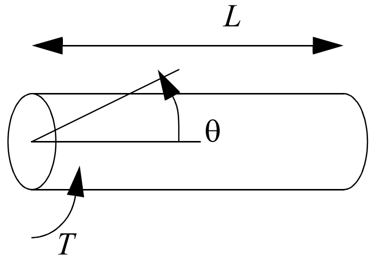
$$J_A = \tilde{J}_A + Ar^2 \quad \text{eqn 4.13}$$

Note: You may notice that when the area moment of inertia is multiplied by the density of the material, the mass moment of inertia is the result. Therefore if you have a table of area moments of inertia, multiplying by density will yield the mass moment of inertia. Keep track of units when doing this.

Figure 4.6 Example: Area moment of inertia

Springs

Twisting a rotational spring will produce an opposing torque. This torque increases as the deformation increases. A simple example of a solid rod torsional spring is shown in Figure 4.7. The angle of rotation is determined by the applied torque, T, the shear modulus, G, the area moment of inertia, J_A, and the length, L, of the rod. The constant parameters can be lumped into a single spring coefficient similar to that used for translational springs.



$$T = \left(\frac{J_A G}{L}\right)\theta \qquad \text{eqn 4.14}$$

$$T = K_S(\Delta\theta) \qquad \text{eqn 4.15}$$

Note: Remember to use radians for these calculations. In fact you are advised to use radians for all calculations. Don't forget to set your calculator to radians also.

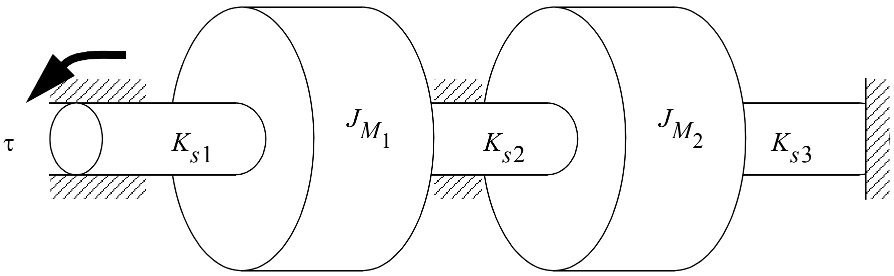
Note: This calculation uses the area moment of inertia.

Figure 4.7 A solid torsional spring

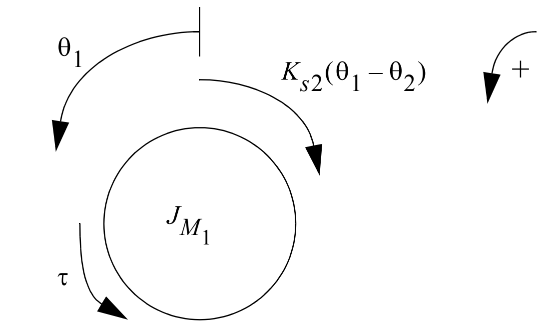
The spring constant for a torsional spring will be relatively constant, unless the material is deformed outside the linear elastic range, or the geometry of the spring changes significantly.

When dealing with strength of material properties the area moment of inertia is required. The calculation for the area moment of inertia is similar to that for the mass moment of inertia. An example of calculating the area moment of inertia is shown in Figure 5.10--, and based on the previous example in Figure 4.6--. The calculations are similar to those for the mass moments of inertia, except for the formulation of the integration elements. Note the difference between the mass moment of inertia and area moment of inertia for the part. The area moment of inertia can be converted to a mass moment of inertia simply by multiplying by the density. Also note the units.

An example problem with torsional springs is shown in Figure 4.8. There are three torsional springs between two rotating masses. The right hand spring is anchored solidly in a wall, and will not move. A torque is applied to the left hand spring. Because the torsional spring is considered massless the torque will be the same at the other end of the spring, at mass J_1. FBDs are drawn for both of the masses, and forces are summed. (Note: the similarity in the methods used for torsional, and for translational springs.) These equations are then rearranged into state variable equations, and finally put in matrix form.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the left most disk. Leave the results in state variable form.

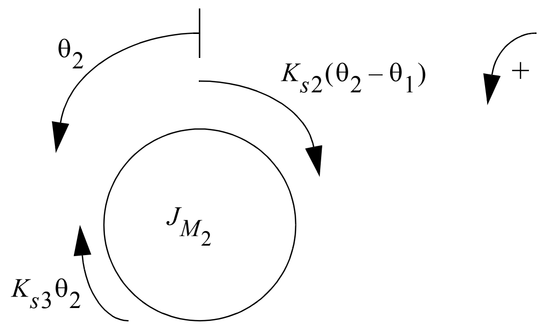


$$\sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_{M_1}\ddot{\theta}_1$$

$$J_{M_1}\ddot{\theta}_1 = -K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\dot{\theta}_1 = \omega_1 \qquad \text{eqn 4.16}$$

$$\dot{\omega}_1 = \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \frac{\tau}{J_{M_1}} \qquad \text{eqn 4.17}$$



$$\sum M = -K_{s2}(\theta_2 - \theta_1) - K_{s3}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\dot{\theta}_2 = \omega_2 \qquad \text{eqn 4.18}$$

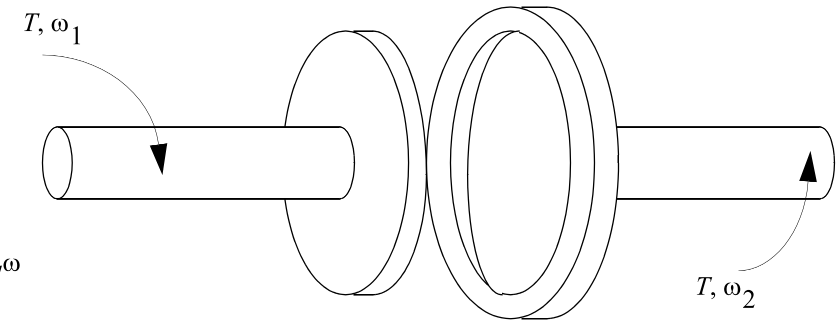
$$\dot{\omega}_2 = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad \text{eqn 4.19}$$

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s2}}{J_{M_1}} & 0 & \frac{K_{s2}}{J_{M_1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & 0 \end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tau}{J_{M_1}} \\ 0 \\ 0 \end{bmatrix}$$

Figure 4.8 Example: A rotational spring

Damping

Rotational damping is normally caused by viscous fluids, such as oils, used for lubrication. It opposes angular velocity with the relationships shown in Figure 4.9. The first equation is used for a system with one rotating and one stationary part. The second equation is used for damping between two rotating parts.

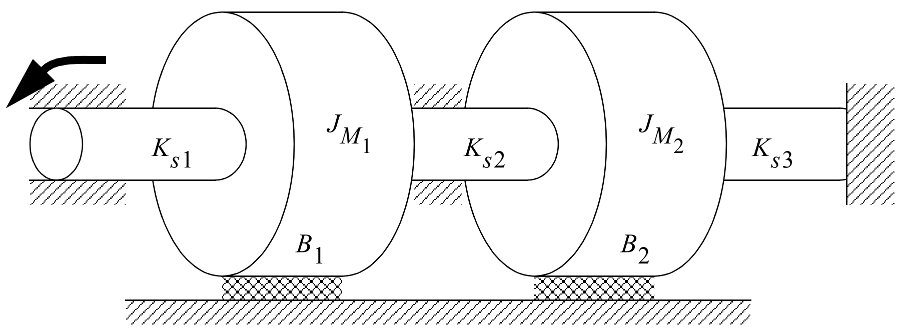


$$T = K_d \omega$$

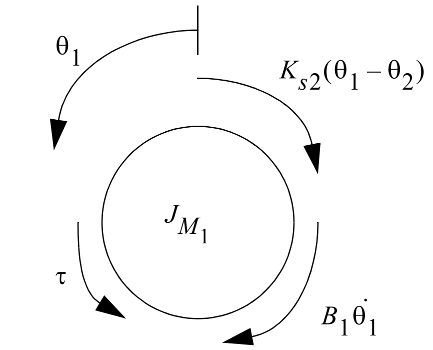
$$T = K_d (\omega_1 - \omega_2)$$

Figure 4.9 The rotational damping equation

The example in Figure 4.8 is extended to include damping in Figure 4.10. The primary addition from the previous example is the addition of the damping forces to the FBDs. In this case the damper coefficients are indicated with ‘B’, but ‘Kd’ could have also been used. The state equations were developed in matrix form. Visual comparison of the final matrices in this and the previous example reveal that the damping terms are the only addition.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the left most disk. Leave the results in state variable form.

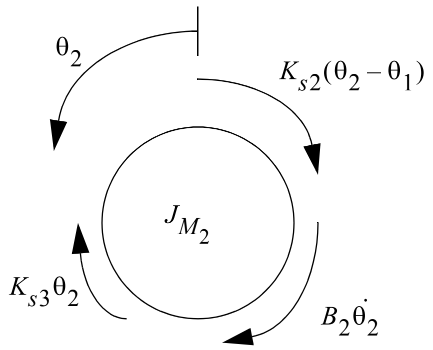


$$+\sum M = \tau - K_{s2}(\theta_1 - \theta_2) - B_1\dot{\theta}_1 = J_{M_1}\ddot{\theta}_1$$

$$J_{M_1}\ddot{\theta}_1 = -B_1\dot{\theta}_1 - K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\dot{\theta}_1 = \omega_1 \qquad \text{eqn 4.20}$$

$$\dot{\omega}_1 = \left(\frac{-B_1}{J_{M_1}}\right)\omega_1 + \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \frac{\tau}{J_{M_1}} \qquad \text{eqn 4.21}$$



$$+\sum M = -K_{s2}(\theta_2 - \theta_1) - B_2\dot{\theta}_2 - K_{s3}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 = \left(\frac{-B_2}{J_{M_2}}\right)\dot{\theta}_2 + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\dot{\theta}_2 = \omega_2 \qquad \text{eqn 4.22}$$

$$\dot{\omega}_2 = \left(\frac{-B_2}{J_{M_2}}\right)\omega_2 + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad \text{eqn 4.23}$$

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s2}}{J_{M_1}} & \frac{-B_1}{J_{M_1}} & \frac{K_{s2}}{J_{M_1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & \frac{-B_2}{J_{M_2}} \end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tau}{J_{M_1}} \\ 0 \\ 0 \end{bmatrix}$$

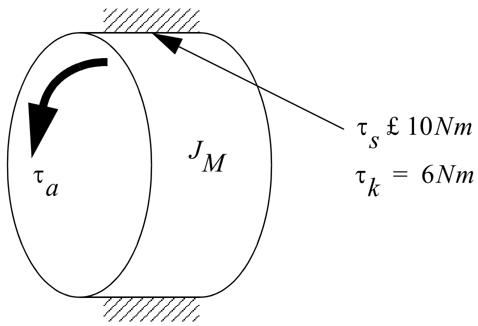
Figure 4.10 Example: A torsional mass spring damper system

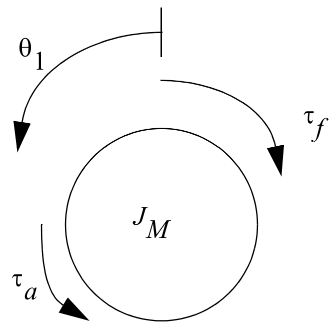
Friction

Friction between rotating components is a major source of inefficiency in machines. It is the result of contact surface materials and geometries. Calculating friction values in rotating systems is more difficult than translating systems. Normally rotational friction will be given as static and kinetic friction torques.

An example problem with rotational friction is shown in Figure 4.11. Basically these problems require that the model be analyzed as if the friction surface is fixed. If the friction force exceeds the maximum static friction the mechanism is then analyzed using the kinetic friction torque. There is friction between the shaft and the hole in the wall. The friction force is left as a variable for the derivation of the state equations. The friction value must be calculated using the appropriate state equation. The result of this calculation and the previous static or dynamic condition is then used to determine the new friction value.

A torque (Ta) is applied to the mass. The mass can rotate if it overcomes friction.





+ $$\sum M = \tau_a - \tau_f = J_M \ddot{\theta}$$

$$\ddot{\theta} = \frac{\tau_a - \tau_f}{J_M}$$ eqn 4.24

$$\dot{\theta} = \omega$$ eqn 4.25

$$\dot{\omega} = \frac{\tau_a}{J_M} + \left(\frac{-\tau_f}{J_M}\right)$$ eqn 4.26

Next, the friction torque must be calculated, and then used to determine the new static case force.

$$\tau_f = \tau_a - J_M \dot{\omega}$$ eqn 4.27

cases:

It was not slipping previously (slip == 0)

if $|\tau_f| > 10Nm$ then

$slip = 1$ $\qquad$ $\tau_f = -6sign(\tau_f)Nm$

else

$\omega = 0$

Slipping previously (slip == 1)

if ($|\omega| < 0.0001$) OR $sign(\omega_{now})$ ¥ $sign(\omega_{previous}) = -1$

$\omega_{now} = 0$ $\qquad$ $slip = 0$

else

$\tau_f = 6sign(\omega)Nm$

Figure 4.11 Example: Friction on a single rotating mass

The friction example in Figure 4.11 can be analyzed using the C program in Figure 4.12. For the purposes of the example some component values are selected and the system is assumed to be at rest initially. The program loops to integrate the state equations. Each loop the friction conditions are checked and then used for a first-order solution to the state equations.

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>

#define MOTION_TOL 0.0000001 /* a minimum value equal to zero */

double sign(double value){ /* a function to help determine direction/sign */
        if(value >= MOTION_TOL) return +1;
        if(value <= -MOTION_TOL) return -1;
        return 0;
}

int main(){
        double          h = 0.01, /* time step */
                        theta, w, /* the state variables */
                        acceleration, /* the acceleration */
                        w_last = 0, /* to store the previous value of w for direction change */
                        Tf, /* the friction test force */
                        Ttest,
                        J = 100, /* the moment of inertia (I picked a value) */
                        Ta_mag = 11, /* the applied torque (I picked a value) */
                        Ta_freq = 2, /* the freq of oscillation */
                        Ta,     /* the applied force */
                        t = 0; /* time */
        int             slip = 0; /* the system starts with no slip */
        FILE            *fp;

        theta = 0; w = 0;/* the initial conditions - starting at rest here */
        acceleration = 0.0;/* set the initial acceleration to zero also */
        if( ( fp = fopen("out.csv", "w")) != NULL){/* open a file to write the results */
                for( t = 0.0; t < 10.0; t += h ){/* loop */
                        Ta = Ta_mag * sin(Ta_freq * t); /* a sinusoidal forcing function to make it 
   interesting */
                        Ttest = Ta - J * acceleration;
                        if(slip == 0){ /* not slipping */
                                if(fabs(Ttest) > 10){ /* starting to slip */
                                        Tf = 6 * sign(Ttest); slip = 1;
                                        acceleration = (Ta - Tf) / J;
                                        w = 0;
                                } else { /* still stuck */
                                        Tf = Ttest; w = 0;
                                        acceleration = 0;
                                }
                        } else { /* slipping */
                                if((fabs(w) < MOTION_TOL) || (fabs(sign(w) - sign(w_last)) >= 2-
   MOTION_TOL)){
                                        slip = 0; w = 0; Tf = Tf; acceleration = 0;
                                } else {
                                        Tf = 6 * sign(w);
                                        acceleration = (Ta - Tf) / J;
                                }
                        }
                        w_last = w;
                        w = w + h * acceleration;
                        theta = theta + h * w;
                        fprintf(fp, "%f,%f,%f,%f,%f,%f\n", t, theta, w, acceleration, Ta, Tf);
                }
        }
        fclose(fp);
}
```

Figure 4.12 Example: A C program for the friction example

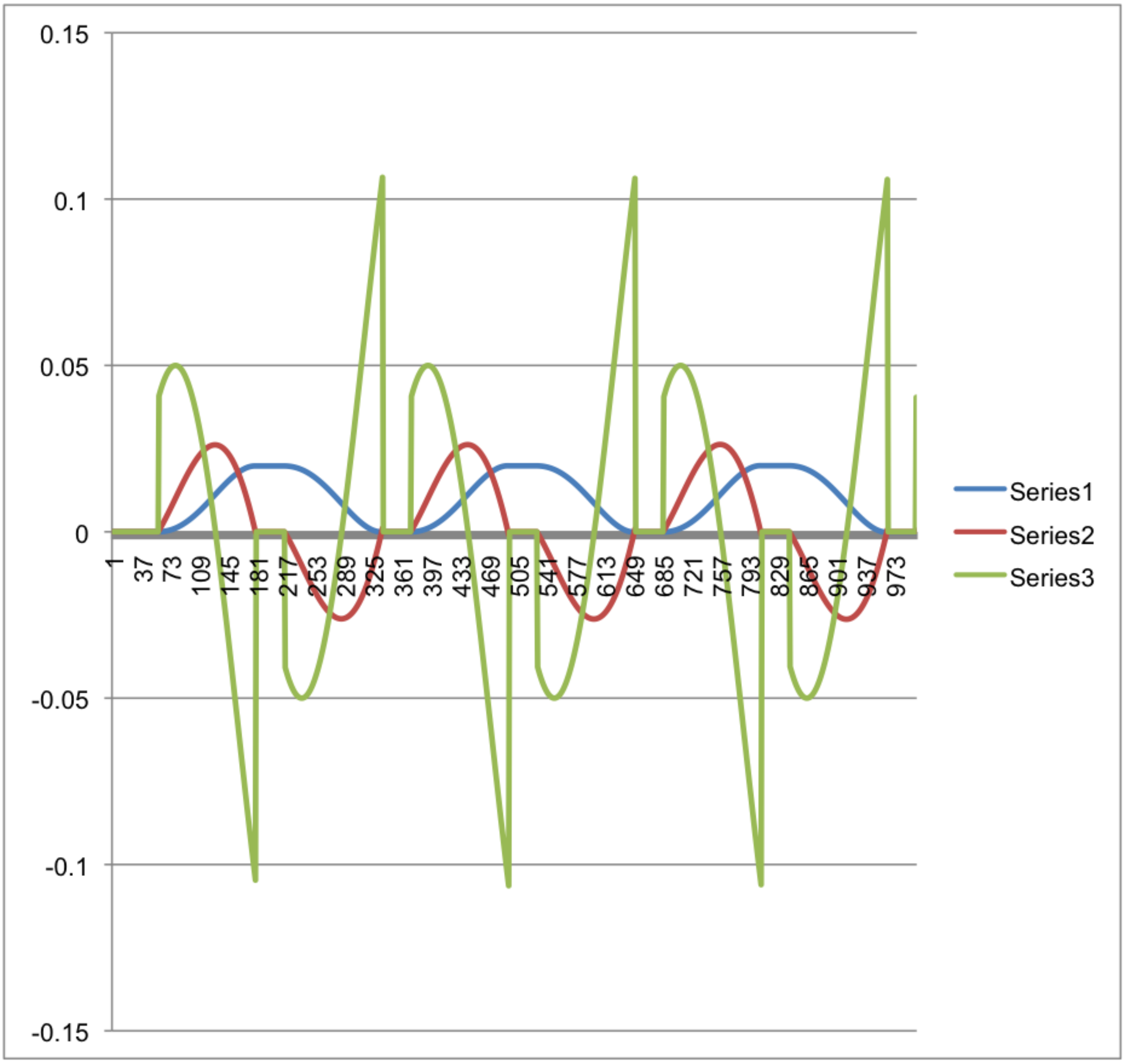


Figure 4.13 Example: Simulation Results for Rotational Friction. Series 1:theta, 2:omega, 3:alpha

Permanent Magnet Electric Motors

DC motors create a torque between the rotor (inside) and stator (outside) that is related to the applied voltage or current. In a permanent magnet motor there are magnets mounted on the stator, while the rotor consists of wound coils. When a voltage is applied to the coils the motor will accelerate. The differential equation for a motor is shown in Figure 4.14, and will be derived in a later chapter. The value of the constant 'K' is a function of the motor design and will remain fixed. The value of the coil resistance 'R' can be directly measured from the motor. The moment of inertia 'J' should include the motor shaft, but when a load is added this should be added to the value of 'J'.

$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J_M}$$

where,

ω = *the angular velocity of the motor*

K = *the motor speed constant*

J_M = *the moment of inertia of the motor and attached loads*

R = *the resistance of the motor coils*

T_{load} = *a torque applied to a motor shaft*

Figure 4.14 Model of a permanent magnet DC motor

The speed response of a permanent magnet DC motor is first-order. The steady-state velocity will be a straight line function of the torque applied to the motor, as shown in Figure 4.15. In addition the line shifts outwards as the voltage applied to the motor increases.

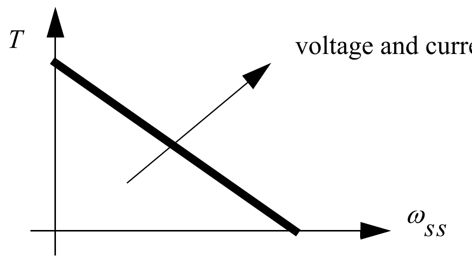


Figure 4.15 Torque speed curve for a permanent magnet DC motor

4.1 Other Topics

The energy and power relationships for rotational components are given in Figure 4.16. These can be useful when designing a system that will store and release energy.

$$E = E_K + E_P \qquad \text{eqn 4.28}$$

$$E_K = J_M\omega^2 \qquad \text{eqn 4.29}$$

$$E_P = T\theta \qquad \text{eqn 4.30}$$

$$P = T\omega \qquad \text{eqn 4.31}$$

Figure 4.16 Energy and power relations for rotation

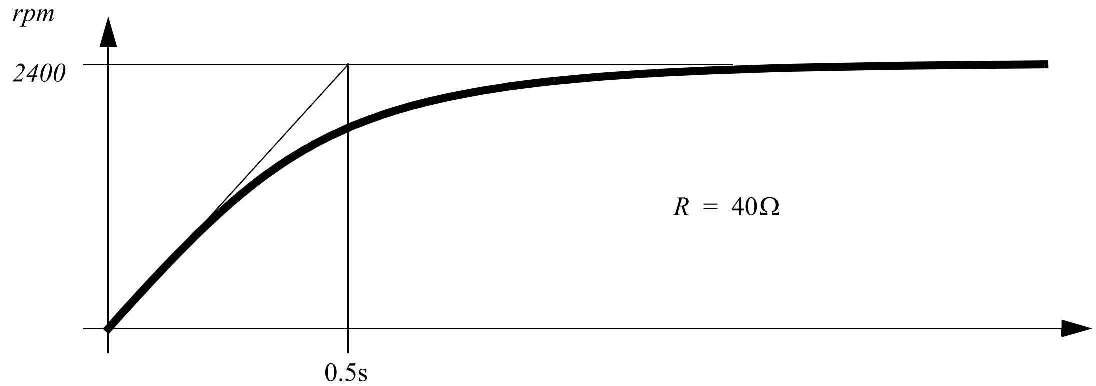
Note: The units for various rotational quantities are listed to the right. They may be used to check equations by doing a unit balance. The unit 'rad' should be ignored as it appears/disappears sporadically.

coefficient	units
K_s	$\frac{Nm}{rad}$
K_d, B	$\frac{Nms}{rad}$
J_M	Kgm^2

4.1 Design Case

A large machine is to be driven by a permanent magnet electric motor. A 20:1 gear box is used to reduce the speed and increase the torque of the motor. The motor drives a 10000kg mass in translation through a rack and pinion gear set. The pinion has a pitch diameter of 6 inches. A 10 foot long shaft is required between the gear box and the rack and pinion set. The mass moves on rails with static and dynamic coefficients of friction of 0.2 and 0.1 respectively. We want to select a shaft diameter that will keep the system critically damped when a voltage step input of 20V is applied to the motor.

To begin the analysis the velocity curve in Figure 4.17 was obtained experimentally by applying a voltage of 15V to the motor with no load attached. In addition the resistance of the motor coils was measured and found to be 40 ohms. The steady-state speed and time constant were used to determine the constants for the motor.



$$\left(\frac{d}{dt}\right)\omega_m + \omega_m\left(\frac{K^2}{J_M R}\right) = V_s\left(\frac{K}{J_M R}\right) - \frac{T_{load}}{J_M}$$

The steady-state velocity can be used to find the value of K.

$$(0) + \left(2400\frac{rot}{min}\right)\left(\frac{K^2}{J_M R}\right) = 15V\left(\frac{K}{J_M R}\right) - (0)$$

$$\left(2400\frac{rot}{min}\frac{1min}{60s}\frac{2\pi rad}{1rot}\right)(K) = 15V$$

$$K = \frac{15V}{120\pi rads^{-1}} = 39.8 \text{ ¥ } 10^{-3}\frac{Vs}{rad}$$

The time constant can be used to find the remaining parameters.

$$\frac{K^2}{J_M R} = \frac{1}{0.5s} = 2s^{-1}$$

$$J = \frac{\left((39.8\times10^{-3})\frac{Vs}{rad}\right)^2}{(40\Omega)(2s^{-1})} = 0.198005\times10^{-4} = (19.8\times10^{-6})Kgm^2$$

$$\left(\frac{d}{dt}\right)\omega_m + \omega_m 2s^{-1} = V_s(50.3V^{-1}s^{-2}rad) - \frac{T_{load}}{(19.8\times10^{-6})Kgm^2}$$

$$\theta_m' = \omega_m \qquad \text{eqn 4.32}$$

$$\omega_m' = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}T_{load} \qquad \text{eqn 4.33}$$

Figure 4.17 Example: Motor speed curve and the derived differential equation

The remaining equations describing the system are developed in Figure 4.18. These calculations are done with the assumption that the inertial effects of the gears and other components are insignificant.

The long shaft must now be analyzed. This will require that angles at both ends be defined, and the shaft be considered as a spring.

$\theta_{gear}, \omega_{gear}$ = *angular position and velocity of the shaft at the gear box*

$\theta_{pinion}, \omega_{pinion}$ = *angular position and velocity of the shaft at the pinion*

$$\theta_{gear} = \frac{1}{20}\theta_m \qquad \omega_{gear} = \frac{1}{20}\omega_m$$

$$T_{shaft} = K_s(\theta_{gear} - \theta_{pinion})$$

The rotation of the pinion is related to the displacement of the rack through the circumferential travel. This ratio can also be used to find the force applied to the mass.

$$x_{mass} = \theta_{pinion}\pi 6in$$

$$T_{shaft} = F_{mass}\left(\frac{6in}{2}\right)$$

$$K_s(\theta_{gear} - \theta_{pinion}) = F_{mass}\left(\frac{6in}{2}\right)$$

$$\sum F_{mass} = F_{mass} = M_{mass}\ddot{x}_{mass}$$

$$\frac{K_s(\theta_{gear} - \theta_{pinion})}{\left(\frac{6in}{2}\right)} = M_{mass}\ddot{\theta}_{pinion}\pi 6in$$

$$\ddot{\theta}_{pinion} = (\theta_{gear} - \theta_{pinion})1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s$$

$$\ddot{\theta}_{pinion} = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s\left(\frac{0.0254in}{1.0m}\right)^2$$

$$\ddot{\theta}_{pinion} = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)(1.141 \times 10^{-9})m^{-2} Kg^{-1} K_s$$

$$\dot{\theta}_{pinion} = \omega_{pinion} \qquad \text{eqn 4.34}$$

$$\dot{\omega}_{pinion} = 57.1 \times 10^{-12} m^{-2} Kg^{-1} K_s\theta_m - 1.141 \times 10^{-9} m^{-2} Kg^{-1} K_s\theta_{pinion} \qquad \text{eqn 4.35}$$

Figure 4.18 Example: Additional equations to model the machine

If the gear box is assumed to have relatively small moment of inertia, then we can say that the torque load on the motor is equal to the torque in the shaft. This then allows the equation for the motor shaft to be put into a useful form, as shown in Figure 4.19. Having this differential equation now allows the numerical analysis to proceed. The analysis involves iteratively solving the equations and determining the point at which the system begins to overshoot, indicating critical damping.

The Tload term is eliminated from equation (2)

$$\dot{\omega}_m = V_s 50.3 V^{-1} s^{-2} rad - \omega_m 2 s^{-1} - 50505 Kg^{-1} m^{-2} K_s (\theta_{gear} - \theta_{pinion})$$

$$\dot{\omega}_m = V_s 50.3 V^{-1} s^{-2} rad - \omega_m 2 s^{-1} - 50505 Kg^{-1} m^{-2} K_s \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)$$

$$\dot{\omega}_m = (V_s 50.3 V^{-1} s^{-2} rad) + \theta_{pinion}(50505 Kg^{-1} m^{-2} K_s) + \omega_m(-2 s^{-1}) + \theta_m(-2525 Kg^{-1} m^{-2} K_s)$$

The state equations can then be put in matrix form for clarity. The units will be eliminated for brevity, but acknowledging that they are consistent.

$$\frac{d}{dt}\begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2525 K_s & -2 & 50505 K_s & 0 \\ 0 & 0 & 0 & 1 \\ 57.1 \times 10^{-12} K_s & 0 & -1.141 \times 10^{-9} K_s & 0 \end{bmatrix} \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} + \begin{bmatrix} 0 \\ V_s 50.3 \\ 0 \\ 0 \end{bmatrix}$$

The state equations for the system are then analyzed using a computer for the parameters below to find the Ks value that gives a response that approximates critical damping for a step input from 0 to 10V.

Ks (rad/Nm)	Overshoot (rad)
100	

Figure 4.19 Example: Numerical analysis of system response

These results indicate that a spring value of XXX is required to have the system behave as if it is critically damped. (Note: Clearly this system is not second order, but in the absence of another characteristics we approximate it as second order.)

4.1 Summary

- The basic equations of motion were discussed.
- Mass and area moment of inertia are used for inertia and springs.
- Rotational dampers and springs.
- A design case was presented.

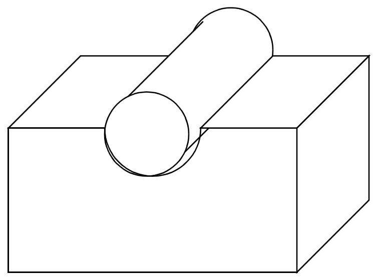
4.2 Problems With Solutions

Problem 4.1 Using the initial position and velocity of a rotating mass, find the state (i.e. position and velocity) 5 seconds later.

$$\theta_0 = 1 rad \qquad \omega_0 = 2\frac{rad}{s} \qquad \alpha = 3\frac{rad}{s^2}$$

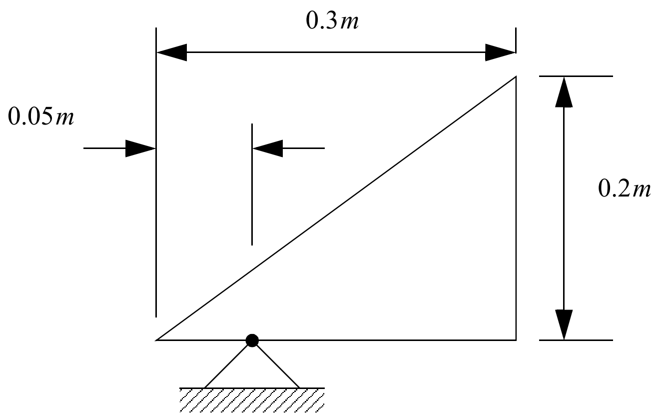
Problem 4.2 Find the second area and mass moments of inertia for a rectangle about the y-y axes. Assume that the rectangle is centered on the origin.

Problem 4.3 The 20cm diameter 10 kg cylinder is sitting in a depression that is effectively frictionless. The cylinder starts at rest. If a torque of 10 Nm is applied, what will the angular velocity be after 5s?

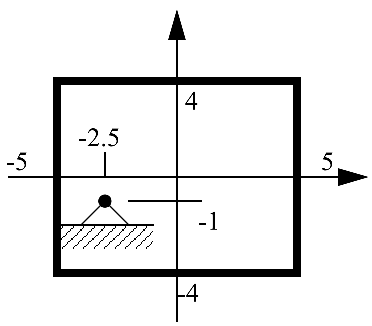


Problem 4.4 Find the polar moments of inertia of area and mass for a round cross section with known radius and mass per unit area. How are they related?

Problem 4.5 Find the mass moment of inertia for the plate as shown. The total mass is 10Kg and it is pinned to rotate near the lower left corner. You may choose to use any solution method, but all steps must be described.



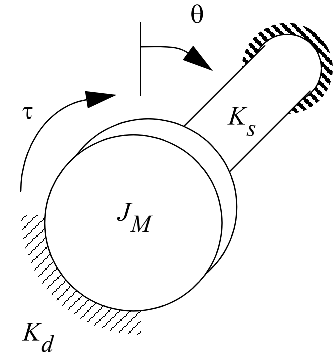
Problem 4.6 The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia WITHOUT using the parallel axis theorem.



Problem 4.7 For a 1/2" 1020 steel rod that is 1 yard long, find the torsional spring coefficient.

Problem 4.8 If a wheel (J_M=5kg m^2) is turning at 150 rpm and the damper coefficient is 1Nms/rad, what is the deceleration?

Problem 4.9 Find the response as a function of time (i.e. solve the differential equation to get a function of time.). Assume the system starts undeflected and at rest.



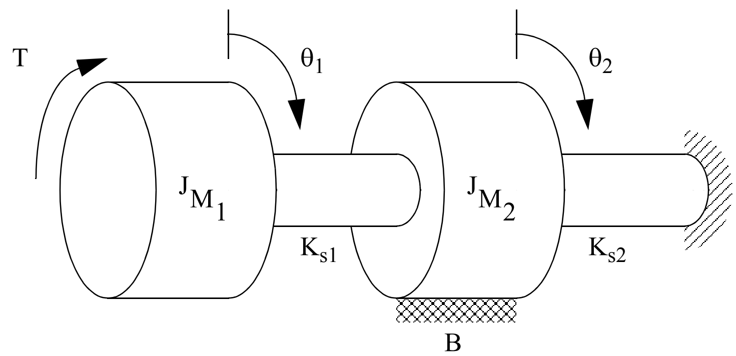
$$\tau = 10Nm$$

$$J_M = 1Kgm^2$$

$$K_d = 3\frac{Ns}{m}$$

$$K_s = 9\frac{N}{m}$$

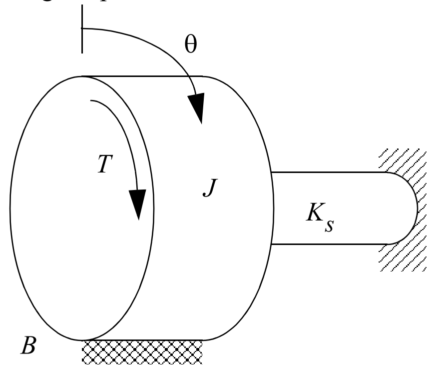
Problem 4.10 Draw the FBDs and write the differential equations for the mechanism below. The right most shaft is fixed in a wall.



Problem 4.11 The rotational spring is connected between a mass 'J', and the wall where it is rigidly held. The mass has an applied torque 'T', and also experiences damping 'B'.
a) Derive the differential equation for the rotational system shown.
b) Put the equation in state variable form (using variables) and then plot the position (not velocity) as a function of time for the first 5 seconds with your calculator using the parameters below. Assume the system starts at rest.

$$K_s = 10\frac{Nm}{rad} \qquad B = 1\frac{Nms}{rad}$$

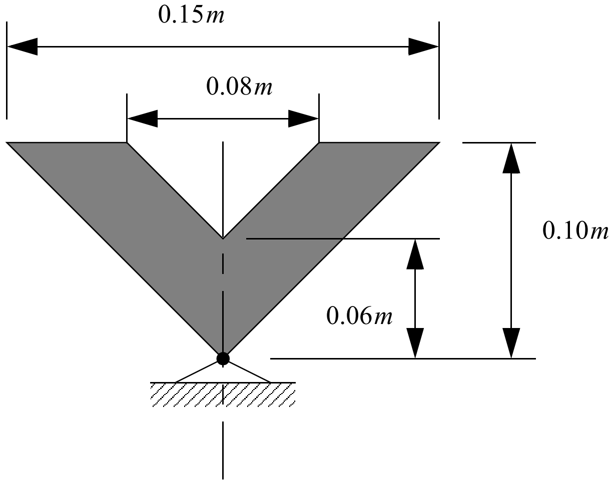
$$J_M = 1Kgm^2 \qquad T = 10Nm$$



c) A differential equation for the rotating mass with a spring and damper is given below. Solve the differential equation to get a function of time. Assume the system starts at rest.

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

Problem 4.12 Calculate the area and mass moments of inertia for the 1kg plate in the figure.



4.3 Problem Solutions

Answer 4.1

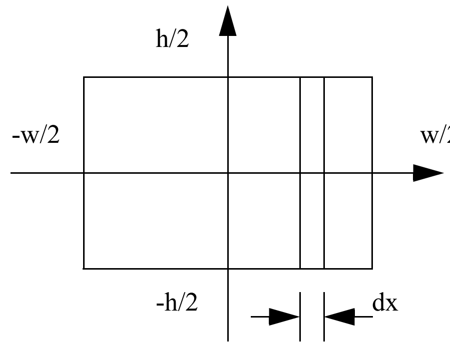
$$\theta(5) = 48.5 rad$$

$$\omega(5) = 17\frac{rad}{s}$$

Answer 4.2

Given:

M = mass
h = height
w = width



For Area:

$$I_{yy} = \int_{-\frac{w}{2}}^{\frac{w}{2}} (x^2) dA = \int_{-\frac{w}{2}}^{\frac{w}{2}} (x^2)(h) dx = \left.\frac{hx^3}{3}\right|_{-\frac{w}{2}}^{\frac{w}{2}} = \frac{h}{3}\left(\left(\frac{w}{2}\right)^3 - \left(-\frac{w}{2}\right)^3\right) = h\frac{w^3}{12}$$

For Mass:

$$\rho = \frac{M}{A} = \frac{M}{hw}$$

$$dM = \rho dA = \frac{M}{hw} h dx = \frac{M}{w} dx$$

$$I_{yy} = \int_{-\frac{w}{2}}^{\frac{w}{2}} (x^2) dM = \int_{-\frac{w}{2}}^{\frac{w}{2}} (x^2)\left(\frac{M}{w}\right) dx = \left.\frac{Mx^3}{3w}\right|_{-\frac{w}{2}}^{\frac{w}{2}} = \frac{M}{3w}\left(\left(\frac{w}{2}\right)^3 - \left(-\frac{w}{2}\right)^3\right) = M\frac{w^2}{12}$$

Answer 4.3

$$\theta(t) = \iint \left(\frac{\tau}{J}\right)(dt)dt = \int \left(\frac{\tau}{J}t + C_1\right) dt = \frac{\tau}{2J}t^2 + C_1 t + C_2 = \frac{\tau}{2J}t^2 + v_0 t + x_0$$

$$J = \frac{Mr^2}{2} = \frac{10Kg(0.10m)^2}{2} = 0.05Kgm^2$$

Note: The units all cancel but rad is added for angular displacement.

$$\theta(5s) = \frac{10Nm}{2(0.05Kgm^2)}(5s)^2 + 0 + 0 = \frac{10}{2(0.05)} 25 \frac{Ns^2}{Kgm} = 2500 rad$$

$$\omega(5s) = \frac{\tau}{J}t + v_0 = \frac{10Nm}{0.05Kgm^2}(5s) + 0 = 1000 s^{-1} rad$$

Answer 4.4

For area:

$$J_{area} = \int_0^R r^2 dA = \int_0^R r^2(2\pi r dr) = 2\pi\int_0^R r^3 dr = 2\pi\frac{r^4}{4}\bigg|_0^R = \frac{\pi R^4}{2}$$

For mass:

$$\rho = \frac{M}{A} = \frac{M}{\pi R^2}$$

$$J_{mass} = \int_0^R r^2 dM = \int_0^R r^2(\rho 2\pi r dr) = 2\pi\rho\int_0^R r^3 dr = 2\pi\rho\frac{r^4}{4}\bigg|_0^R = \rho\left(\frac{\pi R^4}{2}\right) = \frac{MR^2}{2}$$

The mass moment can be found by multiplying the area moment by the area density.

Answer 4.5

$$\rho = \frac{M}{A} = \frac{10Kg}{0.5(0.2m)(0.3m)} = 333\frac{kg}{m^2}$$

$$\tilde{I}_{yy} = \rho\int_{-0.05}^{0.25} x^2\left(\frac{0.2}{0.3}(x+0.05)\right)dx$$

$$\tilde{I}_{yy} = \frac{2}{3}\rho\int_{-0.05}^{0.25}(x^3+0.05x^2)dx = \frac{2}{3}\rho\left(\frac{x^4}{4}+\frac{0.05x^3}{3}\right)\bigg|_{-0.05}^{0.25}$$

$$\tilde{I}_{yy} = \frac{2}{3}333\left(\frac{0.25^4}{4}+\frac{0.05(0.25)^3}{3}-\frac{(-0.05)^4}{4}-\frac{0.05(-0.05)^3}{3}\right)$$

$$\tilde{I}_{yy} = 0.2754kgm^2$$

$$\tilde{I}_{xx} = \rho\int_0^{0.2} y^2\left(\left(\frac{-0.3}{0.2}\right)(y-0.2)\right)dy$$

$$\tilde{I}_{xx} = (-1.5\rho)\int_0^{0.2}(y^3-0.2y^2)dy = (-1.5\rho)\left(\frac{y^4}{4}-\frac{0.2y^3}{3}\right)\bigg|_0^{0.2}$$

$$\tilde{I}_{xx} = (-1.5)333\left(\frac{0.2^4}{4}-\frac{0.2(0.2)^3}{3}\right) = 0.0666kgm^2$$

$$\tilde{J} = \tilde{I}_{xx}+\tilde{I}_{yy} = 0.2754kgm^2+0.0666kgm^2 = 0.342kgm^2$$

Answer 4.6

$$I_{M_x} = 63.33Kgm^2$$
$$I_{M_y} = 145.8Kgm^2$$
$$J_M = 209.2Kgm^2$$

Answer 4.7

$$K_s = 215\frac{Nm}{rad}$$

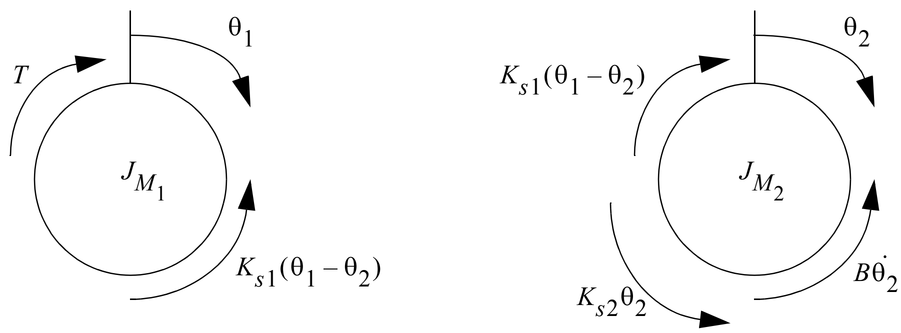
Answer 4.8

$$\ddot{\theta} = -3.141\frac{rad}{s^2}$$

Answer 4.9

$$\theta(t) = \frac{10}{9} + (-1.283)e^{-1.5t}\cos\left(\frac{\sqrt{27}}{2}t - 0.524\right)$$

Answer 4.10



$$+\curvearrowright \quad \sum M_1 = T - K_{s1}(\theta_1 - \theta_2) = J_{M_1}\ddot{\theta}_1$$

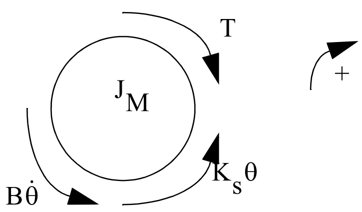
$$\ddot{\theta}_1 + \theta_1\left(\frac{K_{s1}}{J_{M_1}}\right) + \theta_2\left(\frac{-K_{s1}}{J_{M_1}}\right) = \frac{T}{J_{M_1}}$$

$$+\curvearrowright \quad \sum M_2 = K_{s1}(\theta_1 - \theta_2) - B\dot{\theta}_2 - K_{s2}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 + \dot{\theta}_2\left(\frac{B}{J_{M_2}}\right) + \theta_2\left(\frac{K_{s1} + K_{s2}}{J_{M_2}}\right) + \theta_1\left(\frac{-K_{s1}}{J_{M_2}}\right) = 0$$

Answer 4.11 a)

a)



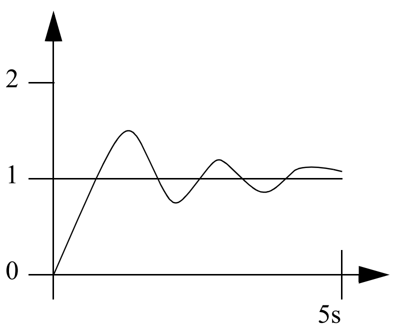
$$\sum M = T - K_s\theta - B\dot{\theta} = J_M\ddot{\theta}$$

$$J_M\ddot{\theta} + B\dot{\theta} + K_s\theta = T$$

$$\ddot{\theta} + \dot{\theta}\frac{B}{J_M} + \theta\frac{K_s}{J_M} = \frac{T}{J_M}$$

b) $\theta' = \omega$

$$\dot{\omega} = \frac{T}{J_M} - \frac{K_s}{J_M}\theta - \frac{B}{J_M}\omega$$



$$\dot{\omega} = \frac{10\text{Nm}}{1\text{Kgm}^2} - \frac{10\frac{\text{Nm}}{\text{rad}}}{1\text{Kgm}^2}\theta - \frac{1\frac{\text{Nms}}{\text{rad}}}{1\text{Kgm}^2}\omega$$

$$\dot{\omega} = \frac{\text{Nm}}{\text{Kgm}^2}\left(10 - \frac{10\theta}{\text{rad}} - \frac{\text{s}}{\text{rad}}\omega\right)$$

$$\dot{\omega} = \frac{\left(\frac{\text{Kgm}}{\text{s}^2}\right)\text{m}}{\text{Kgm}^2}\left(10 - \frac{10\theta}{\text{rad}} - \frac{\text{s}}{\text{rad}}\omega\right)$$

$$\dot{\omega} = s^{-2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

c)

homogeneous:

$$\ddot{\theta} + (1s^{-1})\dot{\theta} + (10s^{-2})\theta = 0$$

guess: $\theta_h = e^{At}$ $\quad \dot{\theta}_h = Ae^{At}$ $\quad \ddot{\theta}_h = A^2e^{At}$

$$A^2e^{At} + (1s^{-1})Ae^{At} + (10s^{-2})e^{At} = 0$$

$$A^2 + (1s^{-1})A + 10s^{-2} = 0 \qquad A = \frac{-1s^{-1} \pm \sqrt{(1s^{-1})^2 - 4(1)(10s^{-2})}}{2(1)}$$

$$A = \frac{-1s^{-1} \pm \sqrt{1s^{-2} - 40s^{-2}}}{2(1)} = (-0.5 \pm j3.123)s^{-1}$$

$$\theta_h = C_1e^{-0.5s^{-1}t}\cos\left(3.123s^{-1}t + C_2\right)$$

particular:

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

guess: $\theta_p = A$ $\quad \dot{\theta}_p = 0$ $\quad \ddot{\theta}_p = 0$

$$(0) + (1s^{-1})(0) + (10s^{-2})(A) = 10s^{-2} \qquad A = \frac{10s^{-2}}{10s^{-2}} = 1$$

$$\theta_p = 1$$

Initial conditions:

$$\theta(t) = C_1e^{-0.5s^{-1}t}\cos\left(3.123s^{-1}t + C_2\right) + 1$$

$$\theta(0) = C_1e^{-0.5s^{-1}0}\cos\left(3.123s^{-1}0 + C_2\right) + 1 = 0$$

$$C_1\cos(C_2) + 1 = 0$$

$$\theta'(t) = -0.5s^{-1}C_1e^{-0.5s^{-1}t}\cos\left(3.123s^{-1}t + C_2\right) - 3.123s^{-1}C_1e^{-0.5s^{-1}t}\sin\left(3.123s^{-1}t + C_2\right)$$

$$\theta'(0) = -0.5s^{-1}C_1(1)\cos(C_2) - 3.123s^{-1}C_1(1)\sin(C_2) = 0$$

$$-0.5\cos(C_2) - 3.123\sin(C_2) = 0$$

$$\frac{\sin(C_2)}{\cos(C_2)} = \frac{-0.5}{3.123} = \tan(C_2) \qquad C_2 = -0.159$$

$$C_1\cos(-0.159) + 1 = 0 \qquad C_1 = \frac{-1}{\cos(-0.159)} = -1.013$$

$$\theta(t) = -1.013e^{-0.5s^{-1}t}\cos(3.123s^{-1}t - 0.159) + 1$$

Answer 4.12

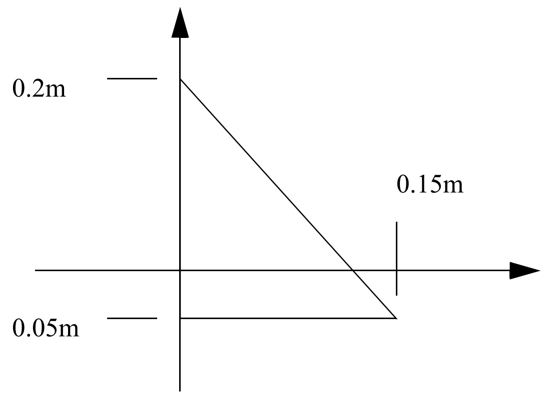
As integrals:

$$\tilde{I}_{xx} = \int_0^{0.10} y^2 dA = \int_0^{0.06} y^2\left(y\left(\frac{0.15}{0.10}\right)\right)dy + \int_{0.06}^{0.10} y^2(0.15-0.08)dy$$

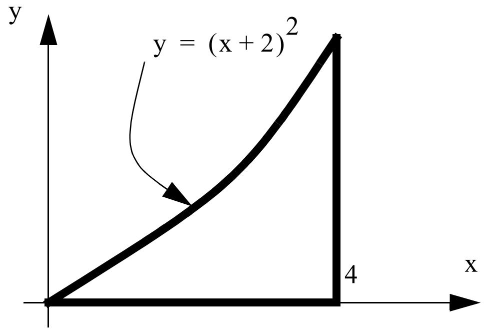
$$\tilde{I}_{yy} = 2\int_0^{\frac{0.15}{2}} x^2 dA = 2\left[\int_0^{\frac{0.08}{2}} x^2(0.06)dx + \int_{\frac{0.08}{2}}^{\frac{0.15}{2}} x^2\left(\left(\frac{0.15}{2}-x\right)\left(\frac{0.10}{0.15}\right)\right)dx\right]$$

4.4 Problems Without Solutions

Problem 4.13 Find the polar mass moment of inertia for the shape below. The total mass is 10kg.



Problem 4.14 For the shape a) find the area of the shape, b) find the centroid of the shape, and c) find the moment of inertia of the shape about the centroid.



4.5 Mass Properties Review

• A set of the basic 2D and 3D geometric primitives are given, and the notation used is described below,

A = contained area

P = perimeter distance

V = contained volume

S = surface area

x, y, z = centre of mass

$\bar{x}, \bar{y}, \bar{z}$ = centroid

I_x, I_y, I_z = moment of inertia of area (or second moment of inertia)

Area Properties:

$$I_x = \int_A y^2 dA = \text{the second moment of inertia about the y-axis}$$

$$I_y = \int_A x^2 dA = \text{the second moment of inertia about the x-axis}$$

$$I_{xy} = \int_A xy dA = \text{the product of inertia}$$

$$J_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y = \text{The polar moment of inertia}$$

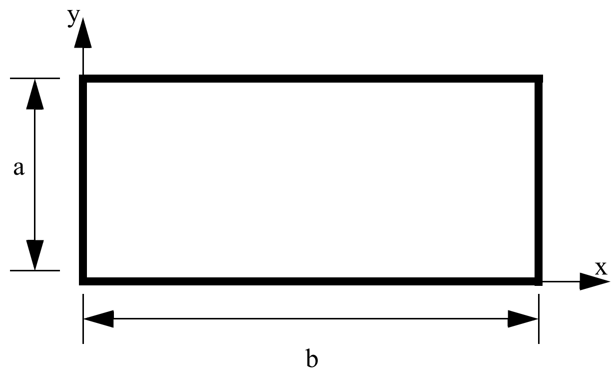
$$\bar{x} = \frac{\int_A x dA}{\int_A dA} = \text{centroid location along the x-axis}$$

$$\bar{y} = \frac{\int_A y dA}{\int_A dA} = \text{centroid location along the y-axis}$$

Rectangle/Square:

$A = ab$

$P = 2a + 2b$



Centroid:

$\bar{x} = \frac{b}{2}$

$\bar{y} = \frac{a}{2}$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \frac{ba^3}{12}$

$\bar{I}_y = \frac{b^3 a}{12}$

$\bar{I}_{xy} = 0$

Moment of Inertia (about origin axes):

$I_x = \frac{ba^3}{3}$

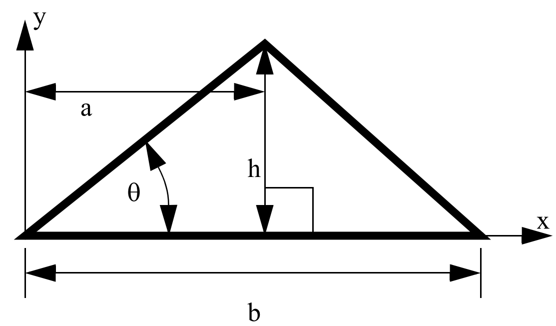
$I_y = \frac{b^3 a}{3}$

$I_{xy} = \frac{b^2 a^2}{4}$

Triangle:

$A = \frac{bh}{2}$

$P =$



Centroid:

$\bar{x} = \frac{a+b}{3}$

$\bar{y} = \frac{h}{3}$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \frac{bh^3}{36}$

$\bar{I}_y = \frac{bh}{36}(a^2 + b^2 - ab)$

$\bar{I}_{xy} = \frac{bh^2}{72}(2a - b)$

Moment of Inertia (about origin axes):

$I_x = \frac{bh^3}{12}$

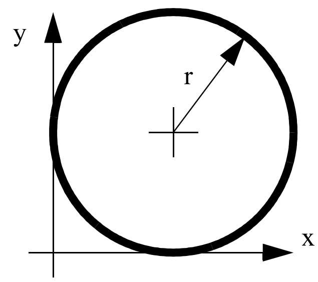
$I_y = \frac{bh}{12}(a^2 + b^2 - ab)$

$I_{xy} = \frac{bh^2}{24}(2a - b)$

Circle:

$A = \pi r^2$

$P = 2\pi r$



Centroid:

$\bar{x} = r$

$\bar{y} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \frac{\pi r^4}{4}$

$\bar{I}_y = \frac{\pi r^4}{4}$

$\bar{I}_{xy} = 0$

Moment of Inertia (about origin axes):

$I_x =$

$I_y =$

$I_{xy} =$

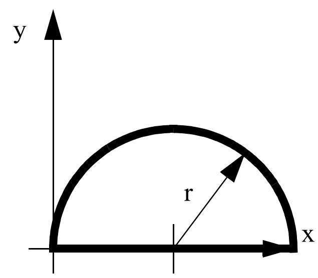
Mass Moment of Inertia (about centroid):

$J_z = \frac{Mr^2}{2}$

Half Circle:

$A = \frac{\pi r^2}{2}$

$P = \pi r + 2r$



Centroid:

$\bar{x} = r$

$\bar{y} = \frac{4r}{3\pi}$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$

$\bar{I}_y = \frac{\pi r^4}{8}$

$I_{xy} = 0$

Moment of Inertia (about origin axes):

$I_x = \frac{\pi r^4}{8}$

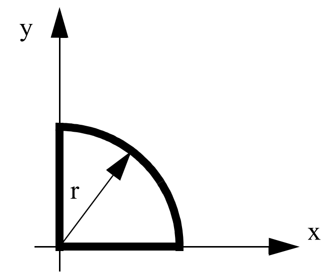
$I_y = \frac{\pi r^4}{8}$

$I_{xy} = 0$

Quarter Circle:

$$A = \frac{\pi r^2}{4}$$

$$P = \frac{\pi r}{2} + 2r$$



Centroid:

$$\bar{x} = \frac{4r}{3\pi}$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\overline{I_x} = 0.05488r^4$$

$$\overline{I_y} = 0.05488r^4$$

$$\overline{I_{xy}} = -0.01647r^4$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r^4}{16}$$

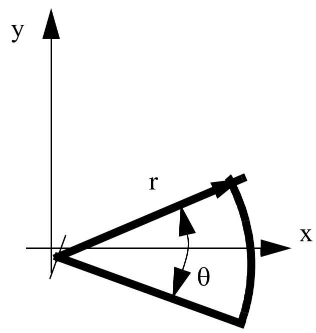
$$I_y = \frac{\pi r^4}{16}$$

$$I_{xy} = \frac{r^4}{8}$$

Circular Arc:

$$A = \frac{\theta r^2}{2}$$

$$P = \theta r + 2r$$



Centroid:

$$\bar{x} = \frac{2r\sin\frac{\theta}{2}}{3\theta}$$

$$\bar{y} = 0$$

Moment of Inertia
(about centroid axes):

$$\overline{I_x} =$$

$$\overline{I_y} =$$

$$\overline{I_{xy}} =$$

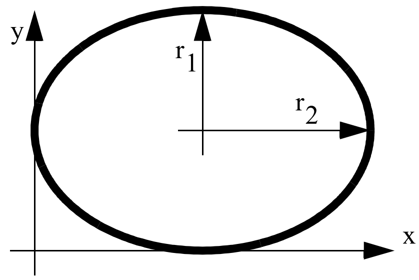
Moment of Inertia
(about origin axes):

$$I_x = \frac{r^4}{8}(\theta - \sin\theta)$$

$$I_y = \frac{r^4}{8}(\theta + \sin\theta)$$

$$I_{xy} = 0$$

Ellipse:



$$A = \pi r_1 r_2$$

$$P = 4r_1\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{\sqrt{r_1^2+r_2^2}}{a}(\sin\theta)^2}\,d\theta$$

$$P \approx 2\pi\sqrt{\frac{r_1^2+r_2^2}{2}}$$

Centroid:

$$\bar{x} = r_2$$

$$\bar{y} = r_1$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = \frac{\pi r_1^3 r_2}{4}$$

$$\overline{I_y} = \frac{\pi r_1 r_2^3}{4}$$

$$\overline{I_{xy}} =$$

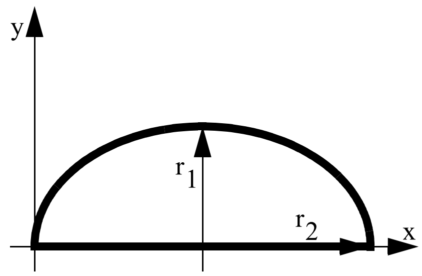
Moment of Inertia (about origin axes):

$$I_x =$$

$$I_y =$$

$$I_{xy} =$$

Half Ellipse:



$$A = \frac{\pi r_1 r_2}{2}$$

$$P = 2r_1\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{\sqrt{r_1^2+r_2^2}}{a}(\sin\theta)^2}\,d\theta + 2r_2$$

$$P \text{ ª } \pi\sqrt{\frac{r_1^2+r_2^2}{2}} + 2r_2$$

Centroid:

$$\bar{x} = r_2$$

$$\bar{y} = \frac{4r_1}{3\pi}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = 0.05488 r_2 r_1^3$$

$$\overline{I_y} = 0.05488 r_2^3 r_1$$

$$\overline{I_{xy}} = -0.01647 r_1^2 r_2^2$$

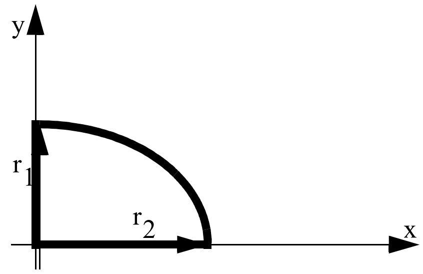
Moment of Inertia (about origin axes):

$$I_x = \frac{\pi r_2 r_1^3}{16}$$

$$I_y = \frac{\pi r_2^3 r_1}{16}$$

$$I_{xy} = \frac{r_1^2 r_2^2}{8}$$

Quarter Ellipse:



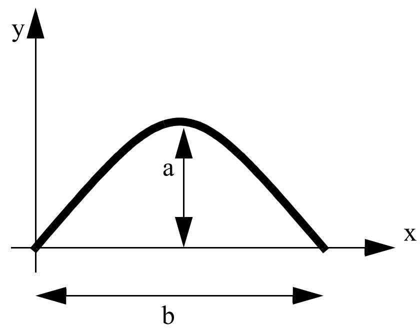
$$A = \frac{\pi r_1 r_2}{4}$$

$$P = r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2}\, d\theta + 2r_2$$

$$P \approx \frac{\pi}{2}\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{4r_2}{3\pi}$	$\overline{I_x} =$	$I_x = \pi r_2 r_1^3$
$\bar{y} = \frac{4r_1}{3\pi}$	$\overline{I_y} =$	$I_y = \pi r_2^3 r_1$
	$\overline{I_{xy}} =$	$I_{xy} = \frac{r_2^2 r_1^2}{8}$

Parabola:



$$A = \frac{2}{3}ab$$

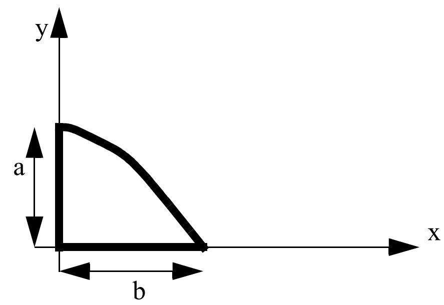
$$P = \frac{\sqrt{b^2 + 16a^2}}{2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{b}{2}$	$\overline{I_x} =$	$I_x =$
$\bar{y} = \frac{2a}{5}$	$\overline{I_y} =$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Parabola:

$$A = \frac{ab}{3}$$

$$P = \frac{\sqrt{b^2 + 16a^2}}{4} + \frac{b^2}{16a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{3b}{8}$	$\overline{I_x} = \frac{8ba^3}{175}$	$I_x = \frac{2ba^3}{7}$
$\bar{y} = \frac{2a}{5}$	$\overline{I_y} = \frac{19b^3a}{480}$	$I_y = \frac{2b^3a}{15}$
	$\overline{I_{xy}} = \frac{b^2a^2}{60}$	$I_{xy} = \frac{b^2a^2}{6}$

• A general class of geometries are conics. This for is shown below, and can be used to represent many of the simple shapes represented by a polynomial.

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Conditions

$A = B = C = 0$	straight line
$B = 0, A = C$	circle
$B^2 - AC < 0$	ellipse
$B^2 - AC = 0$	parabola
$B^2 - AC > 0$	hyperbola

Volume Properties:

$$I_x = \int_V r_x^2 dV = \text{ the moment of inertia about the x-axis}$$

$$I_y = \int_V r_y^2 dV = \text{ the moment of inertia about the y-axis}$$

$$I_z = \int_V r_z^2 dV = \text{ the moment of inertia about the z-axis}$$

$$\bar{x} = \frac{\int_V x dV}{\int_V dV} = \text{ centroid location along the x-axis}$$

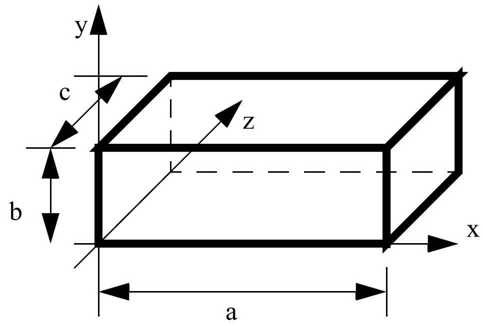
$$\bar{y} = \frac{\int_V y dV}{\int_V dV} = \text{ centroid location along the y-axis}$$

$$\bar{z} = \frac{\int_V z dV}{\int_V dV} = \text{ centroid location along the z-axis}$$

Parallelepiped (box):

$V = abc$

$S = 2(ab + ac + bc)$

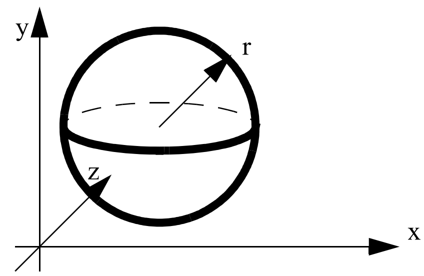


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = \frac{a}{2}$	$\bar{I}_x = \frac{M(a^2 + b^2)}{12}$	$I_x =$	$J_x =$
$\bar{y} = \frac{b}{2}$	$\bar{I}_y = \frac{M(a^2 + c^2)}{12}$	$I_y =$	$J_y =$
$\bar{z} = \frac{c}{2}$	$\bar{I}_z = \frac{M(b^2 + a^2)}{12}$	$I_z =$	$J_z =$

Sphere:

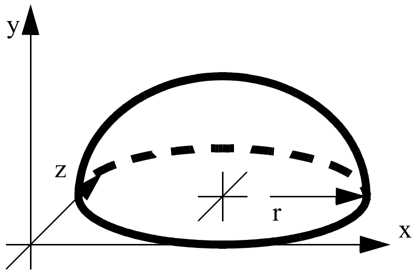
$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = \frac{2Mr^2}{5}$	$I_x =$	$J_x = \frac{2Mr^2}{5}$
$\bar{y} = r$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$	$J_y = \frac{2Mr^2}{5}$
$\bar{z} = r$	$\bar{I}_z = \frac{2Mr^2}{5}$	$I_z =$	$J_z = \frac{2Mr^2}{5}$

Hemisphere:



$$V = \frac{2}{3}\pi r^3$$

S =

Centroid:

$\bar{x} = r$

$\bar{y} = \frac{3r}{8}$

$\bar{z} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \frac{83}{320}Mr^2$

$\bar{I}_y = \frac{2Mr^2}{5}$

$\bar{I}_z = \frac{83}{320}Mr^2$

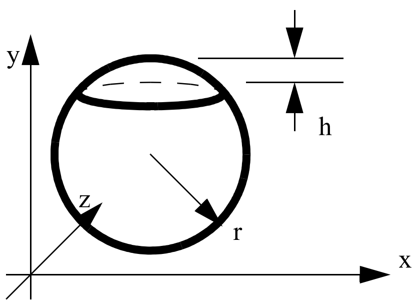
Moment of Inertia (about origin axes):

$I_x =$

$I_y =$

$I_z =$

Cap of Sphere:



$$V = \frac{1}{3}\pi h^2(3r - h)$$

$$S = 2\pi rh$$

Centroid:

$\bar{x} = r$

$\bar{y} =$

$\bar{z} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x =$

$\bar{I}_y =$

$\bar{I}_z =$

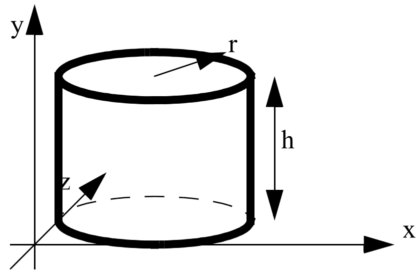
Moment of Inertia (about origin axes):

$I_x =$

$I_y =$

$I_z =$

Cylinder:

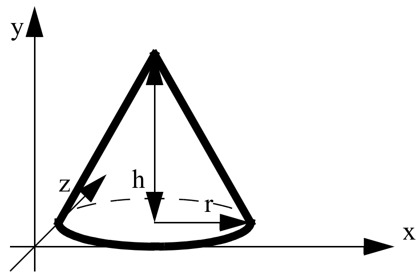


$$V = h\pi r^2$$

$$S = 2\pi rh + 2\pi r^2$$

Centroid:	Moment of Inertia (about centroid axis):	Moment of Inertia (about origin axis):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_x = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_x = \frac{M(3r^2 + h^2)}{12}$
$\bar{y} = \frac{h}{2}$	$\bar{I}_y = \frac{Mr^2}{2}$	$I_y =$	$J_y = \frac{Mr^2}{2}$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_z = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_z = \frac{M(3r^2 + h^2)}{12}$

Cone:



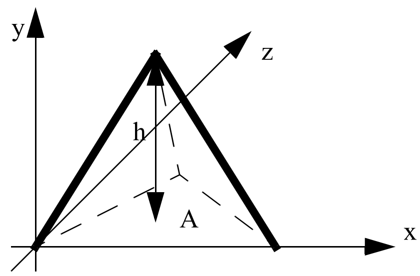
$$V = \frac{1}{3}\pi r^2 h$$

$$S = \pi r\sqrt{r^2 + h^2}$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y = \frac{3Mr^2}{10}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_z =$

Tetrahedron:

$$V = \frac{1}{3}Ah$$

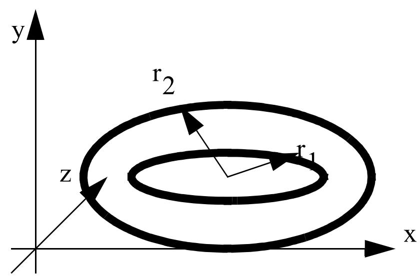


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} =$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y =$	$I_y =$
$\bar{z} =$	$\bar{I}_z =$	$I_z =$

Torus:

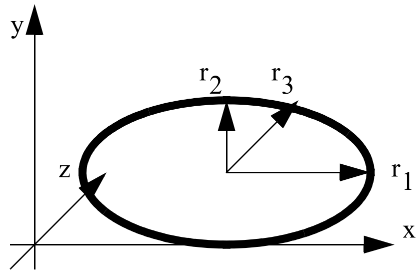
$$V = \frac{1}{4}\pi^2(r_1 + r_2)(r_2 - r_1)^2$$

$$S = \pi^2(r_2^2 - r_1^2)$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \left(\frac{r_2 - r_1}{2}\right)$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_2$	$\bar{I}_z =$	$I_z =$

Ellipsoid:

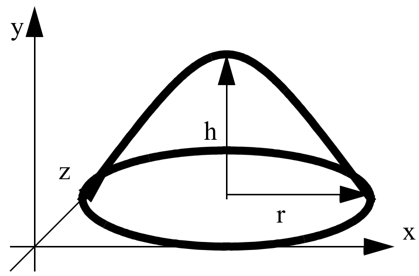


$V = \frac{4}{3}\pi r_1 r_2 r_3$

$S =$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_1$	$\bar{I}_x =$	$I_x =$
$\bar{y} = r_2$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_3$	$\bar{I}_z =$	$I_z =$

Parabaloid:



$V = \frac{1}{2}\pi r^2 h$

$S =$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x =$	$I_x =$
$\bar{y} =$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r$	$\bar{I}_z =$	$I_z =$

5. Mechanisms

Topic 5.1 *Levers, gears and belts.*
Topic 5.2 *Design cases.*
Objective 5.1 *To be able to develop and analyze systems with rotation, translation, and complex components.*

Mechanisms contain multiple components connected together to perform a complex function. The first step in analysis of mechanisms is to break the device into multiple rigid bodies with clearly defined contact points. Components, such as gears, result in more complex interactions.

Contact Points And Joints

A system is built by connecting components together. These connections can be rigid or moving. In solid connections all forces and moments are transmitted and the two pieces act as a single rigid body. In moving connections there is at least one degree of freedom. If we limit this to translation only, there are up to three degrees of freedom, x, y and z. In any direction there is a degree of freedom, a force or moment cannot be transmitted.

When constructing FBDs for a system we must break all of the components into individual rigid bodies. Where the mechanism has been broken the contact forces must be added to both of the separated pieces. Consider the example in Figure 5.1. At joint A the forces are written as two components in the x and y directions. For joint B the force components with equal magnitudes but opposite directions are added to both FBDs.

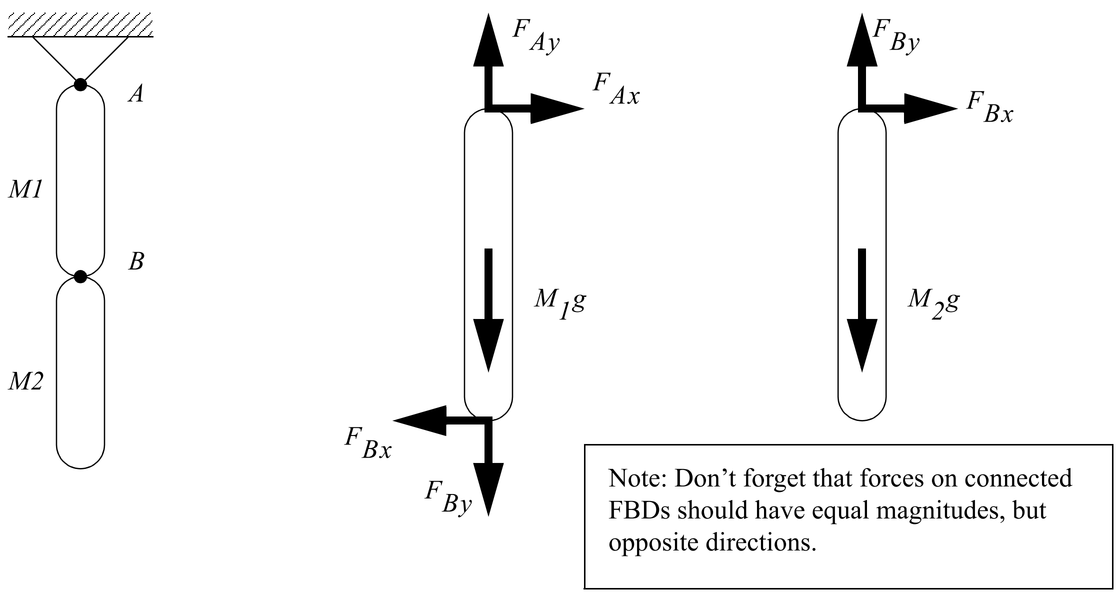
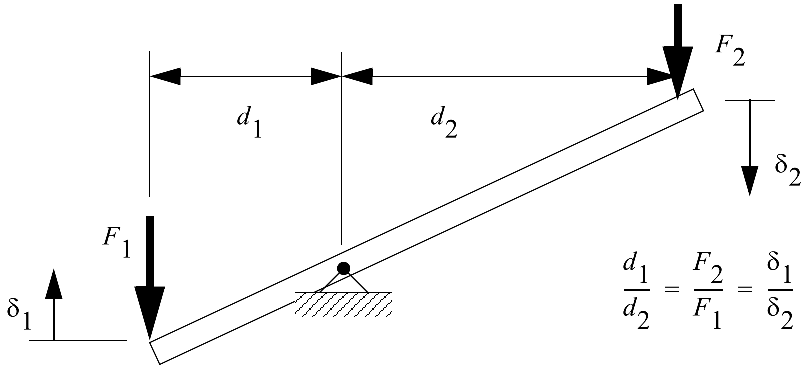


Figure 5.1 FBDs for systems with connected members

Levers

The lever shown in Figure 5.2 can be used to amplify forces or motion. Although theoretically a lever arm could rotate fully, it typically has a limited range of motion. The amplification is determined by the ratio of arm lengths to the left and right of the center.



Note: As the lever rotates the length ratio will be maintained because of similar triangles. This allows the lever to work over a range of angles. The lever above would become ineffective if it was vertical.

Note: The tip deflection can be related to the angle of rotation of the lever if the angle of rotation is small. $\theta = \frac{\delta_1}{d_1} = \frac{\delta_2}{d_2}$

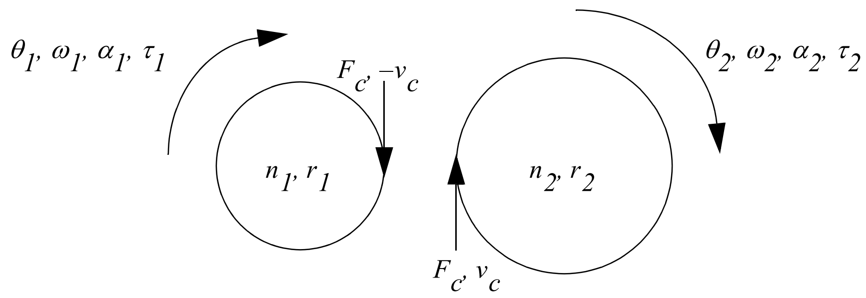
Figure 5.2 Force transmission with a lever

Gears and Belts

While levers amplify forces and motions over limited ranges of motion, gears can rotate indefinitely. Some of the basic gear forms are listed below.

- Spur - Round gears with teeth parallel to the rotational axis.
- Rack - A straight gear (used with a small round gear called a pinion).
- Helical - The teeth follow a helix around the rotational axis.
- Bevel - The gear has a conical shape, allowing forces to be transmitted at angles.

Gear teeth are carefully designed so that they will *mesh* smoothly as the gears rotate. The forces on gears acts at a tangential distance from the center of rotation called the *pitch diameter*. The ratio of motions and forces through a pair of gears is proportional to their radii, as shown in Figure 5.3. The number of teeth on a gear is proportional to the diameter. The gear ratio is used to measure the relative rotations of the shafts. For example a gear ratio of 20:1 would mean the input shaft of the gear box would have to rotate 20 times for the output shaft to rotate once.



$$\tau_1 = F_c r_1 \qquad \tau_2 = F_c r_2 \qquad \frac{n_1}{r_1} = \frac{n_2}{r_2} \qquad \frac{\tau_1}{\tau_2} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

$$v_c = \omega_1 r_1 = -\omega_2 r_2 \qquad \frac{r_2}{r_1} = \frac{-\omega_1}{\omega_2} = \frac{-\alpha_1}{\alpha_2} = \frac{-\Delta\theta_1}{\Delta\theta_2} = \frac{n_2}{n_1}$$

where,

n = number of teeth on respective gears
r = radii of respective gears
Fc = reaction force of contact between gear teeth
vc = tangential velocity of gear teeth
τ = *Torque on gears*

Figure 5.3 Basic Gear Relationships

For lower gear ratios a simple gear box with two gears can be constructed. For higher gear ratios more gears can be added. To do this, compound gear sets are required. In a compound gear set two or more gears are connected on a single shaft, as shown in Figure 5.4. In this example the gear ratio on the left is 4:1, and the ratio for the set on the right is 4:1. Together they give a gear ratio of 16:1.

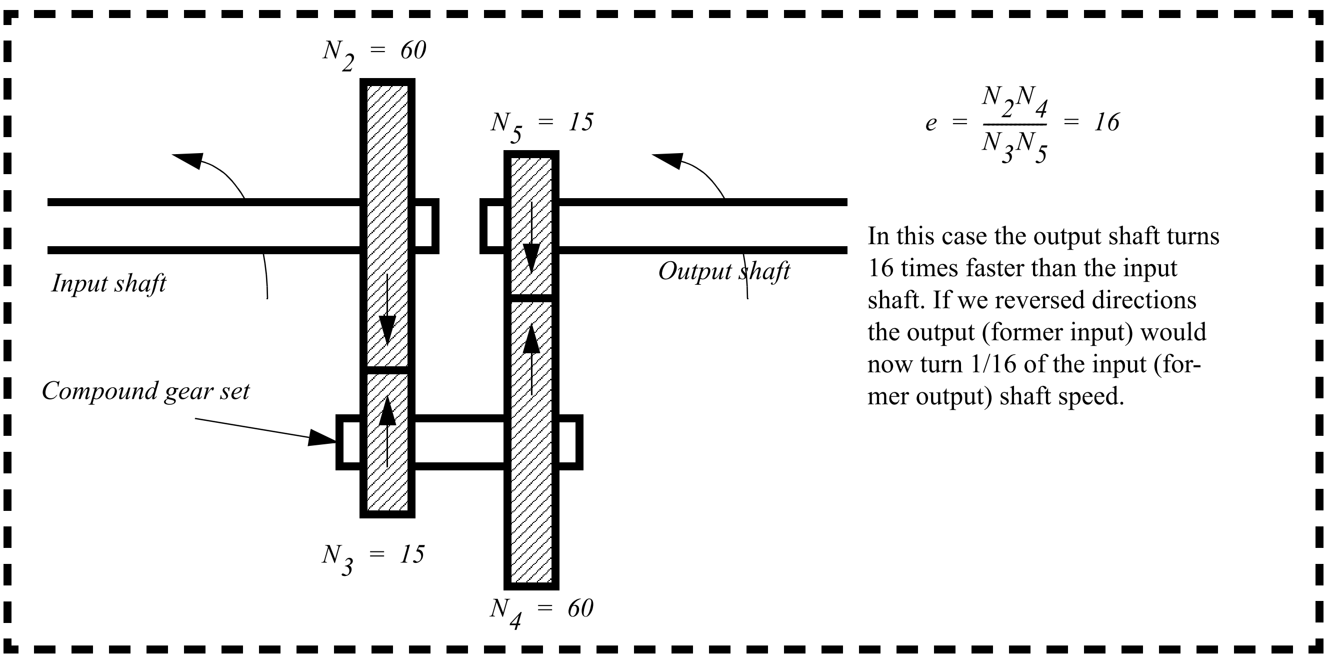
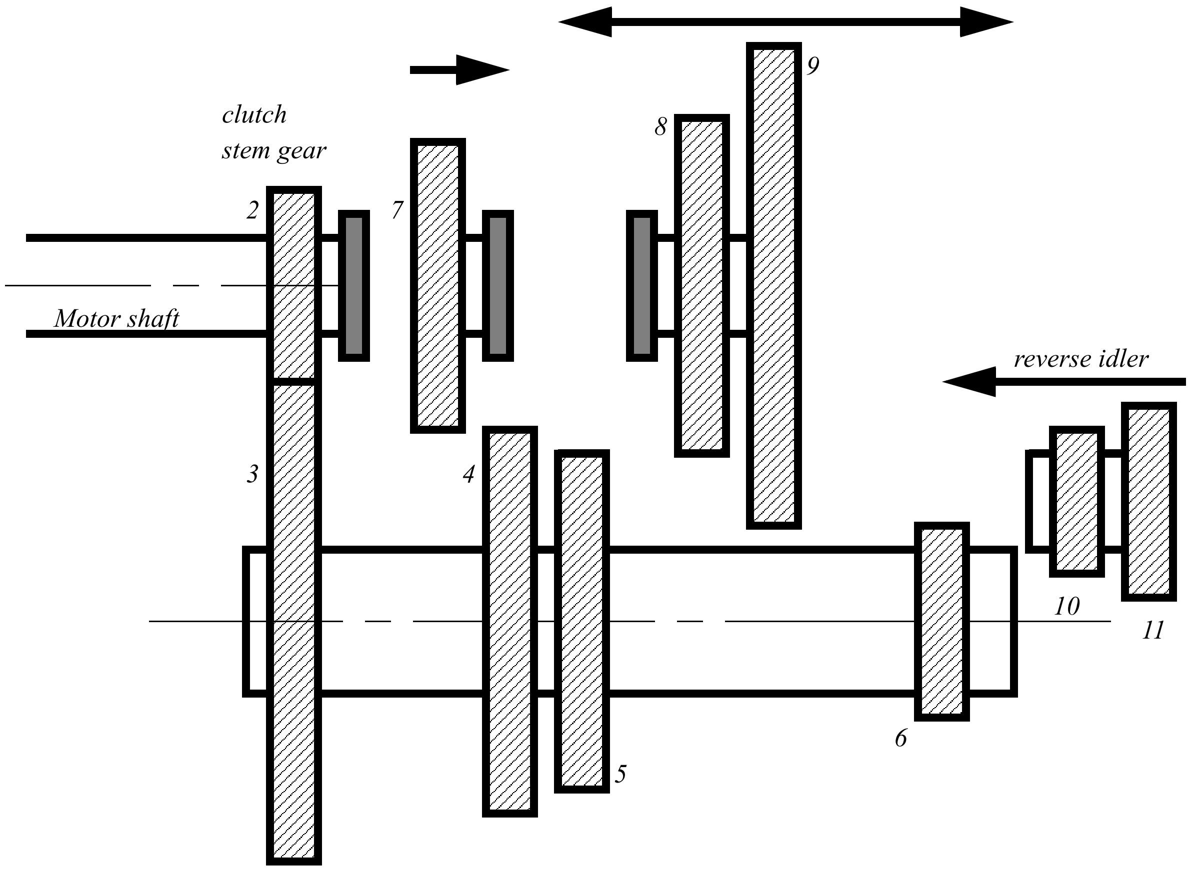


Figure 5.4 Example: A compound gear set

A manual transmission is shown in Figure 5.5. In the transmission the gear ratio is changed by sliding (left-right) some of the gears to change the sequence of gears transmitting the force. Notice that when in reverse an additional compound gear set is added to reverse the direction of rotation.



Speed (gear)	Gear Train
1	2-3-6-9
2	2-3-5-8
3	2-3-4-7
4	bypass gear train
reverse	2-3-6-10-11-9

In this manual transmission the gear shifter will move the gears in and out of contact. At this point all of the needed gears will be meshed and turning. The final step is to engage the last gear in the gear train with the clutch (plate) and this couples the gears to the wheels.

Figure 5.5 Example: A manual transmission [Shigley]

Rack and pinion gear sets are used for converting rotation to translation. A rack is a long straight gear that is driven by a small mating gear called a pinion. The basic relationships are shown in Figure 5.6.

$$T = Fr \qquad V_c = \omega r \qquad \Delta l = r\Delta\theta$$

where,

r = radius of pinion
F = force of contact between gear teeth
Vc = tangential velocity of gear teeth and velocity of rack
T = torque on pinion

Figure 5.6 Relationships for a rack and pinion gear set

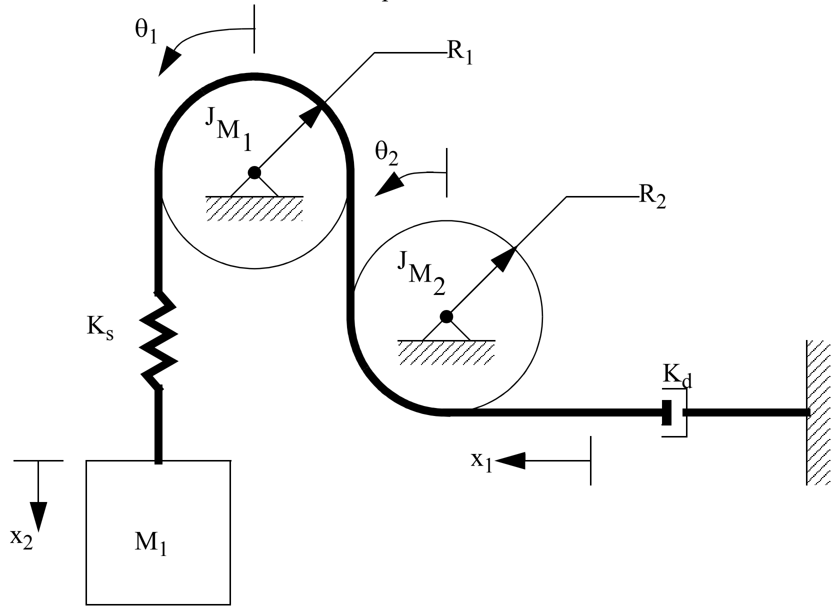
Belt based systems can be analyzed with methods similar to gears (with the exception of teeth). A belt wound around a drum will act like a rack and pinion gear pair. A belt around two or more pulleys will act like gears.

5.1 Summary

- The basic equations of motion were discussed.
- Mass and area moment of inertia are used for inertia.

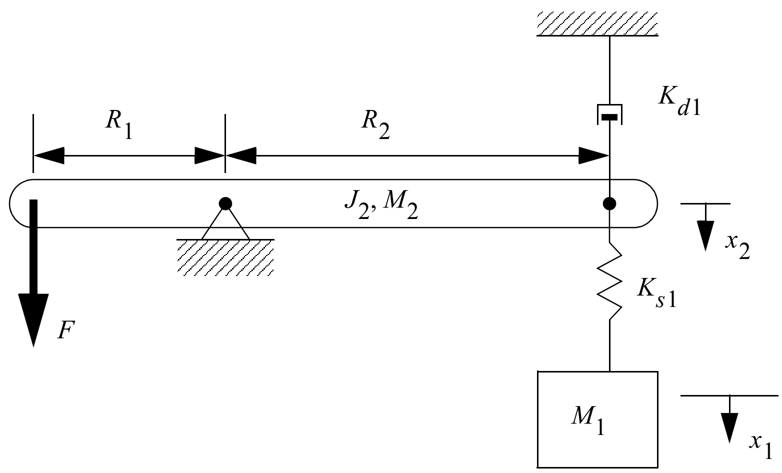
5.2 Problems With Solutions

Problem 5.1 Draw the FBDs and write the differential equations for the mechanism below.



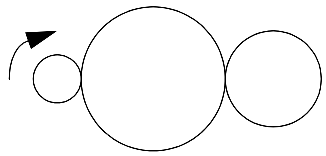
Problem 5.2 Given a lever set to lift a 1000 kg rock - if the lever is 2m long and the distance from the fulcrum to the rock is 10cm, how much force is required to lift it?

Problem 5.3 A lever arm has a force on one side, and a spring damper combination on the other side with a suspended mass. The moment J1 is about the pivot point. For the system a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

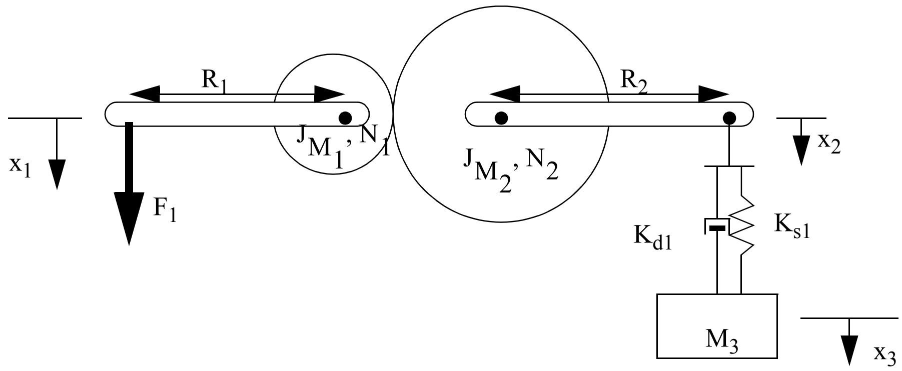


Problem 5.4 A gear train has an input gear with 20 teeth, a center gear that has 100 teeth, and an output gear that has 40 teeth. If the input shaft is rotating at 5 rad/sec what is the rotation speed of the output shaft? What if the center gear is

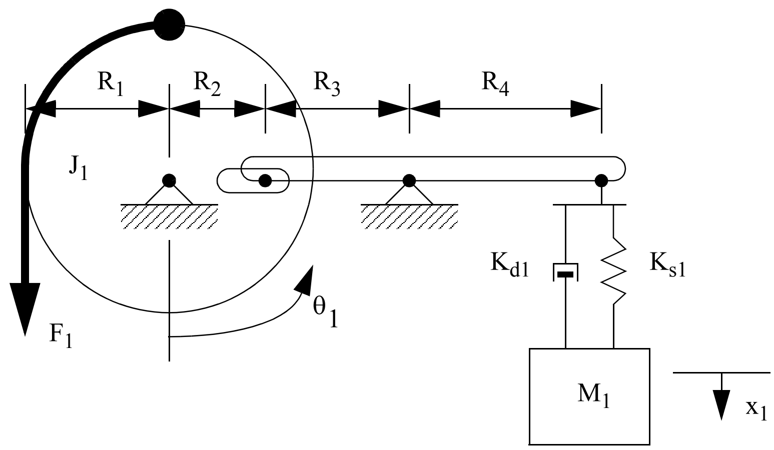
removed?



Problem 5.5 This system consists of two gears with fixed centers of rotation and lever arms. a) Write the differential equations (assume small angular deflections) and b) put the equations in state variable form. All moment of inertia values are about the pivot points.Assume the mass of the R1 and R2 arms is negligible.

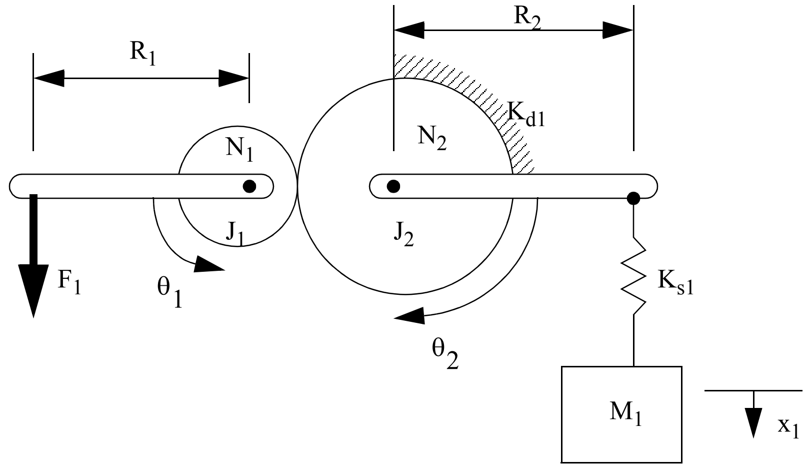


Problem 5.6 A round drum with a slot. The slot drives a lever arm with a suspended mass. A force is applied to a belt over the drum. For the system a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

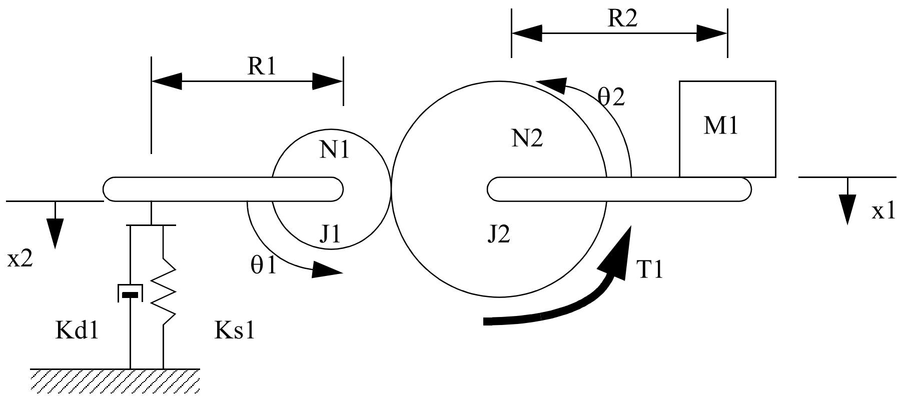


Problem 5.7 Two gears have a force on one side, and a mass on the other, both suspended from moment arms. There is a rotational damping on one of the gears. The moments of inertia are given about the pivot points. For the system a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

Assume that the mass of the arms R1 and R2 are negligible.

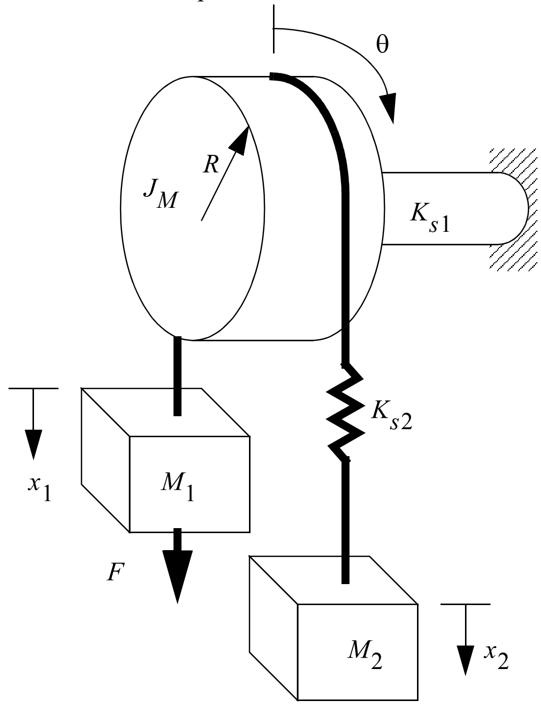


Problem 5.8 Two gears have levers attached. On one side is a mass, the other side a spring damper pair. A torque is applied to one gear. Assume the mass remains in contact with the lever. Both moments of inertia are about the pivot points. For the system a) write the differential equation for the system with theta2 as the output (assume small angular deflections) and b) put the equations in state variable form. The mass of the arms is much lower than the gears and can be ignored.

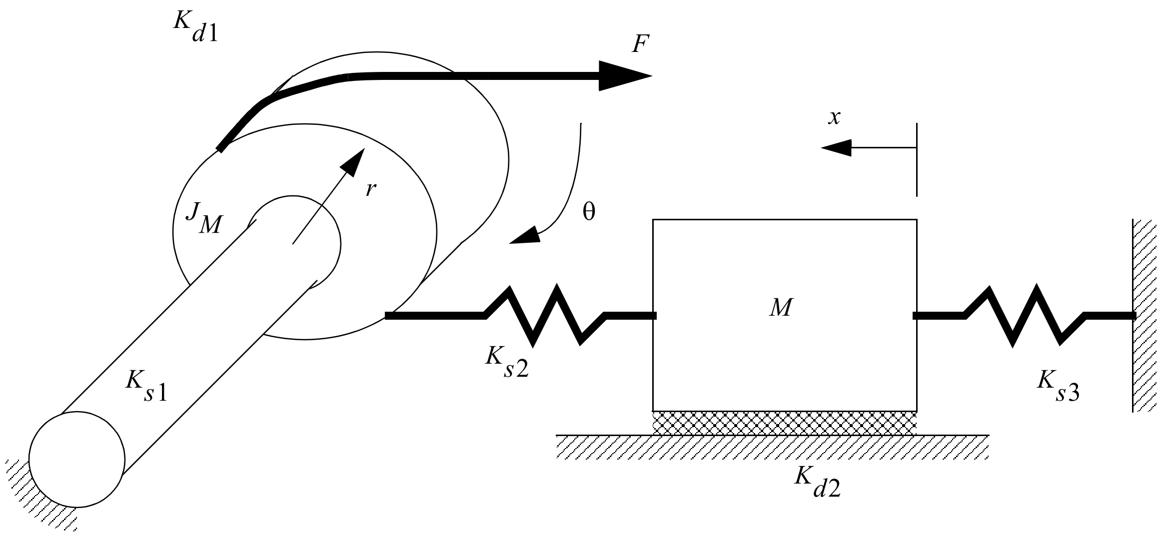


Problem 5.9 The system below consists of two masses hanging by a cable over mass 'JM'. There is a spring in the cable near M2. The cable doesn't slip on 'J'. a) Derive the differential equations for the following system. b) Convert the

differential equations to state variable equations

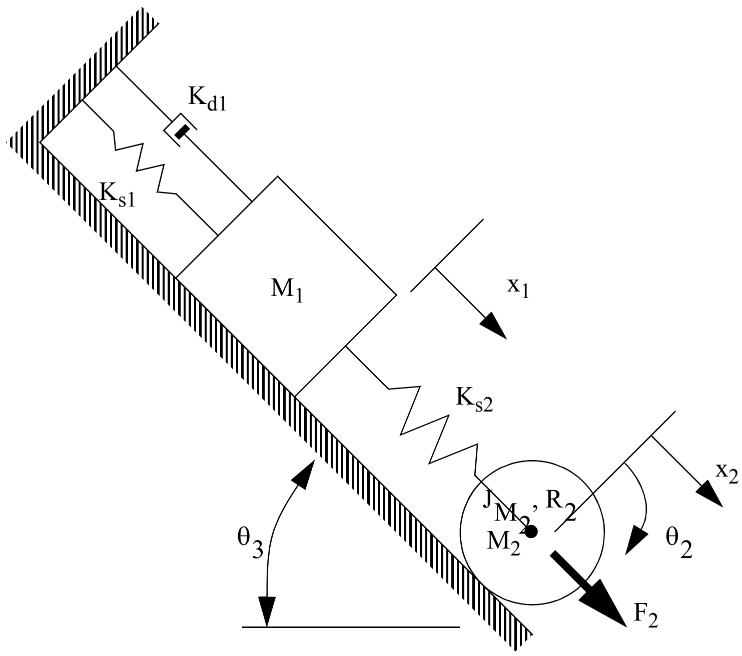


Problem 5.10 Write the state equations for the system to relate the applied force 'F' to the displacement 'x'. Note that the rotating mass also experiences a rotational damping force indicated with Kd1

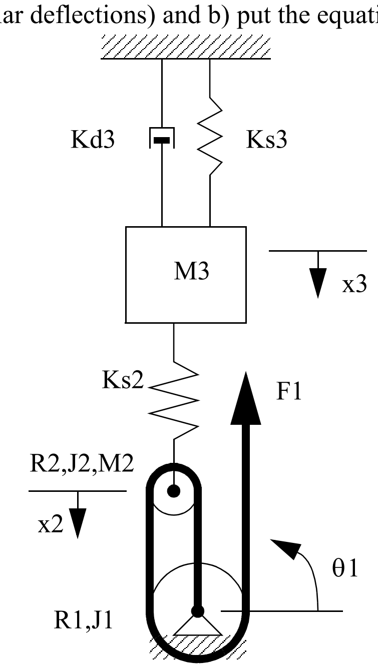


Problem 5.11 A mass slides on a plane with dry kinetic friction (0.3). It is connected to a round mass that rolls and does not slip. For the system a) write the differential equations (assume small angular deflections) and b) put the equations

in state variable form.

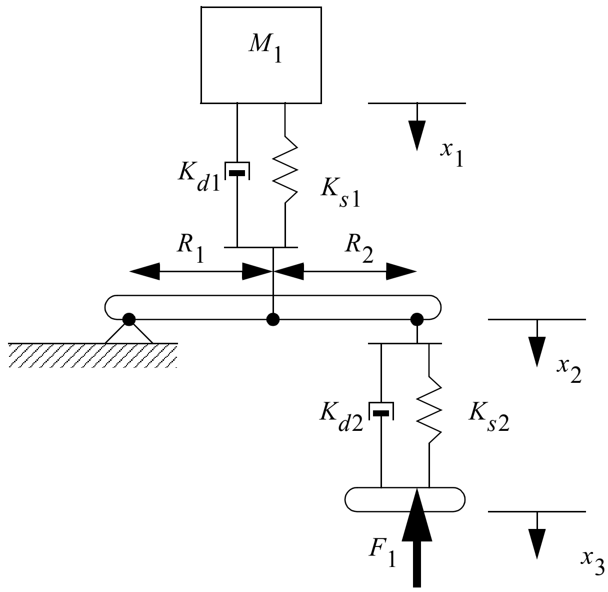


Problem 5.12 A pulley system has the bottom pulley anchored. A mass is hung in the middle of the arrangement with springs and dampers on either side. Assume that the cable is always tight. For the system a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

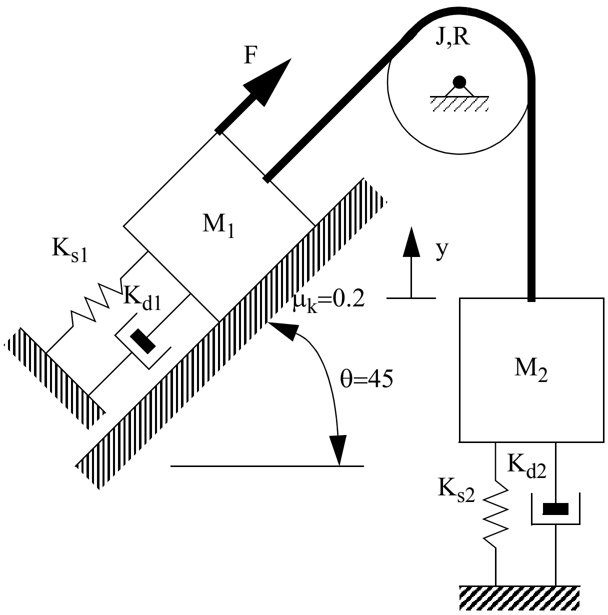


Problem 5.13 A mass is suspended over a lever arm. Forces are applied to the lower side of the moment arm through a spring damper pair. For the system a) write the differential equations (assume small angular deflections) and b) put the

equations in state variable form.



Problem 5.14 Analyze the system pictured below assuming the rope remains tight.



$F = 10N$

$M_1 = 1\text{kg}$ $K_{s1} = 100\frac{N}{m}$

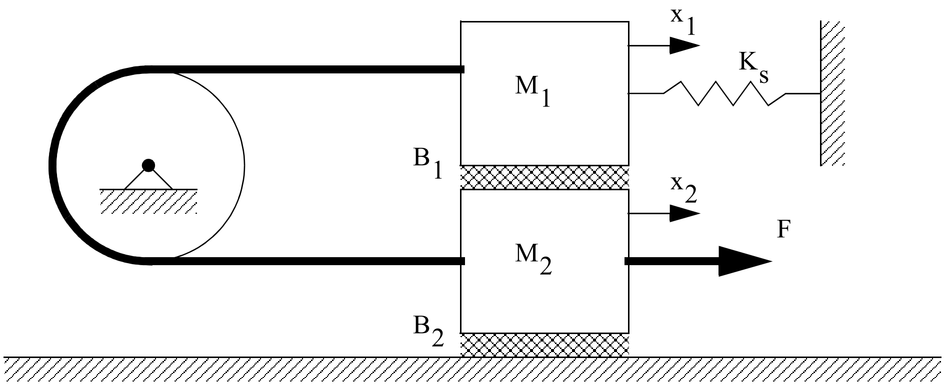
$M_2 = 1\text{kg}$ $K_{s2} = 100\frac{N}{m}$

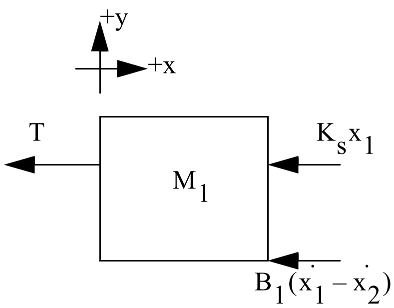
$R = 0.1\text{m}$ $K_{d1} = 50\frac{Ns}{m}$

$J = 10\text{Kgm}^2$ $K_{d2} = 50\frac{Ns}{m}$

a) Draw FBDs and write the differential equations for the individual masses.
b) Write the equations in state variable matrix form.
c) Use Runge-Kutta integration to find the system state after 1 second.

Problem 5.15 The system below is comprised of two masses. There is viscous damping between the masses and between the bottom mass and the floor. The masses are also connected with a cable that is run over a massless and frictionless pulley.



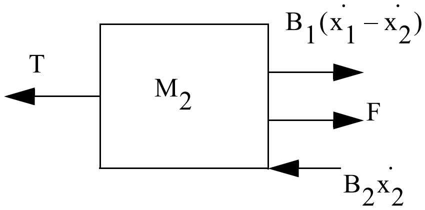


$$\sum F = -T - K_s x_1 - B_1(\dot{x}_1 - \dot{x}_2) = M_1 \ddot{x}_1$$

$$v_1 = \dot{x}_1$$

$$v_2 = \dot{x}_2$$

$$\dot{v}_1 = x_1\left(\frac{-K_s}{M_1}\right) + v_1\left(\frac{-B_1}{M_1}\right) + v_2\left(\frac{B_1}{M_1}\right) + \left(\frac{-T}{M_1}\right)$$

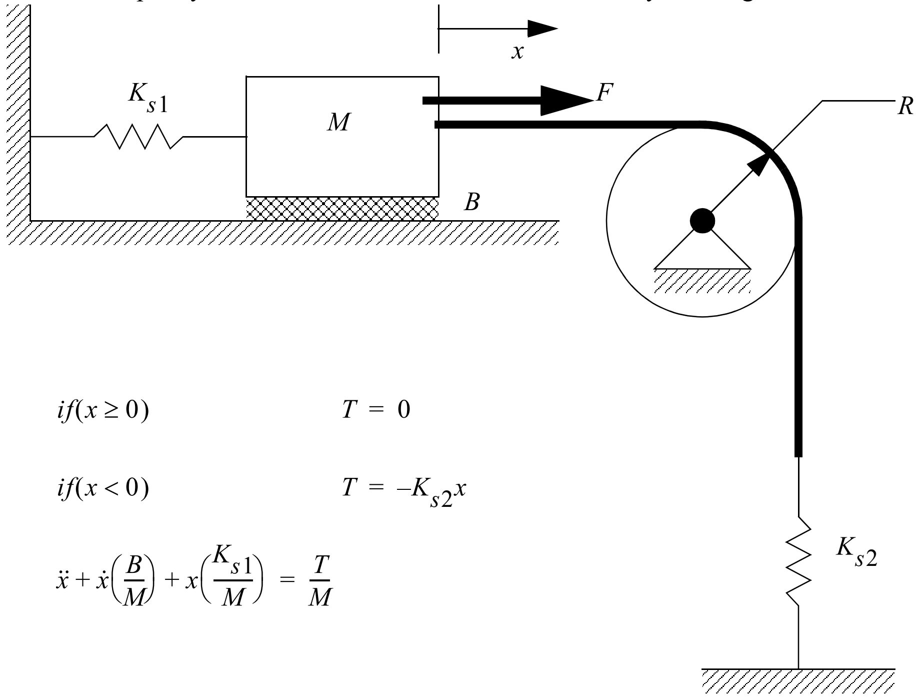


$$\sum F = -T + B_1(\dot{x}_1 - \dot{x}_2) + F - B_2\dot{x}_2 = M_2 \ddot{x}_2$$

$$\dot{v}_2 = v_1\left(\frac{B_1}{M_2}\right) + v_2\left(\frac{-B_1 - B_2}{M_2}\right) + \left(\frac{-T + F}{M_2}\right)$$

a) Write a program to numerically integrate the state equations, assume that the rope always stays solid, even when in compression.
b) Test the program by setting all of the parameters to 1 (i.e. M, Ks, B, F).
c) Calculate the steady state position for part b) by setting derivatives to zero. (Note: the results from steps b and c must agree.)
d) Produce a graph of the system for F = 1 sin (100 t).
e) Modify the program to allow the rope to become slack when in compression.
f) Product a graph using the new program for F = 1 sin (t).

Problem 5.16 Assume that the pulley is massless and frictionless and that the system begins undeflected.



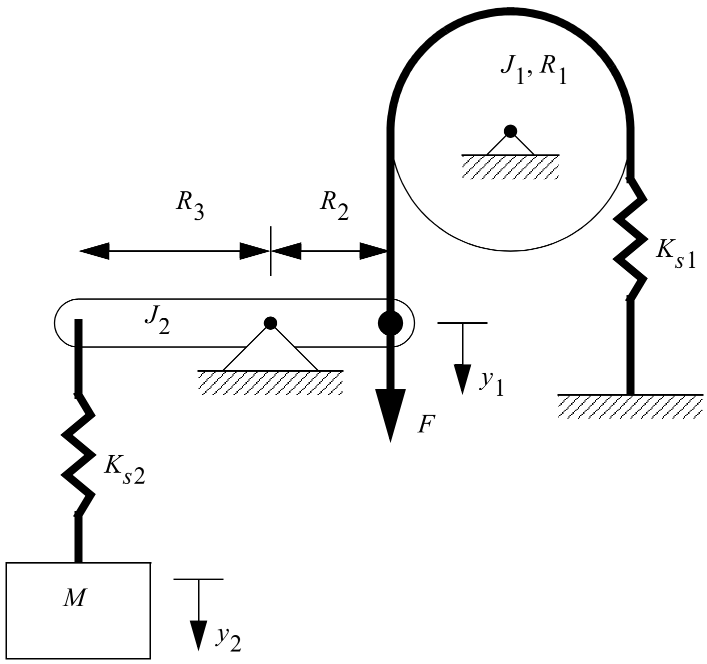
$$if(x \geq 0) \qquad T = 0$$

$$if(x < 0) \qquad T = -K_{s2}x$$

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) + x\left(\frac{K_{s1}}{M}\right) = \frac{T}{M}$$

a) Write a program to numerically integrate the state equations.
b) Test the program by setting all of the parameters to 1 (i.e. M, Ks, B, F).
c) Produce a graph of the system for F = 1 sin (10 t).

Problem 5.17 a) Write the state equations for the following mechanical system. Assume that the cables always remain tight and all deflections are small. For this dynamic case the inertia J2 is significant, but the mass of the beam is not.



b) Find the time response to problem 1 using numerical integration with the values below. You must select a time interval that shows the transient effects. (submit the program in an email and a descriptive graph of the output.)
J1 = 0.001 Kg m^2
J2 = 0.00001 Kg m^2
R1 = 0.20 m
R2 = 0.10 m
R3 = 0.25 m
M = 10Kg

F = 500 N
Ks1 = 1000 N/m
Ks2 = 10000 N/m

Problem 5.18 Analyze the system pictured below assuming the rope remains tight and gravity acts downwards.

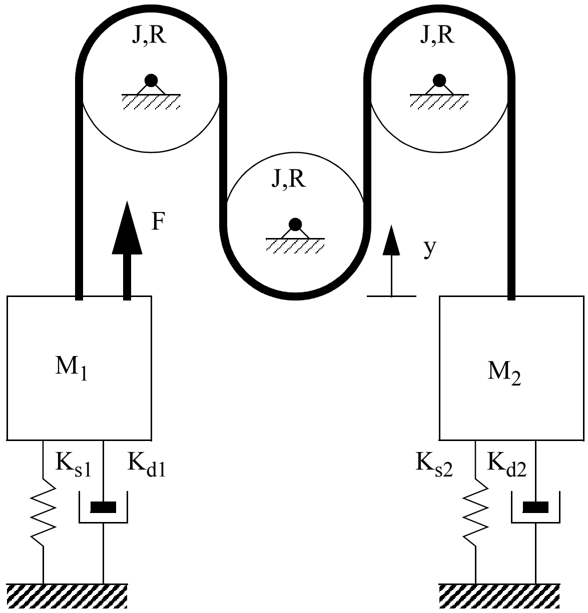
$F = 10N$

$M_1 = 1kg$ $\quad K_{s1} = 100\frac{N}{m}$

$M_2 = 1kg$ $\quad K_{s2} = 100\frac{N}{m}$

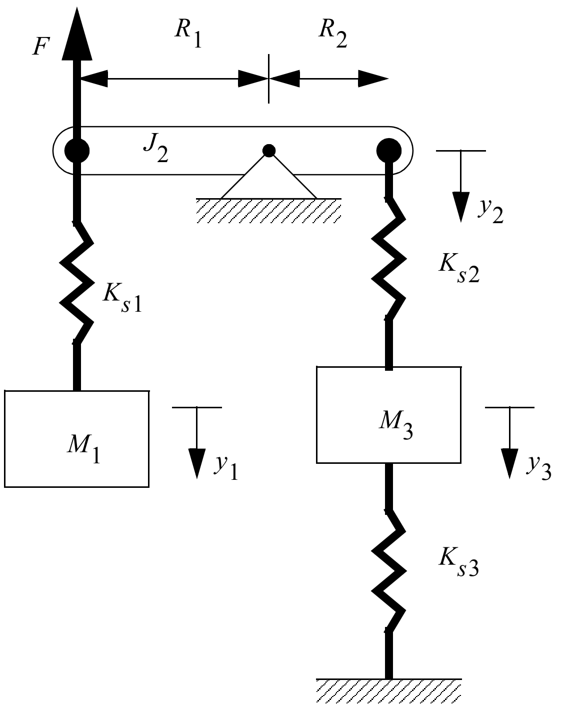
$R = 0.1m$ $\quad K_{d1} = 50\frac{Ns}{m}$

$J = 10Kgm^2$ $\quad K_{d2} = 50\frac{Ns}{m}$



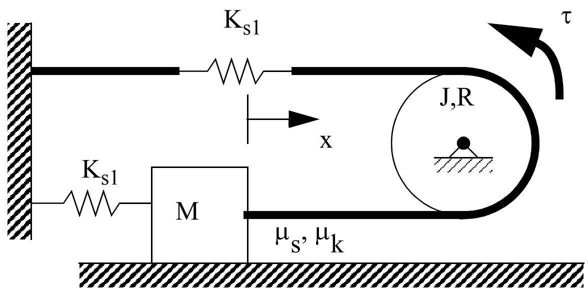
a) Draw FBDs and write the differential equations for the individual masses.
b) Combine the equations and simplify the equations as much as possible.
c) Write the equations in state variable matrix form.
d) Use Runge-Kutta to find the system state after 1 second.

Problem 5.19 Write the state equations for the following mechanical system. Assume that the cables always remain tight and all deflections are small. The moment of inertia value is about the center of rotation.



[[[Note: later add mass to the beam J2]]]

Problem 5.20 Draw FBDs for the following mechanical system. Consider both friction cases.



Problem 5.21 Convert the following state equations to physical systems. Show the method.

$$\dot{x}_1 = v_1$$

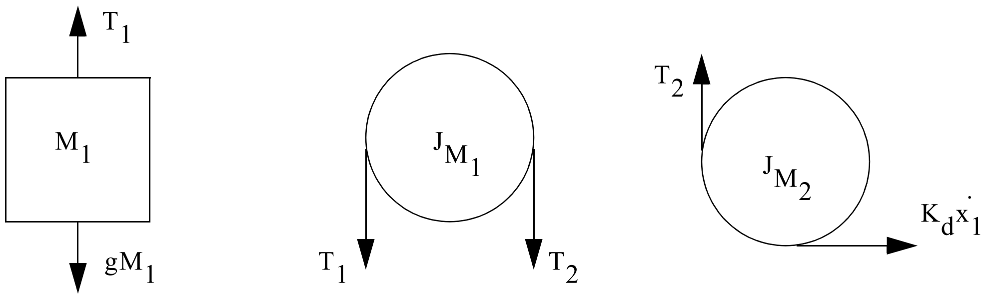
$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + g$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = x_1\left(\frac{-R_2^2 K_{s1}}{J_1}\right) + x_2\left(\frac{R_2^2 K_{s1}}{J_1}\right) + \frac{-FR_1R_2}{J_1}$$

5.3 Problem Solutions

Answer 5.1



if T1,T2,Kdx1 > 0 $\qquad \theta_1 = \frac{-x_1}{R_2} \qquad \theta_2 = \frac{x_1}{R_1} \qquad T_1 = K_s(x_2 - x_1)$

+↑

$$\sum F_y = T_1 - gM_1 = -M_1\ddot{x}_2$$

$$\ddot{x}_2 = g - \frac{T_1}{M_1}$$

+↻

$$\sum M_1 = -T_1R_1 + T_2R_1 = -J_{M_1}\ddot{\theta}_1$$

$$\ddot{\theta}_1 = \frac{T_1R_1 - T_2R_1}{J_{M_1}}$$

+↻

$$\sum M_2 = T_2R_2 - R_2K_d\dot{x}_1 = -J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 + \dot{x}_1\left(\frac{-R_2K_d}{J_{M_2}}\right) = \frac{T_2R_2}{J_{M_2}}$$

6 equations, 6 unknowns

Answer 5.2 $\qquad$ F = 516.3N

Answer 5.3

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + g$$

[[Update to include M2]]

$$\dot{x}_2 = v_2$$

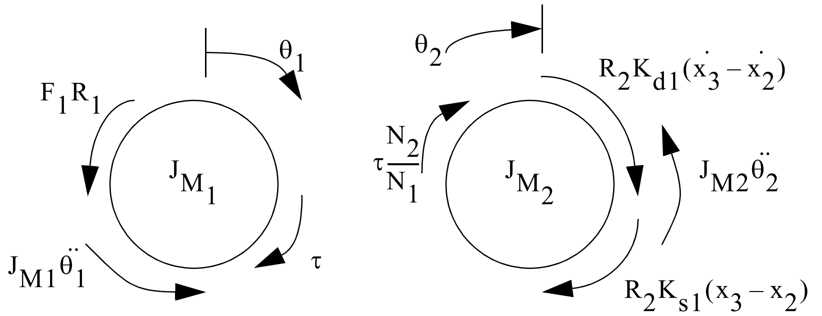
$$\dot{v}_2 = v_2\left(\frac{-R_2^2K_{d1}}{J_2}\right) + x_1\left(\frac{R_2^2K_{s1}}{J_2}\right) + x_2\left(\frac{-R_2^2K_{s1}}{J_2}\right) + \frac{-FR_1R_2}{J_2}$$

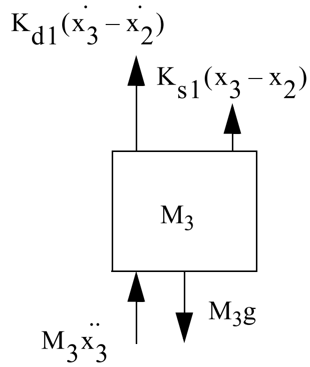
Answer 5.4

Case 1: $\qquad \omega_3 = 2.5\frac{rad}{s}$

Case 2: $\qquad \omega_3 = -2.5\frac{rad}{s}$

Answer 5.5





$$\theta_1 = \frac{-x_1}{R_1} \qquad \theta_2 = \frac{x_2}{R_2}$$

$$\theta_1 N_1 = -\theta_2 N_2 \qquad x_1 = x_2\frac{N_2R_1}{N_1R_2}$$

$$\dot{x}_3 = v_3 \qquad \dot{x}_2 = v_2$$

$$\dot{v}_3 = v_3\left(\frac{-K_{d1}}{M_3}\right) + x_3\left(\frac{-K_{s1}}{M_3}\right) + v_2\left(\frac{K_{d1}}{M_3}\right) + x_2\left(\frac{K_{s1}}{M_3}\right) + g$$

$$-F_1R_1 + \tau - J_{M1}\ddot{\theta}_1 = 0$$

$$\tau = F_1R_1 - \frac{J_{M1}}{R_1}\ddot{x}_1 = F_1R_1 - J_{M1}\frac{N_2}{N_1R_2}\ddot{x}_2$$

$$R_2K_{d1}(\dot{x}_3 - \dot{x}_2) + R_2K_{s1}(x_3 - x_2) - J_{M2}\ddot{\theta}_2 + \tau\frac{N_2}{N_1} = 0$$

$$R_2K_{d1}(\dot{x}_3 - \dot{x}_2) + R_2K_{s1}(x_3 - x_2) - \frac{J_{M2}}{R_2}\ddot{x}_2 + (F_1R_1)\frac{N_2}{N_1} - \left(J_{M1}\frac{N_2}{N_1R_2}\ddot{x}_2\right)\frac{N_2}{N_1} = 0$$

$$\ddot{x}_2\left(\frac{J_{M2}}{R_2} + J_{M1}\frac{N_2^2}{N_1^2R_2}\right) = x_2(-R_2K_{s1}) + v_2(-R_2K_{d1}) + x_3(R_2K_{s1}) + v_3(R_2K_{d1}) - \left(F_1\frac{R_1N_2}{N_1}\right)$$

$$\dot{v}_2 = x_2\left(\frac{-R_2^2N_1^2K_{s1}}{N_1^2J_{M2} + J_{M1}N_2^2}\right) + v_2\left(\frac{-R_2^2N_1^2K_{d1}}{N_1^2J_{M2} + J_{M1}N_2^2}\right) + x_3\left(\frac{R_2^2N_1^2K_{s1}}{N_1^2J_{M2} + J_{M1}N_2^2}\right) +$$

$$+ v_3\left(\frac{R_2^2N_1^2K_{d1}}{N_1^2J_{M2} + J_{M1}N_2^2}\right) + \left(F_1\frac{R_1R_2N_1N_2}{N_1^2J_{M2} + J_{M1}N_2^2}\right)$$

Answer 5.6

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = v_1\left(\frac{-K_{d1}}{M_1}\right) + x_1\left(\frac{-K_{s1}}{M_1}\right) + \omega_1\left(\frac{K_{d1}R_2R_4}{M_1R_3}\right) + \theta_1\left(\frac{K_{s1}R_2R_4}{M_1R_3}\right) + g$$

$$\dot{\theta}_1 = \omega_1$$

$$\dot{\omega}_1 = v_1\left(\frac{K_{d1}R_2R_4}{J_1R_3}\right) + x_1\left(\frac{K_{s1}R_2R_4}{J_1R_3}\right) + \omega_1\left(\frac{-K_{d1}R_2^2R_4^2}{J_1R_3^2}\right) + \theta_1\left(\frac{-K_{s1}R_2^2R_4^2}{J_1R_3^2}\right) + \frac{F_1R_1}{J_1}$$

Answer 5.7

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + \theta_1\left(\frac{R_2 K_{s1} N_1}{M_1 N_2}\right) + g$$

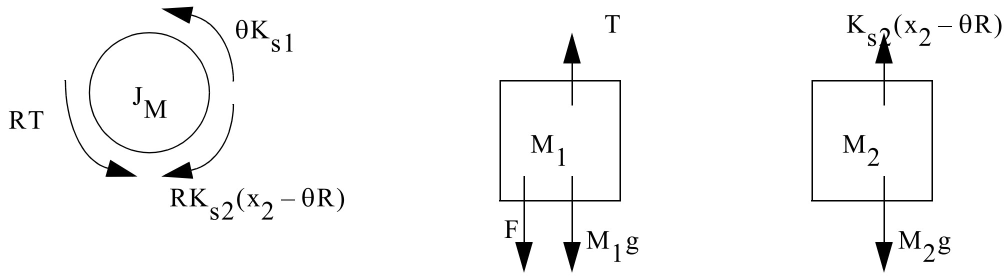
$$\dot{\theta}_1 = \omega_1$$

$$\dot{\omega}_1 = x_1\left(\frac{R_2 K_{s1} N_1 N_2}{J_1 N_2^2 + J_2 N_1^2}\right) + \omega_1\left(\frac{-K_{d1} N_1^2}{J_1 N_2^2 + J_2 N_1^2}\right) + \theta_1\left(\frac{-R_2^2 K_{s1} N_1^2}{J_1 N_2^2 + J_2 N_1^2}\right) + \frac{F_1 R_1 N_2^2}{J_1 N_2^2 + J_2 N_1^2}$$

Answer 5.8

$$\ddot{\theta}_2\left(J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right) + \dot{\theta}_2\left(R_1^2\left(\frac{N_2}{N_1}\right)^2 K_{d1}\right) + \theta_2\left(R_1^2\left(\frac{N_2}{N_1}\right)^2 K_{s1}\right) = \tau - M_1 g R_2$$

Answer 5.9 a)



if(T < 0) T=0

if(T >= 0) $R\theta = -x_1$

$$\sum F_{M1} = T - M_1 g - F = -M_1 \ddot{x}_1$$

$$T = -M_1 \ddot{x}_1 + M_1 g + F = M_1 R\ddot{\theta} + M_1 g + F$$

$$\sum M_J = -RT - \theta K_{s1} + RK_{s2}(x_2 - \theta R) = J_M \ddot{\theta}$$

$$-R(M_1 g + F) - \theta\left(K_{s1} + R^2 K_{s2}\right) + (RK_{s2})x_2 = \left(J_M + R^2 M_1\right)\ddot{\theta}$$

$$\ddot{\theta} + \theta\left(\frac{K_{s1} + R^2 K_{s2}}{J_M + R^2 M_1}\right) + x_2\left(\frac{-RK_{s2}}{J_M + R^2 M_1}\right) = \frac{-R(M_1 g + F)}{J_M + R^2 M_1} \qquad \text{eqn 5.1}$$

$$\sum F_{M2} = K_{s2}(x_2 - \theta R) - M_2 g = -M_2 \ddot{x}_2$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) + \theta\left(\frac{-RK_{s2}}{M_2}\right) = g \qquad \text{eqn 5.2}$$

b)

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \theta\left(\frac{-K_{s1} - R^2 K_{s2}}{J_M + R^2 M_1}\right) + x_2\left(\frac{RK_{s2}}{J_M + R^2 M_1}\right) + \left(\frac{-RM_1 g - RF}{J_M + R^2 M_1}\right)$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \theta\left(\frac{RK_{s2}}{M_2}\right) + x_2\left(\frac{-K_{s2}}{M_2}\right) + g$$

Answer 5.10

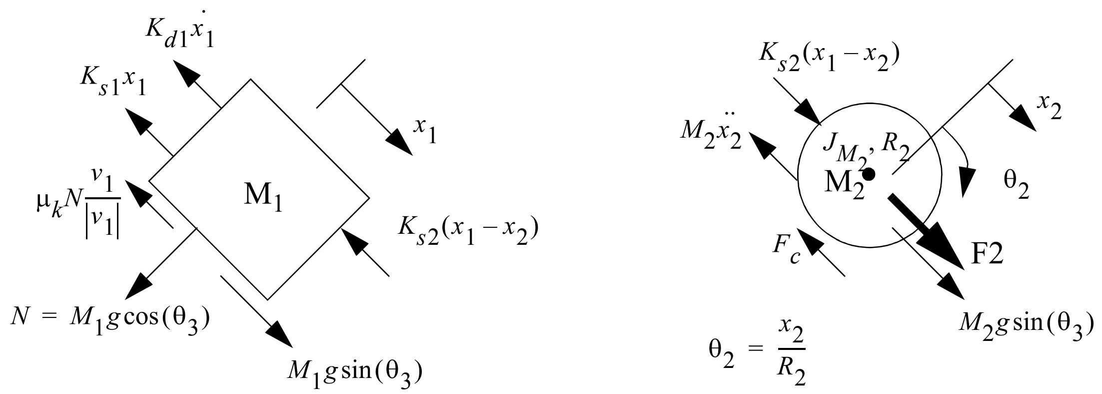
$$\dot{\theta} = \omega$$

$$\dot{\omega} = \theta\left(\frac{-K_{s1} - r^2 K_{s2}}{J_M}\right) + \omega\left(\frac{-K_{d1}}{J_M}\right) + x\left(\frac{K_{s2} r}{J_M}\right) + \frac{Fr}{J_M}$$

$$\dot{x} = v$$

$$\dot{v} = \theta\left(\frac{K_{s2} r}{M}\right) + v\left(\frac{-K_{d2}}{M}\right) + x\left(\frac{-K_{s2} - K_{s3}}{M}\right)$$

Answer 5.11



$$\dot{x}_1 = v_1$$

$$\dot{v}_1 =$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 =$$

Answer 5.12

$$\dot{x}_2 = v_2$$

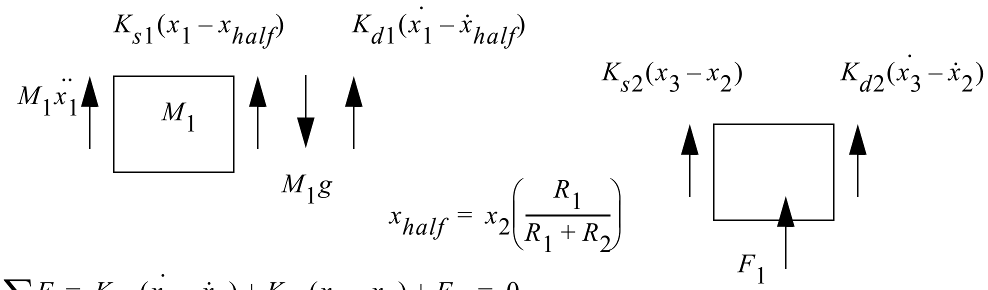
$$\dot{v}_2 = x_3\left(\frac{R_1^2R_2^2(-K_{s2}-K_{s3})}{4J_1R_2^2+J_2R_1^2+R_1^2R_2^2M_2}\right)+x_2\left(\frac{R_1^2R_2^2K_{s2}}{4J_1R_2^2+J_2R_1^2+R_1^2R_2^2M_2}\right)+\frac{R_1^2R_2^2(2F_1+M_2g)}{4J_1R_2^2+J_2R_1^2+R_1^2R_2^2M_2}$$

$$\dot{x}_3 = v_3$$

$$\dot{v}_3 = v_3\left(\frac{-K_{d3}}{M_3}\right)+x_3\left(\frac{-K_{s2}-K_{s3}}{M_3}\right)+x_2\left(\frac{K_{s2}}{M_3}\right)+g$$

Answer 5.13

We are assuming that all of the masses are zero, except M1. Therefor the force F1 is transmitted to mass M1 with no inertial delay. However, the level arm will result in a decrease of displacement proportional to R1 / (R1+R2).



$$x_{half} = x_2\left(\frac{R_1}{R_1 + R_2}\right)$$

$$\sum F = K_{d2}(\dot{x}_3 - \dot{x}_2) + K_{s2}(x_3 - x_2) + F_1 = 0$$

$$\dot{x}_2(-K_{d2}) + x_2(-K_{s2}) + \dot{x}_3(K_{d2}) + x_3(K_{s2}) = -F_1$$

$$\dot{x}_2 - \dot{x}_3 = x_2\left(\frac{-K_{s2}}{K_{d2}}\right) + x_3\left(\frac{K_{s2}}{K_{d2}}\right) + \left(\frac{-F_1}{K_{d2}}\right)$$

$$p = x_2 - x_3$$

$$\dot{p} = x_2\left(\frac{-K_{s2}}{K_{d2}}\right) + x_3\left(\frac{K_{s2}}{K_{d2}}\right) + \left(\frac{-F_1}{K_{d2}}\right) = -\frac{K_{s2}}{K_{d2}}p + \left(\frac{-F_1}{K_{d2}}\right)$$

$$\sum M = K_{s1}R_1(x_1 - x_{half}) + K_{d1}R_1(\dot{x}_1 - \dot{x}_{half}) + K_{s2}(R_1 + R_2)(x_3 - x_2) + K_{d2}(R_1 + R_2)(\dot{x}_3 - \dot{x}_2) = 0$$

$$K_{s1}\frac{R_1}{(R_1 + R_2)}\left(x_1 - x_2\left(\frac{R_1}{R_1 + R_2}\right)\right) + K_{d1}\frac{R_1}{(R_1 + R_2)}\left(\dot{x}_1 - \dot{x}_2\left(\frac{R_1}{R_1 + R_2}\right)\right) + K_{s2}(-p) + K_{d2}(-\dot{p}) = 0$$

$$q = \left(x_1 - x_2\left(\frac{R_1}{R_1 + R_2}\right)\right)\frac{R_1}{(R_1 + R_2)} \qquad x_1 - x_2\left(\frac{R_1}{R_1 + R_2}\right) = q\left(\frac{(R_1 + R_2)}{R_1}\right)$$

$$K_{s1}(q) + K_{d1}(\dot{q}) + K_{s2}(-p) + K_{d2}(-\dot{p}) = 0$$

$$K_{d1}(\dot{q}) + K_{d2}(-\dot{p}) = K_{s1}(-q) + K_{s2}(p)$$

$$\sum F = M_1\ddot{x}_1 + K_{d1}(\dot{x}_1 - \dot{x}_{half}) + K_{s1}(x_1 - x_{half}) - M_1 g = 0$$

$$M_1\ddot{x}_1 + \dot{x}_1(K_{d1}) + x_1(K_{s1}) + \dot{x}_{half}(-K_{d1}) + x_{half}(-K_{s1}) = M_1 g$$

$$M_1\ddot{x}_1 + \dot{x}_1(K_{d1}) + x_1(K_{s1}) + \dot{x}_2\left(\frac{R_1}{R_1 + R_2}\right)(-K_{d1}) + x_2\left(\frac{R_1}{R_1 + R_2}\right)(-K_{s1}) = M_1 g$$

$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + q\left(\frac{(R_1 + R_2)}{M_1 R_1}\right)(K_{d1}) + x_2\left(\frac{R_1}{R_1 + R_2}\right)\left(\frac{K_{s1}}{M_1}\right) + g$$

$$\dot{p} = -\frac{K_{s2}}{K_{d2}}p + \left(\frac{-F_1}{K_{d2}}\right)$$

LOOK AT LATER ------- XXXXXXXXXXXXXXX - SEE NEXT FIGURE FOR ALTERNATE SOLUTION

We are assuming that all of the masses are zero, except M1. Therefor the force F1 is transmitted to mass M1 with no inertial delay. However, the level arm will result in a decrease of displacement proportional to R1 / (R1+R2).

$$M_1\ddot{x}_1 + F_1\left(\frac{R_1 + R_2}{R_1}\right) = 0$$

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = F_1\left(\frac{R_1 + R_2}{R_1 M_1}\right)$$

Although the force is immediately transmitted to M1, the x2 and x3 positions of the other components will not change immediately. To find these positions we begin by summing the forces on Ks1 and Kd1.

$$R_1 K_{s1}\left(x_1 - \frac{R_1}{R_1 + R_2}x_2\right) + R_1 K_{d1}\left(\dot{x}_1 - \frac{R_1}{R_1 + R_2}\dot{x}_2\right) + R_1 F_1\left(\frac{R_1 + R_2}{R_1}\right) = 0$$

$$K_{d1}\left(\frac{R_1}{R_1 + R_2}\dot{x}_2\right) = x_1(K_{s1}) + v_1(K_{d1}) + x_2\left(\frac{-K_{s1}R_1}{R_1 + R_2}\right) + F_1\left(\frac{R_1 + R_2}{R_1}\right)$$

$$\dot{x}_2 = x_1\left(\frac{K_{s1}(R_1 + R_2)}{R_1 K_{d1}}\right) + v_1\left(\frac{R_1 + R_2}{R_1}\right) + x_2\left(\frac{-K_{s1}R_1(R_1 + R_2)}{(R_1 + R_2)R_1 K_{d1}}\right) + F_1\left(\frac{(R_1 + R_2)^2}{R_1^2 K_{d1}}\right)$$

Finally the forces can be summed on the right hand spring and damper.

$$K_{s2}(x_2 - x_3) + K_{d2}(\dot{x}_2 - \dot{x}_3) + F_1 = 0$$

$$\dot{x}_3 = x_2\left(\frac{K_{s2}}{K_{d2}}\right) + x_3\left(\frac{-K_{s2}}{K_{d2}}\right) + v_2 + F_1$$

[[Review for correctness and get rid of x2 and x3 in state equations]]

$$\dot{x}_1 = v_1$$

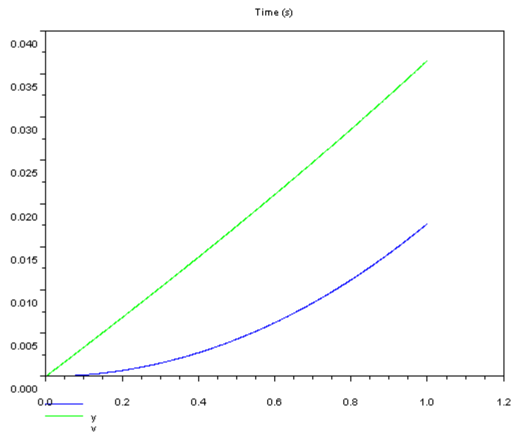
$$\dot{v}_1 = (-F_1)\left(\frac{R_1 + R_2}{M_1 R_1}\right) + g$$

$$\dot{q} = \frac{-K_{s2}}{K_{d2}}q + \frac{F_1}{K_{d2}} \qquad x_2 = (x_1 - p)\left(\frac{R_1 + R_2}{R_1}\right)$$

$$\dot{p} = \frac{-K_{s1}}{K_{d1}}p + F_1\left(\frac{R_1 + R_2}{K_{d1}R_1}\right) \qquad x_3 = (x_1 - p)\left(\frac{R_1 + R_2}{R_1}\right) - q$$

Answer 5.14

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{K_{s1} + K_{s2}}{M_1 + M_2 + \dfrac{J}{R^2}} & \dfrac{K_{d1} + K_{d2}}{M_1 + M_2 + \dfrac{J}{R^2}} \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{M_1 g\left(\mu_k \cos\theta \frac{y}{|y|} + \sin\theta\right) + F + M_2 g}{M_1 + M_2 + \dfrac{J}{R^2}} \end{bmatrix}$$


Time (s)
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0.0
0.2
0.4
0.6
0.8
1.0
1.2
y
v

```
// An integration for ch5 pp16

Ks1 = 100; // System component values
Ks2 = 100;
Kd1 = 50;
Kd2 = 50;
M1 = 1;
M2 = 1;
F = 10;
R = 0.1;
J = 10;
g = 9.81;
theta = %pi / 4; // 45 degrees

y = 0;         // initial conditions
v = 0;

X=[y, v];
tmp = ((J / R^2) + M1 + M2); // this is used a few times, may as well reduce calcs
A = [0, 1 ; (Ks1 + Ks2) / tmp, (Kd1 + Kd2) / tmp]; // define the A matrix
B = [0, 0; (g * (M1 * sin(theta) + M2) + F) / tmp, (M1 * g * cos(theta)) / tmp]; // B matrix

// the state matrix function
function foo=f(state,t)
 y = state($, 1);
 v = state($, 2);
       tmp = A * [y; v] + B * [1; sign(y)];
       foo = tmp'; // do a transpose to turn it into a row
endfunction

// Set the time length and step size for the integration
steps = 1000;
t_start = 0.0;
t_end = 1.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       F2 = h * f(X($,:) + F1/2.0, t($,:) + h/2.0);
       F3 = h * f(X($,:) + F2/2.0, t($,:) + h/2.0);
       F4 = h * f(X($,:) + F3, t($,:) + h);
       X = [X ; X($,:) + (F1 + 2.0*F2 + 2.0*F3 + F4)/6.0];
end

// print some results to compare
print_steps = 10;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  time = j * h + t_start;
  printf("The value at t=%f s is (y, v) = (%f, %f)\n", time, X(j+1,1), X(j+1,2));
end

// Graph the values
plot2d(t, X, [2, 3], leg="y@v");
xtitle('Time (s)');

// SCILAB OUTPUT
//The value at t=0.000000 s is (y, v) = (0.000000, 0.000000)
//The value at t=0.100000 s is (y, v) = (0.000168, 0.003376)
//The value at t=0.200000 s is (y, v) = (0.000677, 0.006796)
//The value at t=0.300000 s is (y, v) = (0.001529, 0.010264)
//The value at t=0.400000 s is (y, v) = (0.002731, 0.013788)
//The value at t=0.500000 s is (y, v) = (0.004289, 0.017374)
//The value at t=0.600000 s is (y, v) = (0.006209, 0.021032)
//The value at t=0.700000 s is (y, v) = (0.008498, 0.024768)
```

Answer 5.15 a)

$$T_1 = x_1(-K_s) + v_1(-B_1) + v_2(B_1) - (M_1)\dot{v}_1$$

$$T_2 = v_1(B_1) + v_2(-B_1 - B_2) + (F) - (M_2)\dot{v}_2$$

$$x \equiv x_1 = -x_2$$

$$x(-K_s) + v(-B_1) + v(-B_1) - (M_1)\dot{v} = v(B_1) + v(B_1 + B_2) + (F) + (M_2)\dot{v}$$

$$(M_1 + M_2)\dot{v} = x(-K_s) + v(-4B_1 - B_2) + (-F)$$

$$\dot{v} = x\left(\frac{-K_s}{M_1 + M_2}\right) + v\left(\frac{-4B_1 - B_2}{M_1 + M_2}\right) + \left(\frac{-F}{M_1 + M_2}\right)$$

```
M1 = 1; M2 = 1; Ks = 1; B1 = 1; B2 = 1; F = 1

x1_0 = 0; v1_0 = 0; x2_0 = 0; v2_0 = 0
X = [x1 0, v1_0, x2_0, v2_0]
t_start = 0; t_end = 10; n_steps = 100
h = (t_end - t_start) / n_steps
t = [t_start]

rope_tight = 1

function der = state_equations(X, t)
        x1 = X(1)
        v1 = X(2)
        x2 = X(3)
        v2 = X(4)
    if (rope_tight == 1) then
        dx1 = v1
        dv1 = x1*(-Ks)/(M1+M2) + v1*(-4*B1-B2)/(M1+M2) + (-F)/(M1+M2)
        dx2 = dx1
        dv2 = dv1
    else
        dx1 = v1
        dx2 = v2
            dv1 = -x1*Ks/M1 - v1*B1/M1 + v2*B1/M1 -T/M1
            dv2 = v1*B1/M2 + v2*(-B1-B2)/M2 + (-T+F)/M2
    end
        der = [dx1, dv1, dx2, dv2]

endfunction

for i = 1:n_steps
        X = [X; X($, :) + h*state_equations(X($, :), t($))]
    if  rope_tight == 1 then
        X($, 3) = X($, 1) // set x2 and v2 equal to x1 and v1
        X($, 4) = X($, 2)
    end
        t = [t; i*h]
end

plot2d(t, X)
```

b) [[Additional solution details needed]]

c)

$$v_1 = \dot{x}_1 = 0$$

$$v_2 = \dot{x}_2 = 0$$

$$\dot{v}_1 = x_1\left(\frac{-K_s}{M_1}\right) + v_1\left(\frac{-B_1}{M_1}\right) + v_2\left(\frac{B_1}{M_1}\right) + \left(\frac{-T}{M_1}\right)$$

$$0 = x_1\left(\frac{-1}{1}\right) + 0\left(\frac{-1}{1}\right) + 0\left(\frac{1}{1}\right) + \left(\frac{-T}{1}\right) = x_1 - T$$

$$\dot{v}_2 = v_1\left(\frac{B_1}{M_2}\right) + v_2\left(\frac{-B_1 - B_2}{M_2}\right) + \left(\frac{-T + F}{M_2}\right)$$

$$0 = 0\left(\frac{1}{1}\right) + 0\left(\frac{-2}{1}\right) + \left(\frac{-T + 1}{1}\right) = -T + 1$$

$$x_1 = -1 \qquad x_2 = 1$$

d)

e)

$$v_1 = \dot{x}_1$$

$$v_2 = \dot{x}_2$$

$$\dot{v}_1 = x_1\left(\frac{-K_s}{M_1}\right) + v_1\left(\frac{-B_1}{M_1}\right) + v_2\left(\frac{B_1}{M_1}\right) + \left(\frac{-T}{M_1}\right)$$

$$\dot{v}_2 = v_1\left(\frac{B_1}{M_2}\right) + v_2\left(\frac{-B_1 - B_2}{M_2}\right) + \left(\frac{-T+F}{M_2}\right)$$

```
M1 = 1; M2 = 1; Ks = 1; B1 = 1; B2 = 1; F = 1
x1_0 = 0; v1_0 = 0; x2_0 = 0; v2_0 = 0
X = [x1_0, v1_0, x2_0, v2_0]
t_start = 0; t_end = 2; n_steps = 1000
h = (t_end - t_start) / n_steps
t = [t_start]
tight = 1 // start with the cable tight
function der = state_equations(X, t)
        x1 = X(1)
        v1 = X(2)
        x2 = X(3)
        v2 = X(4)
  F = sin(100*t($))
  if tight == 0 then // see if it should be tight
    if (x1 + x2) >= 0 then
      tight = 1
      x1 = (x1 + x2) / 2
      x2 = -x1
      tight = 1
    end
  else // see if it should be loose
    if (v1 + v2) < 0 then // they are moving towards each other
      tight = 0
    else // watch for numerical drift
      x1 = (x1 + x2) / 2
      x2 = -x1
    end
  end
  if  tight == 0 then
    dx1 = v1
    dx2 = v2
    dv1 = -x1*Ks/M1 - v1*B1/M1 + v2*B1/M1
    dv2 = v1*B1/M2 + v2*(-B1-B2)/M2 + (F/M2)
  else
    dx1 = v1
    dv1 = x1*(-Ks)/(M1+M2) + v1*(-4*B1-B2)/(M1+M2) + (-F)/(M1+M2)
    dx2 = - dx1
    dv2 = - dv1
  end

  if tight == 1 then
    T1 = x1*(-Ks) + v1*(-B1) + v2*(B1) + dv1*(-M1)
    T2 = v1*(B1) + v2*(-B1 - B2) + F + dv2*(-M2)
  end
        der = [dx1, dv1, dx2, dv2]
endfunction
for i = 1:n_steps
        X = [X; X($, :) + h*state_equations(X($, :))]
        t = [t; i*h]
end
plot2d(t, X)
```

f)

Answer 5.16 a)

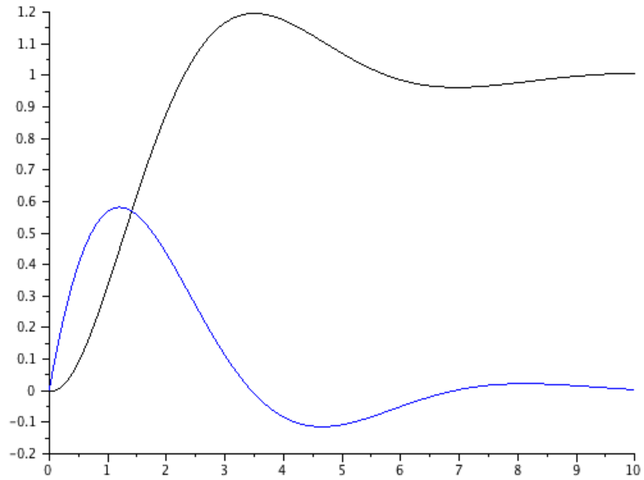
$$if(x \geq 0) \qquad T = 0$$

$$if(x < 0) \qquad T = -K_{s2}x$$

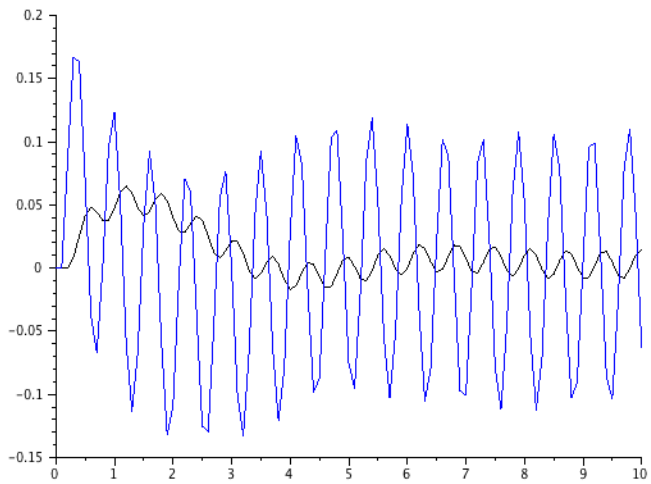
$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) + x\left(\frac{K_{s1}}{M}\right) = \frac{T}{M} + \frac{F}{M}$$

```
M = 1; Ks = 1; B = 1; Ks1 = 1; Ks2 = 2; F = 1

x_0 = 0; v_0 = 0
X = [x_0, v_0]
t_start = 0; t_end = 10; n_steps = 100
h = (t_end - t_start) / n_steps
t = [t_start]

function der = state_equations(X, t)
        x = X(1)
        v = X(2)
        // F = sin(10 * t($)) // Note: uncomment for part c
        if x > 0 then
          T = 0
        else
          T = -Ks2 * x
        end
        dx = v
        dv = x*(-Ks1/M) + v*(-B/M) + T/M + F/M
        der = [dx, dv]
endfunction

for i = 1:n_steps
        X = [X; X($, :) + h*state_equations(X($, :))]
        t = [t; i*h]
end

plot2d(t, X)
```

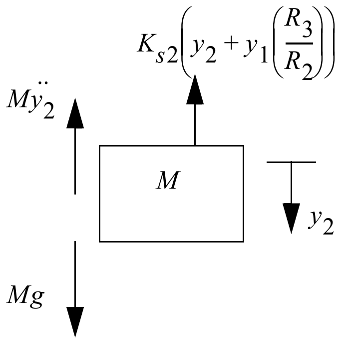
b)


1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
1
2
3
4
5
6
7
8
9
10

c)


0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
1
2
3
4
5
6
7
8
9
10

Answer 5.17 a)

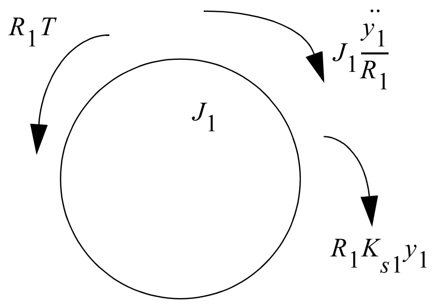


$$\dot{y}_1 = v_1$$

$$\dot{y}_2 = v_2$$

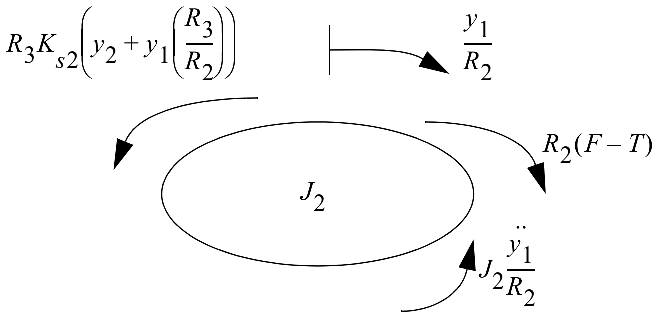
$$M\ddot{y}_2 = -K_{s2}\left(y_2 + y_1\left(\frac{R_3}{R_2}\right)\right) + Mg$$

$$\dot{v}_2 = y_1\left(\frac{-K_{s2}}{M}\frac{R_3}{R_2}\right) + y_2\left(\frac{-K_{s2}}{M}\right) + (g)$$



$$J_1\frac{\ddot{y}_1}{R_1} + R_1K_{s1}y_1 = R_1T$$

$$T = \frac{J_1}{R_1^2}\ddot{y}_1 + K_{s1}y_1$$



$$-R_3K_{s2}\left(y_2 + y_1\left(\frac{R_3}{R_2}\right)\right) + R_2(F-T) - J_2\frac{\ddot{y}_1}{R_2} = 0$$

$$J_2\frac{\ddot{y}_1}{R_2} = -R_3K_{s2}\left(y_2 + y_1\left(\frac{R_3}{R_2}\right)\right) + R_2F - R_2\left(\frac{J_1}{R_1^2}\ddot{y}_1 + K_{s1}y_1\right)$$

$$\ddot{y}_1\left(\frac{J_2}{R_2} + J_1\frac{R_2}{R_1^2}\right) = y_1\left(\frac{-R_3^2K_{s2}}{R_2} - R_2K_{s1}\right) + y_2(-R_3K_{s2}) + (R_2F)$$

$$\ddot{y}_1\left(\frac{J_2R_1^2 + J_1R_2^2}{R_2R_1^2}\right) = y_1\left(\frac{-R_3^2K_{s2} - R_2^2K_{s1}}{R_2}\right) + y_2(-R_3K_{s2}) + (R_2F)$$

$$\ddot{y}_1 = y_1R_1^2\left(\frac{-R_3^2K_{s2} - R_2^2K_{s1}}{J_2R_1^2 + J_1R_2^2}\right) + y_2\left(\frac{-R_3R_2R_1^2K_{s2}}{J_2R_1^2 + J_1R_2^2}\right) + F\left(\frac{R_1^2R_2{}^2}{J_2R_1^2 + J_1R_2^2}\right)$$

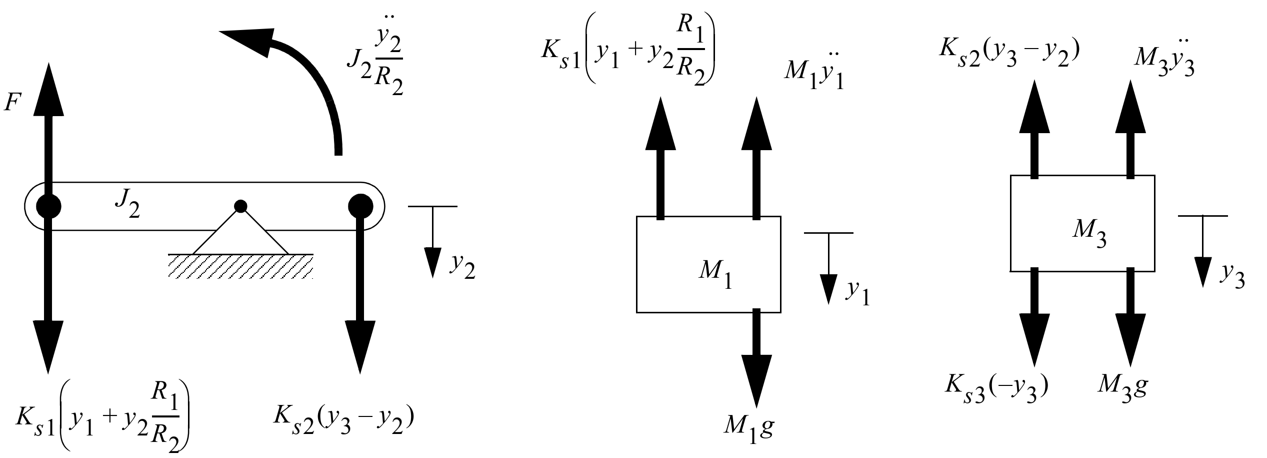
b)

```
// Problem solution

// Constants
J1 = 0.1
J2 = 0.001
R1 = 0.20
R2 = 0.10
R3 = 0.25
M = 10
F = 500
Ks1 = 1000
Ks2 = 10000
g = 9.81

t_start = 0; t_end = 0.1; n_steps = 10000
h = (t_end - t_start) / n_steps
t = [t_start]

y1_0 = 0; v1_0 = 0; y2_0 = 0; v2_0 = 0
X = [y1_0, v1_0, y2_0, v2_0]

A = -Ks2 * R3 / (M * R2)
B = -Ks2 / M
DEN = (J2 * R1 * R1) + (J1 * R2 * R2)
C = R1 * R1 * ((-R3 * R3 * Ks2 - R2 * R2 * Ks1)) / DEN
D = (-R3 * R2 * R1 * R1 * Ks2) / DEN
E = R1 * R1 * R2 * R2 / DEN

function X_diff = state_eqns(X, t)
    y1 = X($, 1)
    v1 = X($, 2)
    y2 = X($, 3)
    v2 = X($, 4)
    dy1 = v1
    dv1 = y1 * A + y2 * B + g
    dy2 = v2
    dv2 = y1 * C + y2 * D + F * E
    X_diff = [dy1, dv1, dy2, dv2]
endfunction

for i = 1: n_steps
    X = [X ; X($, :) + h * state_eqns(X($, :), t($))]
    t = [t ; t($)+h]
end

plot(t, X)
```

Answer 5.18

```
F = 10;
M1 = 1;
M2 = 1;
R = 0.1;
J = 10;
Ks1 = 100;
Ks2 = 100;
Kd1 = 50;
Kd2 = 50;
g = 9.81;

y = 0; // set initial conditions to zero
v = 0;

X = [y, v];

// the state matrix function
function foo=f(state,t)
       foo = [state($, 2), (state($, 2) * (Kd1 + Kd2) + state($, 1) * (Ks1 + Ks2) + (F + M2*g - M1*g))/(-M1 -
   M2 - (3 * J / (R * R)))];
endfunction

// Set the time length and step size for the integration
steps = 100;
t_start = 0.0;
t_end = 1.0;
h = (t_end - t_start) / steps;
t = [t_start];

// Loop for integration
for i=1:steps,
       t = [t ; t($,:) + h];
       F1 = h * f(X($,:), t($,:));
       F2 = h * f(X($, :) + F1 / 2.0, t($, :) + h / 2.0);
       F3 = h * f(X($, :) + F2 / 2.0, t($, :) + h / 2.0);
       F4 = h * f(X($, :) + F3, t($, :) + h);
       X = [X ; X($,:) + (F1 + 2.0 * (F2 + F3) + F4) / 6.0];
end

// print some results to compare
print_steps = 10;
for i = 0:print_steps;
  j = i * (steps / print_steps);
  // time = j * h + t_start;
  time = t(j + 1);
  printf("The value at t=%f s is (y, v) = (%f, %f)\n", time, X(j+1,1), X(j+1,2));
end

// Graph the values
plot2d(t, X, [2, 3], leg="x@v");
xtitle('Time (s)');

// SCILAB OUTPUT
// The value at t=0.000000 s is (y, v) = (0.000000, 0.000000)
// The value at t=0.100000 s is (y, v) = (-0.000017, -0.000333)
// The value at t=0.200000 s is (y, v) = (-0.000066, -0.000664)
// The value at t=0.300000 s is (y, v) = (-0.000149, -0.000993)
// The value at t=0.400000 s is (y, v) = (-0.000265, -0.001321)
// The value at t=0.500000 s is (y, v) = (-0.000414, -0.001647)
// The value at t=0.600000 s is (y, v) = (-0.000594, -0.001971)
// The value at t=0.700000 s is (y, v) = (-0.000808, -0.002292)
// The value at t=0.800000 s is (y, v) = (-0.001053, -0.002611)
// The value at t=0.900000 s is (y, v) = (-0.001330, -0.002927)
// The value at t=1.000000 s is (y, v) = (-0.001638, -0.003240)
```

Answer 5.19



$$\dot{y}_1 = v_1$$

$$\dot{y}_2 = v_2$$

$$\dot{y}_3 = v_3$$

$$R_1F - R_1K_{s1}\left(y_1 + y_2\frac{R_1}{R_2}\right) - J_2\frac{\ddot{y}_2}{R_2} + R_1K_{s2}(y_3 - y_2) = 0$$

$$J_2\frac{\ddot{y}_2}{R_2} = y_1(-R_1K_{s1}) + y_2\left(-R_1K_{s2} - K_{s1}\frac{R_1^2}{R_2}\right) + y_3(R_1K_{s2}) + (R_1F)$$

$$\dot{v}_2 = y_1\left(\frac{-R_1R_2K_{s1}}{J_2}\right) + y_2\left(\frac{-R_1R_2K_{s2} - K_{s1}R_1^2}{J_2}\right) + y_3\left(\frac{R_1R_2K_{s2}}{J_2}\right) + \left(\frac{R_1R_2}{J_2}F\right)$$

$$K_{s1}\left(y_1 + y_2\frac{R_1}{R_2}\right) + M_1\ddot{y}_1 - M_1g = 0$$

$$M_1\ddot{y}_1 = -K_{s1}\left(y_1 + y_2\frac{R_1}{R_2}\right) + M_1g$$

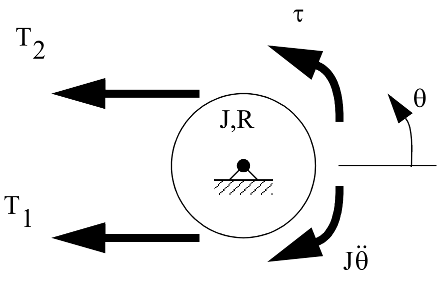
$$\dot{v}_1 = y_1\left(\frac{-K_{s1}}{M_1}\right) + y_2\left(\frac{-K_{s1}R_1}{M_1R_2}\right) + g$$

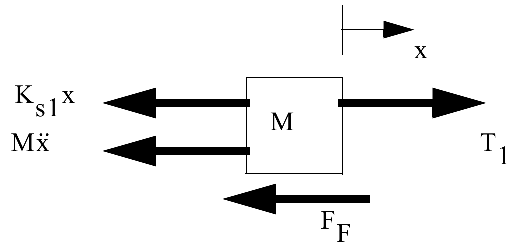
$$K_{s2}(y_3 - y_2) + M_3\ddot{y}_3 - K_{s3}(-y_3) - M_3g = 0$$

$$M_3\ddot{y}_3 = y_2(K_{s2}) + y_3(-K_{s3} - K_{s2}) + M_3g$$

$$\dot{v}_3 = y_2\left(\frac{K_{s2}}{M_3}\right) + y_3\left(\frac{-K_{s3} - K_{s2}}{M_3}\right) + g$$

Answer 5.20





if v <> 0 then

$$F_F = \frac{v}{|v|}\mu_k M g$$

if v == 0 then

$$|F_F| = \mu_s M g$$

if theta > 0 then

$$T_2 = 0$$

if theta <= 0 then

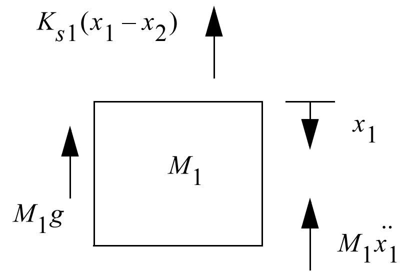
$$T_2 = -K_{s1} R\theta$$

Answer 5.21

$$\ddot{x}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + g$$

$$M_1\ddot{x}_1 + x_1(K_{s1}) + x_2(-K_{s1}) = M_1 g$$

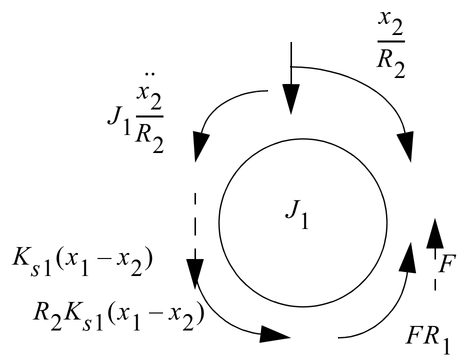
$$M_1\ddot{x}_1 + K_{s1}(x_1 - x_2) - M_1 g = 0$$



$$\ddot{x}_2 = x_1\left(\frac{-R_2^2 K_{s1}}{J_1}\right) + x_2\left(\frac{R_2^2 K_{s1}}{J_1}\right) + \frac{-FR_1R_2}{J_1}$$

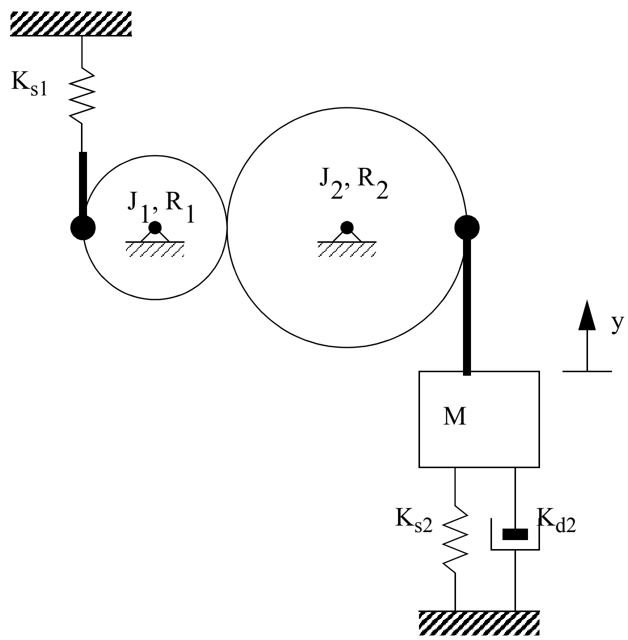
$$J_1\ddot{x}_2 + x_1(R_2^2 K_{s1}) + x_2(-R_2^2 K_{s1}) = -FR_1R_2$$

$$J_1\frac{\ddot{x}_2}{R_2} + R_2K_{s1}(x_1 - x_2) + FR_1 = 0$$



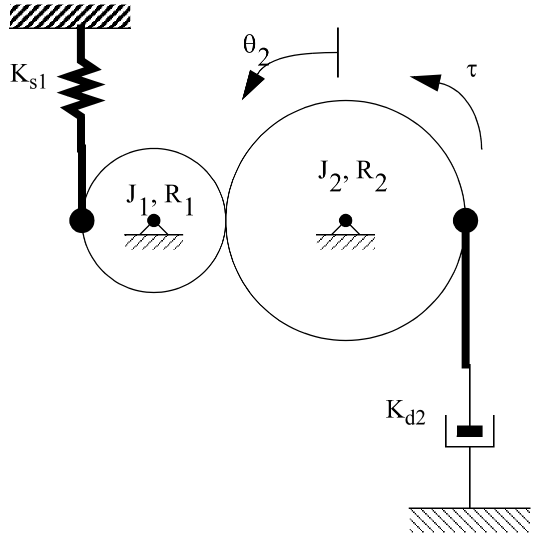
5.1 Problems Without Solutions

Problem 5.22 Draw FBDs for the following mechanical system containing two gears.

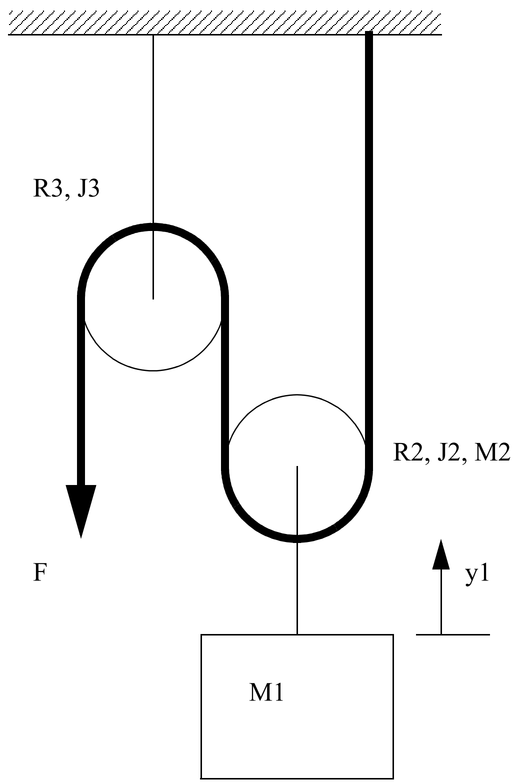


Problem 5.23 Write the state equation(s) for the following mechanical system of two gears. Assume that the cables always

remain tight and all deflections are small.

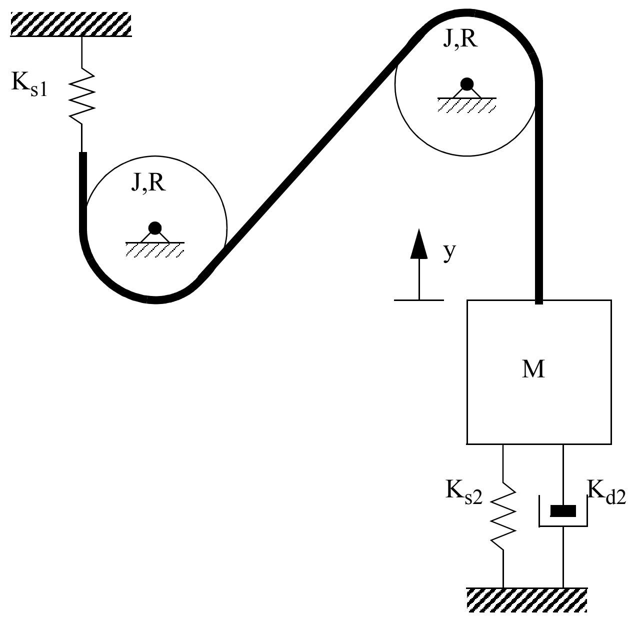


Problem 5.24 Develop state equation(s) for the following system. Assume the rope remains tight.

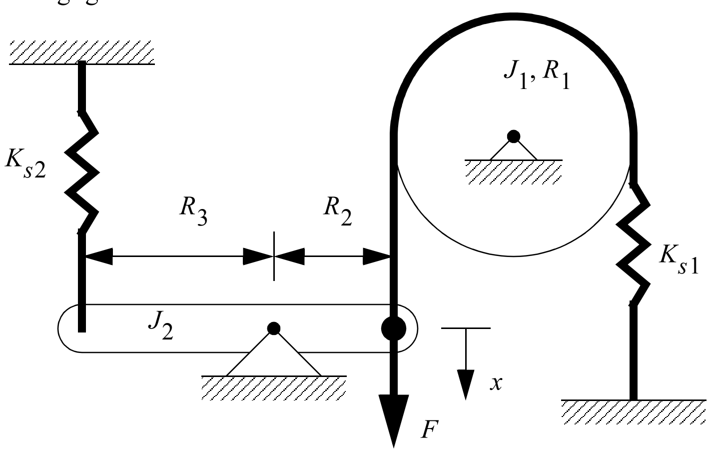


Problem 5.25 a) Draw FBDs for the following mechanical system. b) Develop a differential equation of motion for the system

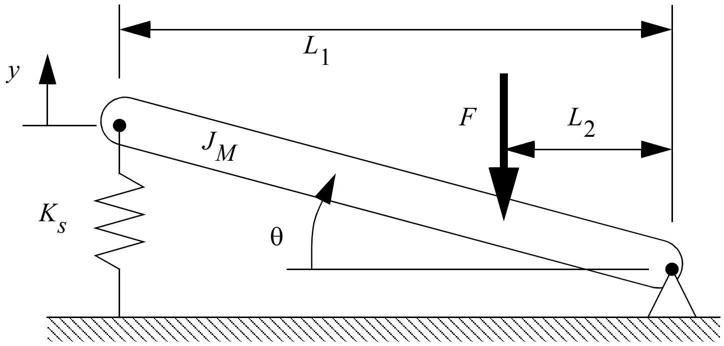
below assuming that the cable always remains tight.



Problem 5.26 Write the state equations for the following mechanical system. Assume that the cables always remain tight and all deflections are small. Both moment of inertia values are about the pivot points. The mass of the beam should be considered negligible.



Problem 5.27 A simplified model of a cam shaft driven lever is shown below. The moment of inertia is about the pivot point.



[[[Note: add mass to the beam in the future]]]

a) Determine the equation of motion (differential equation) for the system as a function of theta.
b) Assume that L1 = 10.0cm and L2 = 4.0cm. Given an applied force of F=400N resulting in a deflection of y=-1.0cm, calculate the spring coefficient Ks.
c) Explicitly solve the differential equation to find theta as a function of time. The result should be left variable form.
d) Provide the system model as a state variable matrix.

e) Select a value for Jm that results in a natural frequency of 4Hz using the values provided in step b).

Problem 5.28 Draw FBDs for a system described with the equations below.

$$\ddot{\theta}_1 + \theta_1\left(\frac{K_{s1}}{J_{M_1}}\right) + \theta_2\left(\frac{-K_{s1}}{J_{M_1}}\right) = \frac{T}{J_{M_1}}$$

$$\ddot{\theta}_2 + \dot{\theta}_2\left(\frac{B}{J_{M_2}}\right) + \theta_2\left(\frac{K_{s1} + K_{s2}}{J_{M_2}}\right) + \theta_1\left(\frac{-K_{s1}}{J_{M_2}}\right) = 0$$

6. Input-Output Equations and Transfer Functions

Topic 6.1 *The differential operator, input-output equations.*
Topic 6.2 *Design case - vibration isolation.*
Topic 6.3 *The transfer function for representing system modules.*
Objective 6.1 *To be able to develop input-output equations for mechanical systems.*
Objective 6.2 *To be able to represent a system module with a transfer function.*

To solve a set of differential equations we have two choices, solve them numerically or symbolically. For a symbolic solution the system of differential equations must be manipulated into a single differential equation. In this chapter we will look at methods for manipulating differential equations into useful forms.

6.1 The Differential Operator

The differential operator 'd/dt' can be written in a number of forms. In this book there have been two forms used thus far, d/dt x and x-dot. For convenience we will add a third, 'D'. The basic definition of this operator, and related operations are shown in Figure 6.1. In basic terms the operator can be manipulated as if it is a normal variable. Multiplying by 'D' results in a derivative, dividing by 'D' results in an integral. The first-order axiom can be used to help solve a first-order differential equation.

basic definition

$$\frac{d}{dt}x = Dx \qquad \frac{d^n}{dt^n} = D^n \qquad \frac{1}{D}x = \int x\,dt$$

algebraic manipulation

$$Dx + Dy = D(x + y)$$

$$Dx + Dy = Dy + Dx$$

$$Dx + (Dy + Dz) = (Dx + Dy) + Dz$$

simplification

$$\frac{D^n x}{D^m} = D^{n-m}x$$

$$\frac{x(D + a)}{(D + a)} = x$$

first-order axiom

$$\frac{x(t)}{(D + a)} = e^{-at}\left(\int x(t)e^{at}dt + C\right)$$

Figure 6.1 General properties of the differential operator

Note:

$$\dot{x} + Ax = y(t)$$

$$xD + Ax = y(t)$$

$$e^{At}xD + e^{At}Ax = e^{At}y(t)$$

$$e^{At}xD = e^{At}y(t)$$

$$e^{At}xD = e^{At}y(t)$$

$$e^{At}x = \int e^{At}y(t)dt$$

$$x = e^{-At}\left(\int e^{At}y(t)dt\right)$$

$$\frac{y(t)}{D+A} = e^{-At}\left(\int e^{At}y(t)dt\right)$$

$$xD + Ax = y(t)$$

$$x(D + A) = y(t)$$

$$x = \frac{y(t)}{D+A}$$

$$\frac{d}{dt}(e^{at}x) = e^{at}\frac{d}{dt}x + ae^{at}x$$

$$De^{at}x = e^{at}Dx + ae^{at}x$$

Figure 6.2 Proof of the first-order axiom

Figure 6.3 contains an example of the manipulation of a differential equation using the 'D' operator. The solution begins by replacing the 'd/dt' terms with the 'D' operator. After this the equation is rearranged to simplify the expression. Notice that the manipulation follows the normal rules of algebra.

$$\left(\frac{d}{dt}\right)^2 x + \frac{d}{dt}x + 5x = 5y$$

$$D^2 x + Dx + 5x = 5y$$

$$x(D^2 + D + 5) = 5y$$

$$x = \frac{5y}{D^2 + D + 5}$$

$$x = y\left(\frac{5}{D^2 + D + 5}\right)$$

Figure 6.3 Example: A simplification with the differential operator

An example of the solution of a first-order differential equation is given in Figure 6.4. This begins by replacing the differential operator and rearranging the equation. The first-order axiom is then used to obtain the solution. The initial conditions are then used to calculate the coefficient values.

Given, $\frac{d}{dt}x + 5x = 3t$ $\qquad x(0) = 10$

$$Dx + 5x = 3t$$

$$x = \frac{3t}{D+5}$$

$$x = e^{-5t}\left(\int e^{5t} 3t dt + C\right)$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + C\right)$$

guess, $\frac{d}{dt}\left(te^{5t} - \frac{e^{5t}}{5}\right)$

$$= 5te^{5t} + e^{5t} - e^{5t}$$

$$= 5te^{5t}$$

$$\frac{d}{dt}\left(te^{5t} - \frac{e^{5t}}{5}\right) = 5e^{5t}t$$

$$\frac{te^{5t} - \frac{e^{5t}}{5}}{5} = \int e^{5t} t dt$$

Initial conditions,

$$x(0) = (1)\left(3\left(\frac{(0)(1) - \left(\frac{1}{1}\right)}{5}\right) + C\right) = 10$$

$$C = 10.6$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + 10.12\right)$$

$$x = 0.6\left(t - \frac{1}{5}\right) + 10.12e^{-5t}$$

$$x = 0.6t - 0.12 + 10.12e^{-5t}$$

Figure 6.4 Example: A a solution for a first-order system

6.2 Input-Output Equations

A typical system will be described by more than one differential equation. These equations can be solved to find a single differential equation that can then be integrated. The basic technique is to arrange the equations into an input-output form, such as that in Figure 6.5. These equations will have only a single output variable, and these are always shown on the left hand side. The input variables (there can be more than one) are all on the right hand side of the equation, and act as the non-homogeneous forcing function.

e.g.

$$2\dddot{y}_1 + \ddot{y}_1 + \dot{y}_1 + 4y_1 = \dot{u}_1 + u_1 + 3u_2 + \ddot{u}_3 + u_3$$

$$\ddot{y}_2 + 6\dot{y}_2 + y_2 = u_1 + 3\ddot{u}_2 + \dot{u}_2 + 0.5u_2 + \dot{u}_3$$

where,

$y = outputs$
$u = inputs$

Figure 6.5 Developing input-output equations

A sample derivation of an input-output equation from a system of differential equations is given in Figure 6.6. This begins by replacing the differential operator and combining the equations to eliminate one of the output variables. The solution ends by rearranging the equation to input-output form.

Given the differential equations.

$$\dot{y}_1 = -3y_1 + 2y_2 + u_1 + 2\dot{u}_2 \qquad \text{eqn 6.1}$$

$$\dot{y}_2 = 2y_1 + y_2 + \dot{u}_1 \qquad \text{eqn 6.2}$$

Find the input-output equations,

(1)
$$Dy_1 = -3y_1 + 2y_2 + u_1 + 2Du_2$$

$$y_1(D+3) = 2y_2 + u_1 + 2Du_2$$

$$y_2 = y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2$$

(2)
$$Dy_2 = 2y_1 + y_2 + Du_1$$

$$y_2(D-1) = 2y_1 + Du_1$$

$$\left(y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2\right)(D-1) = 2y_1 + Du_1$$

$$y_1\left(\frac{D^2+2D-3}{2} - 2\right) - 0.5Du_1 + 0.5u_1 - D^2u_2 + Du_2 = Du_1$$

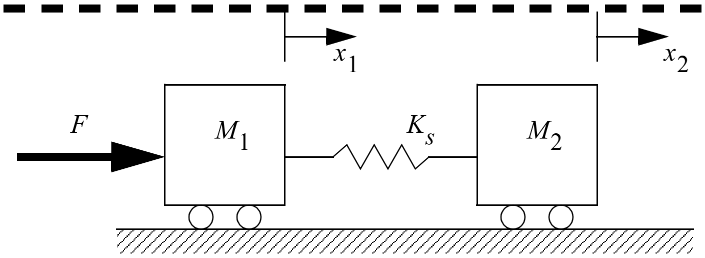
$$0.5D^2y_1 + Dy_1 - 3.5y_1 = Du_1 + 0.5Du_1 - 0.5u_1 + D^2u_2 - Du_2$$

$$\boxed{0.5y_1'' + y_1' - 3.5y_1 = 1.5u_1' - 0.5u_1 + u_2'' - u_2'}$$

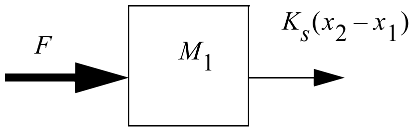
Figure 6.6 Example: An input output equation

Using Input-Output Equations to Develop State Equations

In some instances we will want to numerically integrate an input-output equation. The example starting in Figure 6.7 shows the development of an input-output equation for two freely rolling masses joined by a spring. The final equation has a derivative on the right hand side that would prevent it from being analyzed in many cases. In particular if the input force ‘F’ was a step function the first derivative would yield an undefined (infinite) value that could not be integrated.

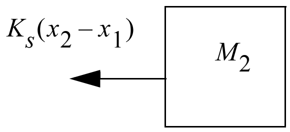


Equations of motion can be derived for these masses.



$$\sum F_x = F + K_s(x_2 - x_1) = M_1 D^2 x_1$$

$$x_1(M_1 D^2 + K_s) = F + K_s x_2$$



$$\sum F_x = -K_s(x_2 - x_1) = M_2 D^2 x_2$$

$$K_s x_1 = x_2(M_2 D^2 + K_s)$$

The equations can be combined to eliminate x2.

$$x_1(M_1 D^2 + K_s) = F + K_s\left(\frac{K_s x_1}{M_2 D^2 + K_s}\right)$$

$$x_1((M_1 D^2 + K_s)(M_2 D^2 + K_s) - K_s^2) = F(M_2 D^2 + K_s)$$

$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2) + K_s^2 - K_s^2) = F(M_2 D^2 + K_s)$$

$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2)) = F(M_2 D^2 + K_s)$$

$$\left(\frac{d}{dt}\right)^4 x_1 M_1 M_2 + \left(\frac{d}{dt}\right)^2 x_1 K_s(M_1 + M_2) = \left(\frac{d}{dt}\right)^2 F M_2 + F K_s$$

Figure 6.7 Example: Writing an input-output equation as a differential equation

The equation is then converted to state variable form, including a step to calculate a second derivative of the input, as shown in Figure 6.8.

This can then be written in state variable form by creating dummy variables for integrating the function 'F'.

$$\left(\frac{d}{dt}\right)x_1 = v_1$$

$$\left(\frac{d}{dt}\right)v_1 = a_1$$

$$\left(\frac{d}{dt}\right)a_1 = d_1$$

$$\left(\frac{d}{dt}\right)d_1 M_1 M_2 + a_1 K_s (M_1 + M_2) = a_F M_2 + F K_s$$

$$\left(\frac{d}{dt}\right)d_1 = a_1\left(\frac{-K_s(M_1 + M_2)}{M_1 M_2}\right) + a_F\left(\frac{1}{M_1}\right) + F\left(\frac{K_s}{M_1 M_2}\right)$$

The approximate value of the second derivative of a unit time step can be calculated using the time step.

$$a_{F_0} = \frac{1}{T^2} \qquad \text{(when the time step starts)}$$

$$a_{F_1} = -\frac{1}{T^2} \qquad \text{(after the first time step)}$$

$$v_F = T a_{F_0} + T a_{F_1} = T\left(\frac{1}{T^2}\right) + T\left(-\frac{1}{T^2}\right) = 0 \qquad \text{(to verify)}$$

$$x_F = \frac{T^2}{2}a_F + \frac{T^2}{2}a_F = \frac{T^2}{2}\left(\frac{1}{T^2}\right) + \frac{T^2}{2}\left(\frac{1}{T^2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

These equations can then be written in matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ a_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-K_s(M_1 + M_2)}{M_1 M_2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ v_1 \\ a_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{M_1} & \frac{K_s}{M_1 M_2} \end{bmatrix}\begin{bmatrix} a_F \\ F \end{bmatrix}$$

Figure 6.8 Example: Writing state equations for equations with derivatives

6.1 Transfer Functions

Input-output equations can also be written as a transfer function. The basic form for a transfer function is shown below in Figure 6.9. The general form calls for output over input on the left hand side. The right hand side is comprised of constants and the 'D' operator. In the example 'x' is the output, while 'F' is the input.

The general form → An example

$$\frac{output}{input} = f(D) \qquad \longrightarrow \qquad \frac{x}{F} = \frac{4 + D}{D^2 + 4D + 16}$$

Figure 6.9 Example: A transfer function

If both sides of the example were inverted then the output would become ‘F’, and the input ‘x’. This ability to invert a transfer function is called reversibility. In reality many systems are not reversible.

There is a direct relationship between transfer functions and differential equations. This is shown for the second-order differential equation in Figure 6.10. The homogeneous equation (the left hand side) ends up as the denominator of the transfer function. The non-homogeneous solution ends up as the numerator of the expression.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f}{M}$$

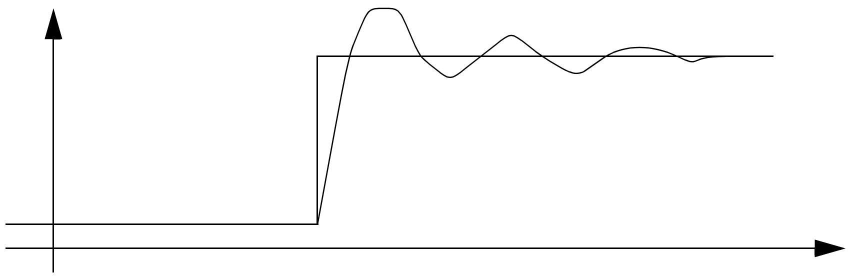
$$xD^2 + 2\zeta\omega_n xD + \omega_n^2 x = \frac{f}{M}$$

$$x(D^2 + 2\zeta\omega_n D + \omega_n^2) = \frac{f}{M}$$

$$\frac{x}{f} = \frac{\left(\frac{1}{M}\right)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

particular (numerator)

homogeneous (denominator)



ω_n Natural frequency of system - Approximate frequency of control system oscillations.

ξ Damping factor of system - If < 1 then underdamped, and the system will oscillate. If =1 critically damped. If > 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.

Figure 6.10 The relationship between transfer functions and differential equations for a mass-spring-damper example

The transfer function for a first-order differential equation is shown in Figure 6.11. As before the homogeneous and non-homogeneous parts of the equation becomes the denominator and the numerator of the transfer function.

$$\dot{x} + \frac{1}{\tau}x = f$$

$$xD + \frac{1}{\tau}x = f$$

$$x\left(D + \frac{1}{\tau}\right) = f$$

$$\frac{x}{f} = \frac{1}{D + \frac{1}{\tau}}$$

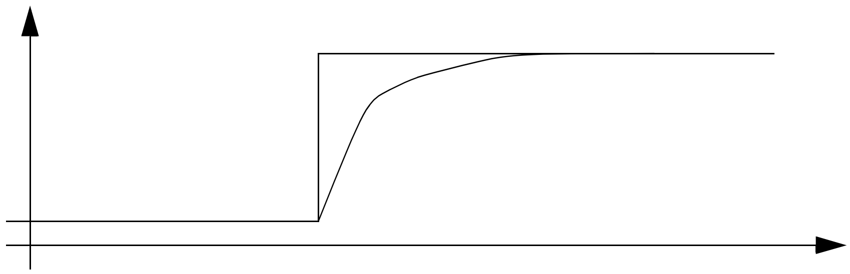


Figure 6.11 A first-order system response

Integrating Input-Output Equations with Input Derivatives

A transfer function is in a form suitable for using normal integration techniques. If the non-homogeneous part does not includes derivatives, then the techniques presented in previous chapters can be used. If the equation does include derivatives of the inputs, the solution can be found using superposition by reducing the differential equation to separate input parts, or decomposing it into partial fractions.

$$\frac{x}{F} = \frac{4D + 5}{D^2 + 3D + 2}$$

The transfer function can be converted to a differential equation.

$$\ddot{x} + \dot{x}(3) + x(2) = \dot{F}(4) + F(5)$$

the problem term

Note: The D in the numerator (top) requires special care - as shown in this example.

Figure 6.12 Example of Non-Trivial ODE

Summation of Input Components

The effect of the derivatives on the input side of the equation is an adjusted value of the initial conditions. For example the equation shown in Figure 6.12 may have initial conditions of zero at t=0-, but the derivative of F will result in a non-zero velocity at t=0+. In simple terms this does not affect the solution of the problem, but requires the calculation of a new set of initial conditions at t=0+, and these are used to find the coefficients of the homogenous equation. It is important to keep in mind that the initial conditions are not at zero, but just before. The input derivatives have an effect that starts at t=0.

$$\frac{x}{F} = \frac{4D + 5}{D^2 + 3D + 2}$$

The solution begins by reducing the transfer function to a differential equation, and then into two differential equations with results that can be summed because of the principle of superposition (it is a linear system). Assume F is a unit step function

$$\ddot{x} + \dot{x}(3) + x(2) = \dot{F}(4) + F(5)$$

homogeneous:

$$R = 1, 2$$

$$x_h = C_1 e^{-t} + C_2 e^{-2t}$$

particular x(t=0+):

i.e., $\frac{d}{dt}u(t) = 0 \qquad t \neq 0$

$$x_{2p} = C_3 \qquad 0 + 0(3) + C_3(2) = 0(4) + 1(5) \qquad C_3 = 2.5$$

Figure 6.13 Example: The Regular Solution to the Differential Equation

Part. x(t=0):

At time t=0, the step function will have a non-zero value for the derivative. This term can be isolated by multiplying through by dt and then taking the limit as dt approaches 0.

$$\left(\frac{d}{dt}\right)^2 x_1 + \left(\frac{d}{dt}\right)x_1(3) + x_1(2) = \left(\frac{d}{dt}\right)F(4) + F(5)$$

$$\left(\frac{d}{dt}\right)x_1 + x_1(3) + (dt)x_1(2) = F(4) + (dt)F(5)$$

$$\left(\frac{d}{dt}\right)x_1 + x_1(3) = F(4)$$

$$x_1 + dt(x_1(3)) = (dt)F(4)$$

As dt equals zero the left hand side of the equation will become x1.

$$x_1 = (dt)(4) \qquad \left(\frac{d}{dt}\right)x_1 = 4$$

Init. Conditions for x1 (assume all zero):

$$x(0^-) = 0 \qquad x(0) = 0 \qquad \dot{x}(0^-) = 0 \qquad \dot{x}(0) = 4$$

$$x = C_1 e^{-t} + C_2 e^{-2t} + 2.5 \qquad \dot{x} = -C_1 e^{-t} - 2(C_2 e^{-2t})$$

$$0 = C_1 + C_2 + 2.5 \qquad 4 = -C_1 - 2C_2$$

$$-2.5 + 4 = (C_1 - C_1) + (C_2 - 2C_2) = -C_2 \qquad C_2 = -1.5$$

$$C_1 = 3 - 4 = -1$$

$$x = (-e^{-t}) + (-1.5)e^{-2t} + 2.5$$

Figure 6.14 Example: Calculation of Initial Conditions and Final Solution

This method will allow the solution of the differential equation using methods that are very similar to those used before. The only complication is the need to determine the initial conditions for t=0. The following method provides a slightly longer, but more routine approach to solving these problems.

Partial Fractions for Integration

The equation is shown in Figure 6.15 can be solved by decomposing it into a simpler set of differential equations. The method is shown in Figure 6.15. It begins by reducing the transfer function to a partial fraction form. Once in this form each of the partial fraction terms can be converted to a separate differential equation. Each of these differential equations can then be solved separately. The total response for the system is then found by summing the individual responses. Note: this is permitted using the principle of superposition because the system is linear.

$$\frac{x}{F} = \frac{4D+5}{D^2+3D+2}$$

The solution begins by reducing the homogeneous equation with partial fractions.

$$\frac{x}{F} = \frac{4D+5}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2} = \frac{D(A+B)+(2A+B)}{(D+1)(D+2)}$$

$$\begin{array}{rl} & 5 = 2A+B \\ - & 4 = A+B \\ \hline & 1 = A \end{array} \qquad B = 3$$

$$\frac{x}{F} = \frac{1}{D+1} + \frac{3}{D+2}$$

The partial fraction form has given two discrete pieces.

$$x = F\left(\frac{1}{D+1}\right) + F\left(\frac{3}{D+2}\right)$$ or

$$x = x_1 + x_2 \qquad x_1 = F\left(\frac{1}{D+1}\right) \qquad x_2 = F\left(\frac{3}{D+2}\right)$$

$$\dot{x}_1 + x_1 = F \qquad \dot{x}_2 + 2x_2 = 3F$$

The solution begins by evaluating the homogeneous equations. The homogeneous solution for these can be found separately and then added. The addition is only possible because the system is linear.

$$x_{h_1} = C_1 e^{-t} \qquad x_{h_2} = C_2 e^{-2t} \qquad x_h = x_{h_1} + x_{h_2} = C_1 e^{-t} + C_2 e^{-2t}$$

Figure 6.15 Example: Integrating an input-output equation

The input is a unit step function (i.e. F=1). The particular solution is found by examining the two equations separately and then summing the results.

For, $\dot{x}_1 + x_1 = 1$

$$x_{p_1} = A \qquad \dot{x}_{p_1} = 0$$

$$0 + A = 1 \qquad \therefore A = 1$$

For, $\dot{x}_2 + 2x_2 = 3$

$$x_{p_1} = A \qquad \dot{x}_{p_1} = 0$$

$$0 + 2A = 3 \qquad \therefore A = \frac{3}{2}$$

$$x_p = x_{p_1} + x_{p_1} = 1 + \frac{3}{2} = \frac{5}{2}$$

(IMPORTANT NEW STEP) Next we must determine the initial conditions. Although we will assume the system starts undeflected at rest.

For t = 0+ $\dot{x}_1 + x_1 = 1 \qquad x_1 = 0 \qquad \therefore \dot{x}_1 = 1$

For t = 0+ $\dot{x}_2 + 2x_2 = 3 \qquad x_2 = 0 \qquad \therefore \dot{x}_2 = 3$

$$\dot{x} = \dot{x}_1 + \dot{x}_2 = 1 + 3 = 4$$

Figure 6.16 Example: Integrating an input-output equation

The initial conditions can then be used to find the values of the coefficients. It will be assumed that the system starts undeflected and at rest.

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{5}{2}$$

$$x(0) = C_1 e^{-0} + C_2 e^{-0} + \frac{5}{2} = 0 \qquad C_1 = -C_2 - \frac{5}{2}$$

$$\dot{x}(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$\dot{x}(0) = -C_1 - 2C_2 = 4$$

$$-\left(-C_2 - \frac{5}{2}\right) - 2C_2 = 4 \qquad C_2 = -\frac{3}{2}$$

$$C_1 = -\left(-\frac{3}{2}\right) - \frac{5}{2} = -1$$

The final equation can then be written.

$$x(t) = -e^{-t} - \frac{3}{2}e^{-2t} + \frac{5}{2}$$

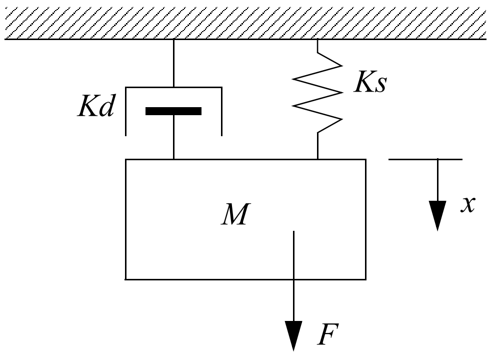
Figure 6.17 Example: Integrating an input-output equation (cont'd)

The downside to this method is that it requires the solution of a partial fraction, and multiple (although simpler) differential equations. The advantage is that it is very routine and does not require any special manipulation.

6.2 Design Case

The classic mass-spring-damper system is shown in Figure 6.18. In this example the forces are summed to provide an equation. The differential operator is replaced, and the equation is manipulated into transfer function form. The transfer function is given in two different forms because the system is reversible and the output could be either 'F' or 'x'.

Aside: Keep in mind that the mathematical expression 'F/x' is a ratio between input (displacement action) and output (reaction force). When shown with differentials it is obvious that the ratio is not simple, and is a function of time. Also keep in mind that if we were given a force applied to the system it would become the input (action force) and the output would be the displacement (resulting motion). To do this all we need to do is flip the numerators and denominators in the transfer function.



$$\sum F_y = -F + K_d\frac{dx}{dt} + K_s x = -M\frac{d^2x}{dt^2}$$

$$F = M\frac{d^2x}{dt^2} + K_d\frac{dx}{dt} + K_s x$$

$$F = MD^2x + K_dDx + K_sx$$

$$\frac{F}{x} = MD^2x + K_dDx + K_sx$$

or

$$\frac{x}{F} = \frac{1}{MD^2 + K_dD + K_s}$$

Aside: An important concept that is ubiquitous yet largely unrecognized is the use of functional design. We look at parts of systems as self contained modules that use inputs to produce outputs. Some systems (such a mechanisms) are reversible, others are not (consider a internal combustion engine, turning the crank does not produce gasoline). An input is typically something we can change, an output is the resulting change in a system. For the example above ‘F’ over ‘x’ implies that we are changing the input ‘x’, and there is some change in ‘F’. We know this could easily be reversed mathematically and practically.

Figure 6.18 Example: A transfer function for a mechanical system

Mass-spring-damper systems are often used when doing vibration analysis and design work. The first stage of such analysis involves finding the actual displacement for a given displacement or force. A system experiencing a sinusoidal oscillating force is given in Figure 6.19. Numerical values are substituted and the homogeneous solution to the equation is found.

Given the component values input force,

$$M = 1Kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad F = 5\sin(6t)N$$

The differential equation for the mass-spring damper system can be written.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

The homogeneous solution can be determined.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 0$$

$$A = \frac{-0.5\frac{Ns}{m} \pm \sqrt{\left(0.5\frac{Ns}{m}\right)^2 - 4(1Kg)\left(2\frac{N}{m}\right)}}{2(1Kg)}$$

$$A = 0.5(-0.5 \pm \sqrt{0.25 - 8})s^{-1}$$

$$A = (-0.25 \pm 1.392j)s^{-1}$$

$$x_h = C_1e^{-0.25t}\cos(1.392t + C_2)$$

Figure 6.19 Example: Explicit analysis of a mechanical system

The solution continues in Figure 6.20 where the particular solution is found and put in phase shift form.

The particular solution can now be found with a guess.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

$$x_p = A\sin 6t + B\cos 6t$$

$$x_p' = 6A\cos 6t - 6B\sin 6t$$

$$x_p'' = -36A\sin 6t - 36B\cos 6t$$

$$-36A\sin 6t - 36B\cos 6t + 0.5(6A\cos 6t - 6B\sin 6t) + 2(A\sin 6t + B\cos 6t) = 5\sin(6t)$$

$$-36B + 3A + 2B = 0 \longrightarrow A = \frac{34}{3}B$$

$$-36A - 3B + 2A = 5$$

$$-34\left(\frac{34}{3}B\right) - 3B = 5$$

$$B = \frac{5}{\frac{-34(34)}{3} - 3} = -0.01288 \qquad A = \frac{34}{3}(-0.01288) = -0.1460$$

$$x_p = (-0.1460)\sin 6t + (-0.01288)\cos 6t$$

$$x_p = \frac{\sqrt{(-0.1460)^2 + (-0.01288)^2}}{\sqrt{(-0.1460)^2 + (-0.01288)^2}}((-0.1460)\sin 6t + (-0.01288)\cos 6t)$$

$$x_p = 0.1466(-0.9961\sin 6t - 0.08788\cos 6t)$$

$$x_p = 0.1466\sin\left(6t + \operatorname{atan}\left(\frac{-0.9961}{-0.08788}\right)\right)$$

$$x_p = 0.1466\sin(6t + 1.483)$$

Figure 6.20 Example: Explicit analysis of a mechanical system (continued)

The system is assumed to be at rest initially, and this is used to find the constants in the homogeneous solution in Figure 6.21. Finally the displacement of the mass is used to find the force exerted through the spring on the ground. In this case there are two force frequency components at 1.392rad/s and 6rad/s. The steady-state force at 6rad/s will have a magnitude of 0.2932N. The transient effects have a time constant of 4 seconds (1/0.25), and should be negligible within a few seconds of starting the machine.

The particular and homogeneous solutions can now be combined.

$$x = x_h + x_p = C_1 e^{-0.25t}\cos(1.392t + C_2) + 0.1466\sin(6t + 1.483)$$

$$x' = -0.25 C_1 e^{-0.25t}\cos(1.392t + C_2) - 1.392(C_1 e^{-0.25t}\sin(1.392t + C_2)) + 6(0.1466\cos(6t + 1.483))$$

The initial conditions can be used to find the unknown constants.

$$0 = C_1 e^0 \cos(0 + C_2) + 0.1466\sin(0 + 1.483)$$

$$C_1\cos(C_2) = -0.1460$$

$$C_1 = \frac{-0.1460}{\cos(C_2)}$$

$$0 = -0.25 C_1 e^0 \cos(0 + C_2) - 1.392(C_1 e^0 \sin(0 + C_2)) + 6(0.1466\cos(0 + 1.483))$$

$$0 = -0.25 C_1\cos(C_2) - 1.392(C_1\sin(C_2)) + 0.07713$$

$$0 = -0.25\left(\frac{-0.1460}{\cos(C_2)}\cos(C_2)\right) - 1.392\left(\frac{-0.1460}{\cos(C_2)}\sin(C_2)\right) + 0.07713$$

$$0 = 0.0365 + (0.2032)\tan(C_2) + 0.07713$$

$$C_2 = \operatorname{atan}\left(\frac{0.0365 + 0.07713}{-0.2032}\right) = -0.5099$$

$$C_1 = \frac{-0.1460}{\cos(-0.5099)} = -0.1673$$

$$x = (-0.1673 e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

The displacement can then be used to calculate the force transmitted to the ground, assuming the spring is massless.

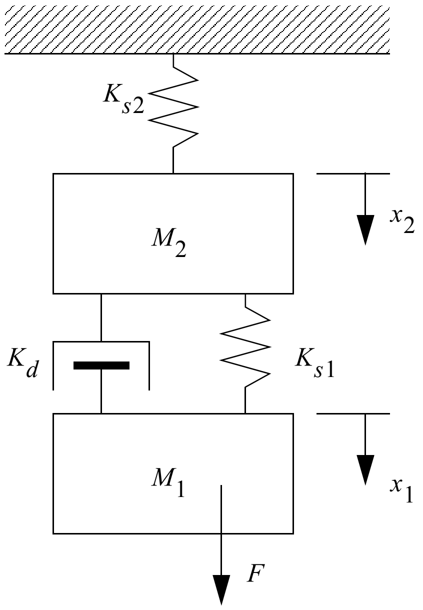
$$F = K_s x$$

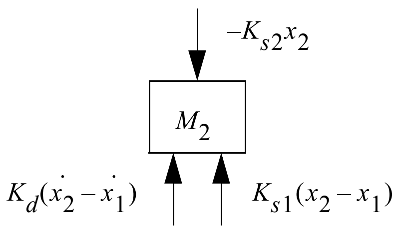
$$F = \left(2\frac{N}{m}\right)(-0.1673 e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

$$F = (-0.3346 e^{-0.25t}\cos(1.392t - 0.5099) + 0.2932\sin(6t + 1.483))N$$

Figure 6.21 Example: Explicit analysis of a mechanical system (continued)

A decision has been made to reduce the vibration magnitude transmitted to the ground to 0.1N. This can be done by adding a mass-spring isolator, as shown in Figure 6.22. In the figure the bottom mass-spring-damper combination is the original system. The mass and spring above have been added to reduce the vibration that will reach the ground. Values must be selected for the mass and spring. The design begins by developing the differential equations for both masses.



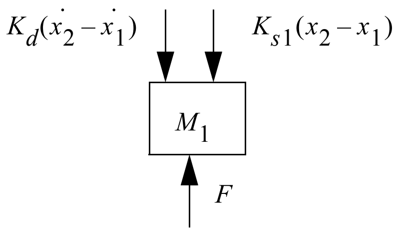


$$\sum F = -K_{s2}x_2 - K_d(\dot{x}_2 - \dot{x}_1) - K_{s1}(x_2 - x_1) = M_2\ddot{x}_2$$

$$-K_{s2}x_2 - K_d(x_2D - x_1D) - K_{s1}(x_2 - x_1) = M_2x_2D^2$$

$$x_2(-K_{s2} - K_dD - K_{s1} - M_2D^2) = x_1(-K_dD - K_{s1})$$

$$x_1 = x_2\left(\frac{K_{s2} + K_dD + K_{s1} + M_2D^2}{K_dD + K_{s1}}\right) \qquad \text{eqn 6.3}$$



$$\sum F = K_d(\dot{x}_2 - \dot{x}_1) + K_{s1}(x_2 - x_1) - F = M_1\ddot{x}_1$$

$$K_d(x_2D - x_1D) + K_{s1}(x_2 - x_1) - F = M_1x_1D^2$$

$$x_1(-K_dD - K_{s1} - M_1D^2) + x_2(K_dD + K_{s1}) = F \qquad \text{eqn 6.4}$$

Figure 6.22 Example: Vibration isolation system

For the design we are only interested in the upper spring, as it determines the force on the ground. An input-output equation for that spring is developed in Figure 6.23. The given values for the mass-spring-damper system are used. In addition a value for the upper mass is selected. This is arbitrarily chosen to be the same as the lower mass. This choice may need to be changed later if the resulting spring constant is not practical.

The solution begins by combining equations (1) and (2) and inserting the.numerical values for the lower mass, spring and damper. We can also limit the problem by selecting a mass value for the upper mass.

$$x_2\left(\frac{K_{s2} + K_d D + K_{s1} + M_2 D^2}{K_d D + K_{s1}}\right)(-K_d D - K_{s1} - M_1 D^2) + x_2(K_d D + K_{s1}) = F$$

$$M_1 = 1Kg \qquad K_{s1} = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad M_2 = 1Kg$$

$$x_2\left(\frac{K_{s2} + 0.5D + 2 + D^2}{0.5D + 2}\right)(-0.5D - 2 - D^2) + x_2(0.5D + 2) = F$$

$$x_2(D^2 + 0.5D + 2 + K_{s2})(D^2 - 0.5D - 2) + x_2(0.5D + 2)^2 = F(0.5D + 2)$$

$$x_2(D^4(-1) + D^2(K_{s2}) + D^1(-0.5K_{s2}) + (-2K_{s2})) = F(0.5D + 2)$$

This can now be converted back to a differential equation and combined with the force.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)F + 2F$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)5\sin(6t) + 2(5)\sin(6t)$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

Figure 6.23 Example: Developing an input output equation

This particular solution of the differential equation will yield the steady-state displacement of the upper mass. This can then be used to find the needed spring coefficient.

The transfer function can be broken into pieces using a partial fraction expansion.

$$\frac{x_2}{F} = \frac{(-0.5)D + (-2)}{D^4(1) + D^2(-K_{s2}) + D^1(0.5K_{s2}) + (2K_{s2})} = \frac{MD + N}{D^2 + SD + Q} + \frac{OD + P}{D^2 + TD + U}$$

$$\frac{x_2}{F} = \frac{D^3(M + O) + D^2(N + MT + SO + P) + D(MU + NT + OQ + PS) + (NU + PQ)}{D^4(1) + D^3(S + T) + D^2(Q + U + ST) + D(SU + QT) + (QU)}$$

$M + O = 0 \qquad \therefore M = -O \qquad S + T = 0 \qquad \therefore S = -T$

$N + (-O)T + (-T)O + P = 0 \qquad \therefore N - 2OT + P = 0 = 0$

$MU + NT + OQ + PS = -0.5 \qquad \therefore (-O)U + NT + OQ + P(-T) = -0.5$

$Q + U + ST = S + T \qquad \therefore Q + U + (-T)T = -T + T$

$NU + PQ = -2$

$SU + QT = 0.5K_{s2} \qquad QU = 2K_{s2}$

[[check contents for correctness and complete]]

Figure 6.24 Example: Developing an input-output equation

The particular solution begins with a guess.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

$$x_p = A\sin 6t + B\cos 6t$$

$$\left(\frac{d}{dt}\right)x_p = 6A\cos 6t - 6B\sin 6t$$

$$\left(\frac{d}{dt}\right)^2 x_p = -36A\sin 6t - 36B\cos 6t$$

$$\left(\frac{d}{dt}\right)^3 x_p = -216A\cos 6t + 216B\sin 6t$$

$$\left(\frac{d}{dt}\right)^4 x_p = 1296A\sin 6t + 1296B\cos 6t$$

$$\sin(6t)(-1296A - 36AK_{s2} + 0.5K_{s2}6B - 2K_{s2}A) = 10\sin(6t)$$

$$A(-1296 - 38K_{s2}) + B(3K_{s2}) = 10$$

$$B = \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}$$

$$\cos(6t)(-1296B - 36BK_{s2} + (-0.5)K_{s2}6A - 2K_{s2}B) = 15\cos(6t)$$

$$A(-3K_{s2}) + B(-1296 - 38K_{s2}) = 15$$

$$A(-3K_{s2}) + \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}(-1296 - 38K_{s2}) = 15$$

$$A(-9K_{s2}^2) + (10 + A(1296 + 38K_{s2}))(-1296 - 38K_{s2}) = 45K_{s2}$$

$$A\left(\frac{-9K_{s2}^2}{(-1296 - 38K_{s2})}\right) + A(1296 + 38K_{s2}) = \frac{45K_{s2}}{(-1296 - 38K_{s2})} - 10$$

$$A = \frac{\frac{45K_{s2}}{(-1296 - 38K_{s2})} - 10}{\frac{-9K_{s2}^2}{(-1296 - 38K_{s2})} + (1296 + 38K_{s2})}$$

$$A = \frac{45K_{s2} + 10(1296 + 38K_{s2})}{-9K_{s2}^2 + (1296 + 38K_{s2})(-1296 - 38K_{s2})}$$

$$A = \frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 6.25 Example: Finding the particular solution

The value for B can then be found.

$$B = \frac{10 + \left(\frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)(1296 + 38K_{s2})}{3K_{s2}}$$

$$B = \frac{10(-1453K_{2s}^2 - 98496K_{2s} - 1679616) + (425K_{2s} + 12960)(1296 + 38K_{s2})}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{1620K_{2s}^2 + 2.097563\times10^8 K_{2s}}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 6.26 Example: Finding the particular solution (cont'd)

Finally the magnitude of the particular solution is calculated and set to the desired amplitude of 0.1N. This is then used to calculate the spring coefficient.

$$amplitude = \sqrt{A^2 + B^2}$$

$$0.1 = \sqrt{\left(\frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2 + \left(\frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2}$$

A value for the spring coefficient was then found using Mathcad to get a value of 662N/m.

$K := 1$

given

$$\sqrt{\left[\frac{425\cdot K + 12960}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2 + \left[\frac{540\cdot K + 69918768}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2} = 0.1$$

$\text{find}(K) = 661.68$

Figure 6.27 Example: Calculation of the spring coefficient

6.1 Summary

- The differential operator can be manipulated algebraically
- Equations can be manipulated into input-output forms and solved as normal differential equations

6.2 Problems With Solutions

Problem 6.1 Find the input-output form for the following equations.

$$\dot{y} + y + x = 3$$

$$\dot{x} + x + y = 0$$

Problem 6.2 Find the input-output form for the following equations.

$$\ddot{x}_1 + \dot{x}_1 + 2x_1 - \dot{x}_2 - x_2 = 0$$

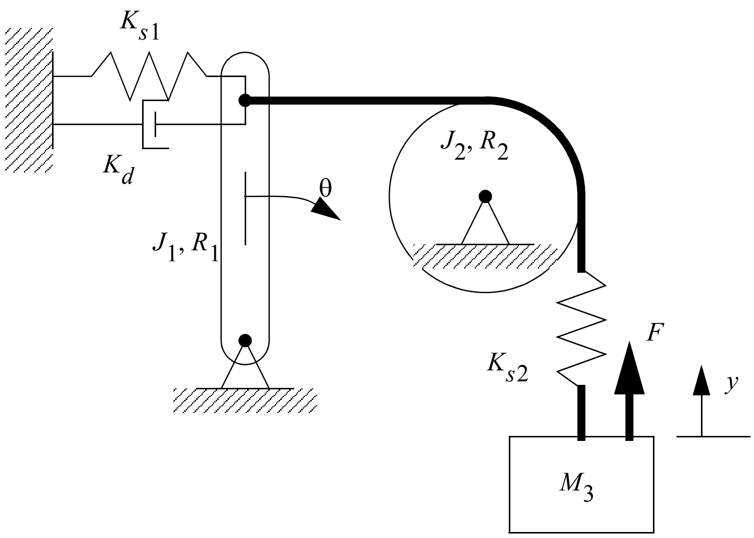
$$-\dot{x}_1 - x_1 + \ddot{x}_2 + \dot{x}_2 + x_2 = F$$

Problem 6.3 Find the input-output equations for the differential equations below if both ‘x’ and ‘y’ are outputs.

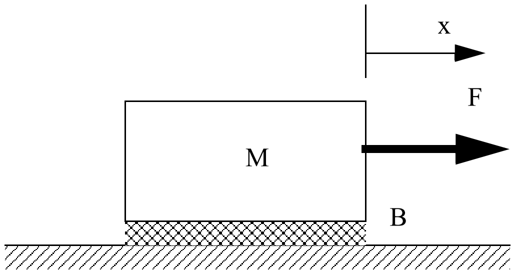
$$\dot{y} + y + 5x = 3u$$

$$\dot{x} + x + y = 7v$$

Problem 6.4 For the system below find the a) state and b) input-output equations. The cable always remains tight, and all deflections are small. Assume that the value of J2 is negligible. The input is the force F and the output is the angle ‘theta’.

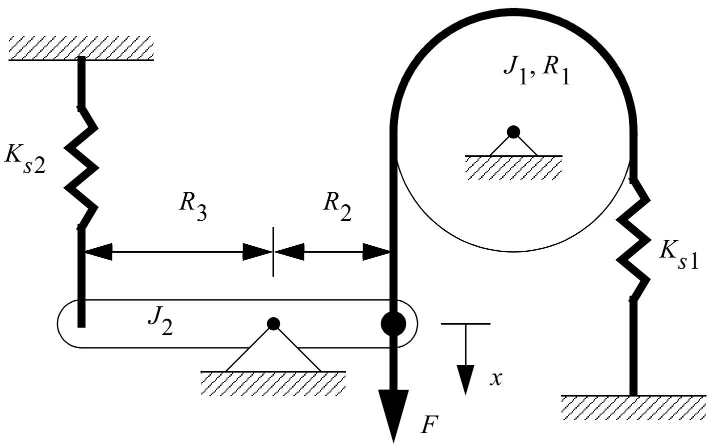


Problem 6.5 Develop the input-output equation and transfer function for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the mass to accelerate.

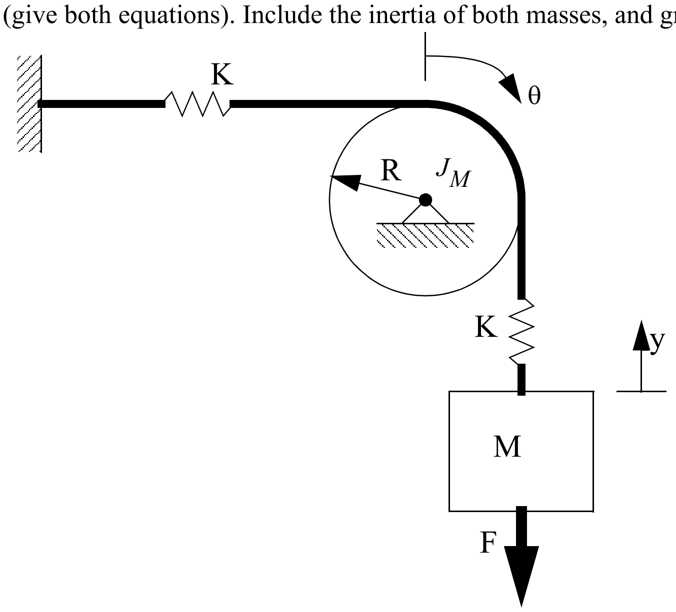


Problem 6.6 Assume that the cables always remain tight and all deflections are small. Find the transfer function for the system

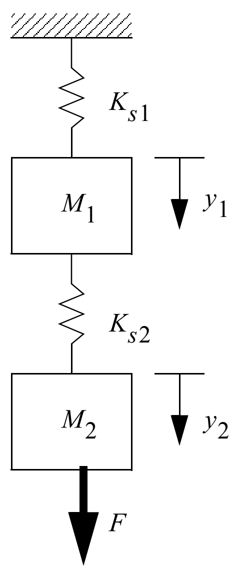
below.



Problem 6.7 Write the input-output equations for the mechanical system below. The input is force 'F', and the output is 'y' or the angle theta (give both equations). Include the inertia of both masses, and gravity for mass 'M'.



Problem 6.8 For the system pictured below find the input-output equation for y2 considering the effect of gravity.



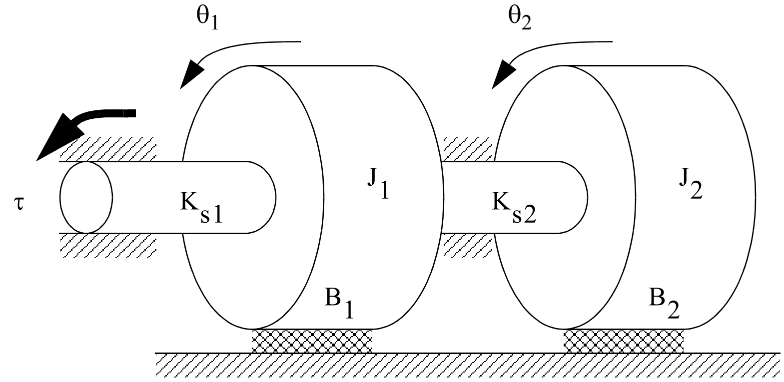
Problem 6.9 The following differential equations were converted to the matrix form shown. Use Cramer's rule (or equivalent) to find an input-output equation for 'y'.

$$\ddot{y} + 2x = \frac{F}{10}$$

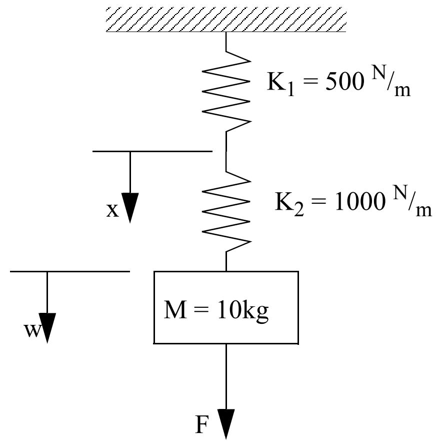
$$7\dot{y} + 4y + 9\ddot{x} + 3x = 0$$

$$\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \frac{F}{10} \\ 0 \end{bmatrix}$$

Problem 6.10 Find the input-output equations for the systems below. Here the input is the torque on the left hand side.

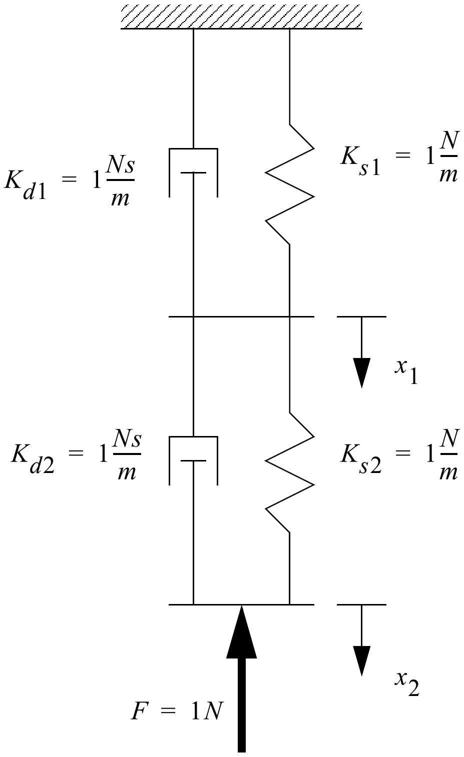


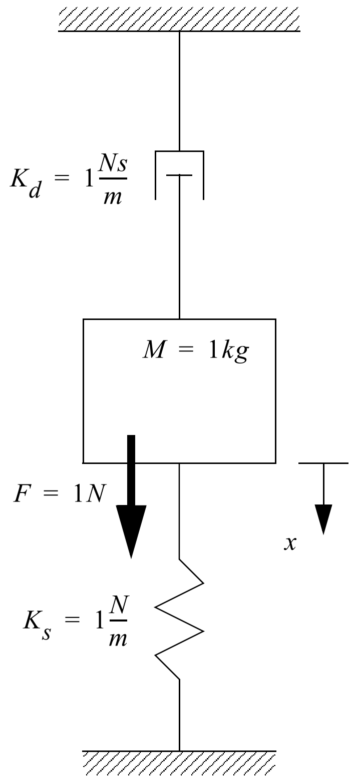
Problem 6.11 The applied force 'F' is the input to the system, and the output is the displacement 'x'.



a) Find x(t), given F(t) = 10N for t >= 0 seconds.
b) Using numerical methods, find the steady-state response for an applied force of F(t) = 10cos(t + 1) N ?
c) Solve the differential equation to find the explicit response for an applied force of F(t) = 10cos(t + 1) N ?
d) Set the acceleration to zero and find an approximate solution for an applied force of F(t) = 10cos(t + 1) N. Compare the solution to the previous solutions.

Problem 6.12 a) Write the differential equations for the system below. Solve the equations for *x* assuming that the system is at rest and undeflected before t=0. Also assume that gravity is present.





b) State whether each system is first or second-order. If the system if first-order find the time constant. If it is second-order find the natural frequency and damping ratio.

6.3 Problem Solutions

Answer 6.1

$$\ddot{x} + 2\dot{x} = -3$$
$$\ddot{y} + 2\dot{y} = 3$$

Answer 6.2

$$\left(\frac{d}{dt}\right)^4 x_1 + 2\left(\frac{d}{dt}\right)^3 x_1 + 3\left(\frac{d}{dt}\right)^2 x_1 + \left(\frac{d}{dt}\right) x_1 + x_1 = \left(\frac{d}{dt}\right) F + F$$

$$\left(\frac{d}{dt}\right)^4 x_2 + 2\left(\frac{d}{dt}\right)^3 x_2 + 3\left(\frac{d}{dt}\right)^2 x_2 + \left(\frac{d}{dt}\right) x_2 + x_2 = \left(\frac{d}{dt}\right)^2 F + \left(\frac{d}{dt}\right) F + 2F$$

Answer 6.3

$$\ddot{y} + 2\dot{y} - 4y = 3u - 35v + 3\dot{u}$$

$$\ddot{x} + 2\dot{x} - 4x = -3u + 7v + 7\dot{v}$$

Answer 6.4 a)

a)
$$\dot{y} = v$$

$$\dot{v} = \theta\left(\frac{-K_{s2}R_1}{M_3}\right) + y\left(\frac{-K_{s2}}{M_3}\right) + \left(\frac{F - M_3 g}{M_3}\right)$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \omega\left(\frac{-K_{d1}R_1^2}{J_1}\right) + \theta\left(\frac{-K_{s1}R_1^2 - K_{s2}R_1^2}{J_1}\right) + y\left(\frac{-R_1 K_{s2}}{J_1}\right)$$

b)
$$\left(\frac{d}{dt}\right)^4\theta + \left(\frac{d}{dt}\right)^3\theta\left(\frac{-K_{d1}R_1^2 M_3}{-J_1 M_3}\right) + \left(\frac{d}{dt}\right)^2\theta\left(\frac{-M_3 K_{s1}R_1^2 - K_{s2}J_1 - M_3 K_{s2}R_1^2}{-J_1 M_3}\right)$$

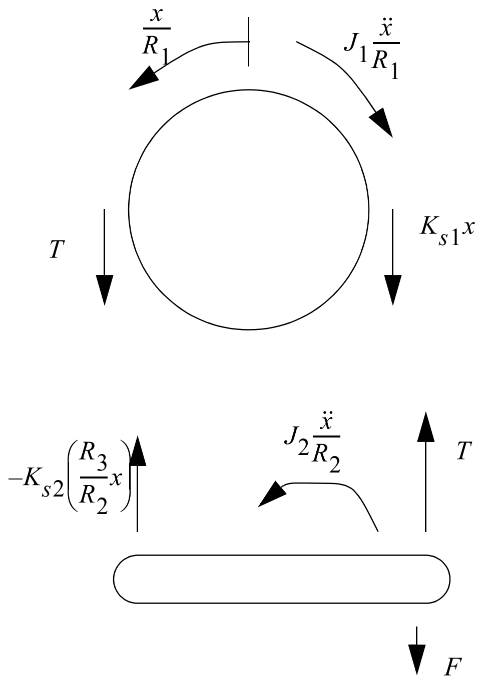
$$+ \left(\frac{d}{dt}\right)^1\theta\left(\frac{-K_{s1}K_{d1}R_1^2}{-J_1 M_3}\right) + \theta\left(\frac{-K_{s1}K_{s2}R_1^2 - K_{s2}^2 R_1^2 + K_{s2}R_1}{-J_1 M_3}\right) = \frac{F - M_3 g}{-J_1 M_3}$$

[[note: this term is incorrect]]

Answer 6.5

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) = \frac{F}{M} \qquad \frac{x}{F} = \frac{1}{D(B + DM)}$$

Answer 6.6



$$\sum M = J_1\frac{\ddot{x}}{R_1} + (K_{s1}x)R_1 - (T)R_1 = 0$$

$$\therefore T = J_1\frac{\ddot{x}}{R_1^2} + (K_{s1}x)$$

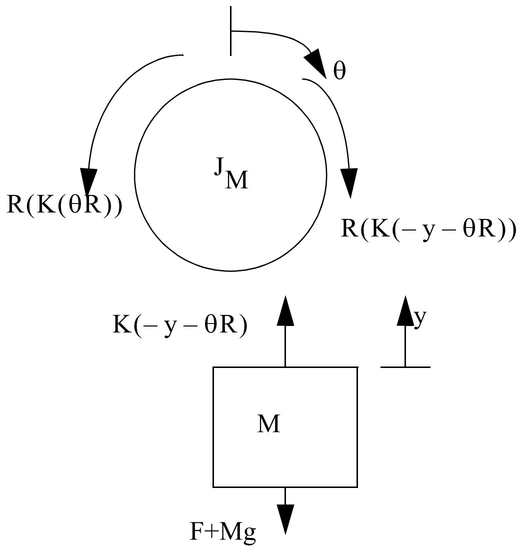
$$\sum M = \left(K_{s2}\left(\frac{R_3}{R_2}x\right)\right)R_3 - J_2\frac{\ddot{x}}{R_2} - TR_2 + FR_2 = 0$$

$$\therefore -K_{s2}\left(\frac{R_3^2}{R_2}x\right) - J_2\frac{\ddot{x}}{R_2} - \left(J_1\frac{\ddot{x}}{R_1^2} + (K_{s1}x)\right)R_2 + FR_2 = 0$$

$$\therefore \ddot{x}\left(\frac{J_2}{R_2} + J_1\frac{R_2}{R_1^2}\right) + x\left(K_{s2}\left(\frac{R_3^2}{R_2}\right) + K_{s1}R_2\right) = FR_2$$

$$\frac{x}{F} = \frac{R_2^2}{D^2\left(J_1\frac{R_2^2}{R_1^2} + J_2\right) + (K_{s1}R_2^2 + K_{s2}R_3^2)}$$

Answer 6.7



$$\sum M = -R(K(\theta R)) + R(K(-y-\theta R)) = J_M\ddot{\theta}$$

$$R^2K\theta + RKy + R^2K\theta = -J_M\ddot{\theta}$$

$$\theta\left(\frac{2R^2K + J_MD^2}{-RK}\right) = y$$

$$\sum F = K(-y-\theta R) - F - Mg = M\ddot{y}$$

$$K(y+\theta R) + F + Mg = -MyD^2$$

$$\theta(KR) + F + Mg = y(-MD^2 - K)$$

$$\theta(KR) + y(MD^2 + K) = -F - Mg$$

for the theta output equation;

$$\theta(KR) + \theta\left(\frac{2R^2K + J_MD^2}{-RK}\right)(MD^2 + K) = -F - Mg$$

$$\theta(-K^2R^2) + \theta\left(2R^2K + J_MD^2\right)(MD^2 + K) = FKR + MgKR$$

$$\theta\left(-K^2R^2 + 2R^2MKD^2 + 2R^2K^2 + J_MMD^4 + J_MKD^2\right) = FKR + MgKR$$

$$\left(\frac{d}{dt}\right)^4\theta(J_MM) + \left(\frac{d}{dt}\right)^2\theta\left(2R^2MK + J_MK\right) + \theta(K^2R^2) = FKR + MgKR$$

$$\left(\frac{d}{dt}\right)^4\theta + \left(\frac{d}{dt}\right)^2\theta\left(\frac{2R^2K}{J_M} + \frac{K}{M}\right) + \theta\left(\frac{K^2R^2}{J_MM}\right) = F\frac{KR}{J_MM} + \frac{gKR}{J_M}$$

for the y output equation;

$$y\left(\frac{-RK}{2R^2K + J_MD^2}\right)(KR) + y(MD^2 + K) = -F - Mg$$

$$y(-RK)(KR) + y(MD^2 + K)(2R^2K + J_MD^2) = -F(2R^2K + J_MD^2) - Mg(2R^2K + J_MD^2)$$

$$y(MJ_MD^4 + 2R^2K(MD^2) + K(J_MD^2) + R^2K^2) = -F(2R^2K + J_MD^2) - Mg(2R^2K + J_MD^2)$$

$$y\left(D^4 + D^2\left(\frac{2R^2K(M) + K(J_M)}{MJ_M}\right) + \frac{R^2K^2}{MJ_M}\right) = -F\left(\frac{2R^2K}{MJ_M} + \frac{J_MD^2}{MJ_M}\right) - Mg\left(\frac{2R^2K}{MJ_M} + \frac{J_M}{MJ_M}D^2\right)$$

$$y\left(D^4 + D^2\left(\frac{2R^2K}{J_M} + \frac{K}{M}\right) + \frac{R^2K^2}{MJ_M}\right) = -F\left(\frac{2R^2K}{MJ_M} + \frac{D^2}{M}\right) - g\left(\frac{2R^2K}{J_M} + D^2\right)$$

$$\left(\frac{d}{dt}\right)^4 y + \left(\frac{d}{dt}\right)^2 y\left(\frac{2KR^2}{J_M} + \frac{K}{M}\right) + y\left(\frac{R^2K^2}{J_MM}\right) = \left(\frac{d}{dt}\right)^2 F\left(\frac{-1}{M}\right) + F\left(\frac{-2KR^2}{J_MM}\right) + \left(\frac{-2gKR^2}{J_M}\right) + \left(\frac{d}{dt}\right)^2 g$$

Answer 6.8

$$\left(\frac{d}{dt}\right)^4 y_2 + \left(\frac{d}{dt}\right)^2 y_2\left(\frac{M_2(K_{s1}+K_{s2})+M_1K_{s2}}{M_1M_2}\right) + y_2\left(\frac{K_{s1}+K_{s2}}{M_1M_2}\right)$$
$$= \left(\frac{d}{dt}\right)^2 F\frac{1}{M_2} + F\left(\frac{K_{s1}+K_{s2}}{M_1M_2}\right) + \frac{g(M_2(K_{s1}+K_{s2})+M_1K_{s2})}{M_1M_2}$$

Answer 6.9

$$y = \frac{\left|\begin{bmatrix} \frac{F}{10} & (2) \\ 0 & (9D^2+3) \end{bmatrix}\right|}{\left|\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix}\right|} = \frac{\frac{F}{10}(9D^2+3)}{D^2(9D^2+3)-2(7D+4)} = \frac{F(0.9D^2+0.3)}{9D^4+3D^2-14D-8}$$

$$y(9D^4+3D^2-14D-8) = F(0.9D^2+0.3)$$

$$\left(\frac{d}{dt}\right)^4 y(9) + \left(\frac{d}{dt}\right)^2 y(3) + \left(\frac{d}{dt}\right)^1 y(-14) + y(-8) = \left(\frac{d}{dt}\right)^2 F(0.9) + F(0.3)$$

$$\left(\frac{d}{dt}\right)^4 y + \left(\frac{d}{dt}\right)^2 y\left(\frac{1}{3}\right) + \left(\frac{d}{dt}\right)^1 y\left(\frac{-14}{9}\right) + y\left(\frac{-8}{9}\right) = \left(\frac{d}{dt}\right)^2 F\left(\frac{1}{10}\right) + F\left(\frac{1}{30}\right)$$

Answer 6.10

$$\left(\frac{d}{dt}\right)^4 \theta_2 + \left(\frac{d}{dt}\right)^3 \theta_2\left(\frac{J_1B_2+J_2B_1}{J_1J_2}\right) + \left(\frac{d}{dt}\right)^2 \theta_2\left(\frac{J_1K_{s2}+J_2K_{s2}+B_1B_2}{J_1J_2}\right) +$$
$$\left(\frac{d}{dt}\right)\theta_2\left(\frac{B_1K_{s2}+B_2K_{s2}}{J_1J_2}\right) = \tau\frac{K_{s2}}{J_1J_2}$$

$$\left(\frac{d}{dt}\right)^4 \theta_1 + \left(\frac{d}{dt}\right)^3 \theta_1\left(\frac{J_1B_2+J_2B_1}{J_1J_2}\right) + \left(\frac{d}{dt}\right)^2 \theta_1\left(\frac{J_1K_{s2}+J_2K_{s2}+B_1B_2}{J_1J_2}\right) +$$
$$\left(\frac{d}{dt}\right)\theta_1\left(\frac{B_1K_{s2}+B_2K_{s2}}{J_1J_2}\right) = \left(\frac{d}{dt}\right)^2 \tau\left(\frac{1}{J_1}\right) + \left(\frac{d}{dt}\right)\tau\left(\frac{B_2}{J_1J_2}\right) + \tau\left(\frac{K_{s2}}{J_1J_2}\right)$$

Answer 6.11

$$\ddot{x} + x\left(\frac{K_1K_2}{M(K_1+K_2)}\right) = F\left(\frac{K_2}{M(K_1+K_2)}\right) + g\left(\frac{K_2}{K_1+K_2}\right)$$

a) $x(t) = -0.2168\cos(5.774t) + 0.2162$

b) $x(t) = -0.2074\cos(5.774t + 0.01449) + 0.0206\cos(t+1) + 0.1962$

c) $x(t) = 0.02\cos(t+1) + 0.1962$

Answer 6.12

a)

$$x_1(t) = e^{-t} - 1$$

$$x_2(t) = 2e^{-t} - 2$$

$$\tau = 1.0$$

b)

$$x(t) = -12.485e^{-0.5t}\cos(0.866t - 0.524) + 10.81$$

$$\zeta = 0.50 \qquad \omega_n = 1.0$$

6.4 Problems Without Solutions

Problem 6.13 Find the second equation for the example in Figure 6.6 for the output y_2.

Problem 6.14 Given the transfer for a mass-sprig-damper function below;
a) Find the differential equation.
b) Solve the differential equation explicitly for a step input of F(t) = 5 u(t).
c) Use a numerical method to find the response to a step input of F(t) = 5 u(t) from 0 to 10s. The graph should be plotted using Scilab or C/Excel. (Hint: the solutions for b and c should match.)

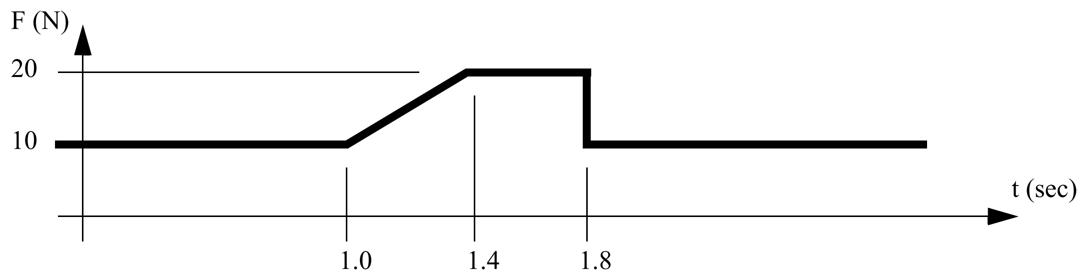
$$\frac{y}{F} = \frac{\frac{1}{M}}{D^2 + D\left(\frac{K_d}{M}\right) + \left(\frac{K_s}{M}\right)}$$

$$M = 100Kg$$

$$K_d = 5000\frac{Ns}{m}$$

$$K_s = 1000\frac{N}{m}$$

d) Find a new value for Kd that makes the system critically damped.
e) Use a numerical method to find the response to the input function below, using your new Kd value. Use the ‘u(t)’ function in your program for the input function. Plot the results from 0 to 5 seconds. (Hint: to check your solutions remember critical damping.)



6.5 References

6.1 Irwin, J.D., and Graf, E.R., Industrial Noise and Vibration Control, Prentice Hall Publishers, 1979.
6.2 Close, C.M. and Frederick, D.K., “Modeling and Analysis of Dynamic Systems, second edition, John Wiley and Sons, Inc., 1995.

7. Electrical Systems

Topic 7.1 *Basic components; resistors, power sources, capacitors, inductors and op-amps.*
Topic 7.2 *Device impedance.*
Topic 7.3 *Example circuits.*
Objective 7.1 *To apply analysis techniques to circuits.*

A voltage is a pull or push acting on electrons. The voltage will produce a current when the electrons can flow through a conductor. The more freely the electrons can flow, the lower the resistance of a material. Most electrical components are used to control this flow.

7.1 Modeling

Kirchoff's voltage and current laws are shown in Figure 7.1. The node current law holds true because the current flow in and out of a node must total zero. If the sum of currents was not zero then electrons would be appearing and disappearing at that node, thus violating the law of conservation of matter. The loop voltage law states that the sum of all rises and drops around a loop must total zero.

$$\sum I_{node} = 0 \qquad \text{node current}$$

$$\sum V_{loop} = 0 \qquad \text{loop current}$$

Figure 7.1 Kirchoff's laws

The simplest form of circuit analysis is for DC circuits, typically only requiring algebraic manipulation. In AC circuit analysis we consider the steady-state response to a sinusoidal input. Finally the most complex is transient analysis, often requiring integration, or similar techniques.

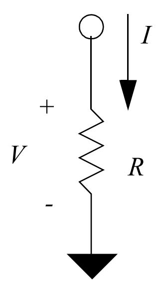
- DC (Direct Current) - find the response for a constant input.
- AC (Alternating Current) - find the steady-state response to an AC input.
- Transient - find the initial response to changes.

There is a wide range of components used in circuits. The simplest components are passive, such as resistors, capacitors and inductors. Active components are capable of changing their behaviors, such as op-amps and transistors. A list of components that will be discussed in this chapter are listed below.

- Resistors - reduce current flow as described with ohm's law
- Voltage/current sources - deliver power to a circuit
- Capacitors - pass current based on current flow, these block DC currents
- Inductors - resist changes in current flow, these block high frequencies
- Op-amps - very high gain amplifiers useful in many forms

Resistors

Resistance is a natural phenomenon found in all materials except superconductors. A resistor will oppose current flow as described by ohm's law in Figure 7.2. The resistance value is assumed to be linear, but in actuality it varies with conductor temperature.

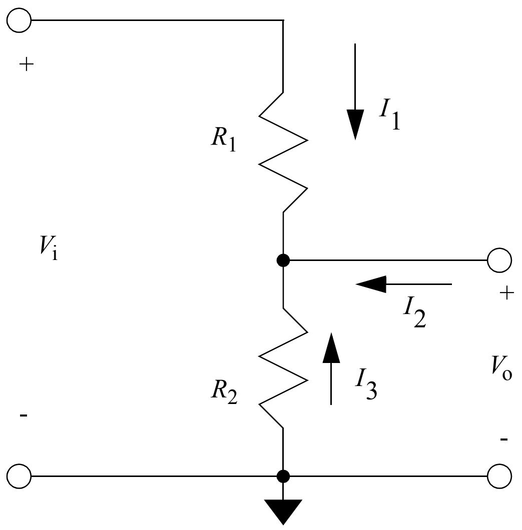


$$I = \frac{V}{R}$$

$$V = IR$$

Figure 7.2 Ohm's law

The voltage divider example in Figure 7.3 illustrates the methods for analysis of circuits using resistors. In this circuit an input voltage is supplied on the left hand side. The output voltage on the right hand side will be some fraction of the input voltage. If the output resistance is very large, no current will flow, and the ratio of output to input voltages is determined by the ratio of the resistance between R1 and R2. To prove this the currents into the center node are summed and set equal to zero. The equations are then manipulated to produce the final relationship.



$$I_1 + I_2 + I_3 = 0$$

$$\left(\frac{V_i - V_o}{R_1}\right) + I_2 + \left(\frac{0 - V_o}{R_2}\right) = 0$$

Assume the output resistance is large, so I_2 is negligible.

$$\left(\frac{V_i - V_o}{R_1}\right) + \left(\frac{-V_o}{R_2}\right) = 0$$

$$V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$V_o\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$\boxed{\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2}}$$

Figure 7.3 A voltage divider circuit

If two resistors are in parallel or series they can be replaced with a single equivalent resistance, as shown in Figure 7.4.

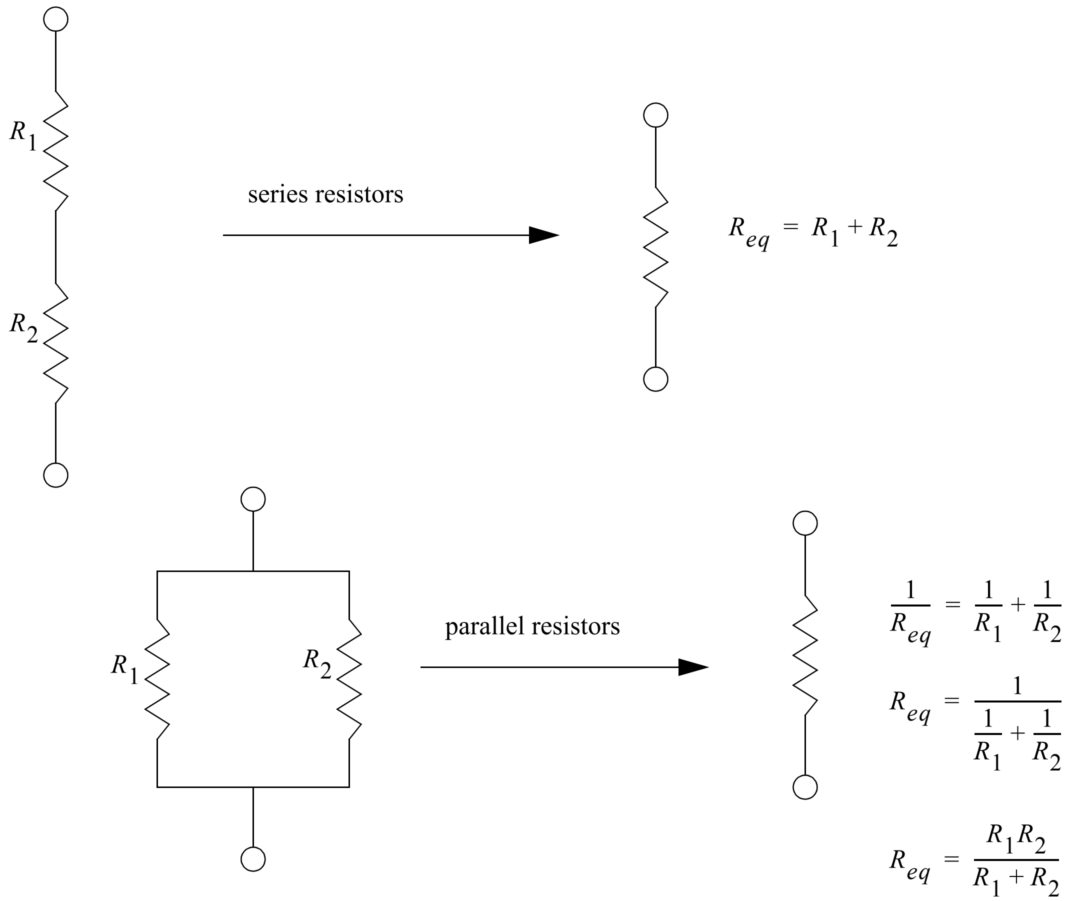


Figure 7.4 Equivalent resistances for resistors in parallel and series

Voltage and Current Sources

A voltage source will maintain a voltage in a circuit, by varying the current as required. A current source will supply a current to a circuit, by varying the voltage as required. The schematic symbols for voltage and current sources are shown in Figure 7.5. The supplies with '+' and '-' symbols provide DC voltages, with the symbols indicating polarity. The symbol with two horizontal lines is a battery. The circle with a sine wave is an AC voltage supply. The last symbol with an arrow inside the circle is a current supply. The arrow indicates the direction of positive current flow.

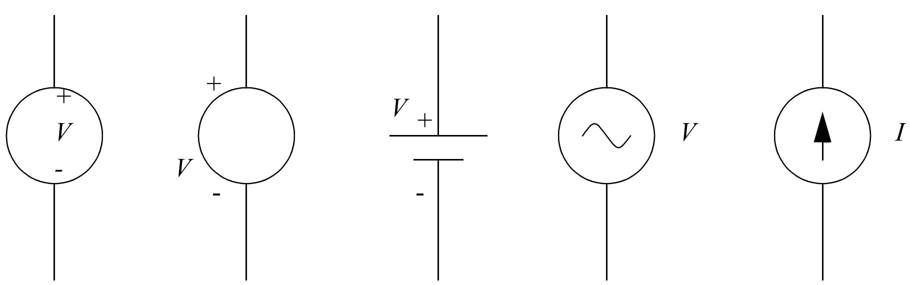
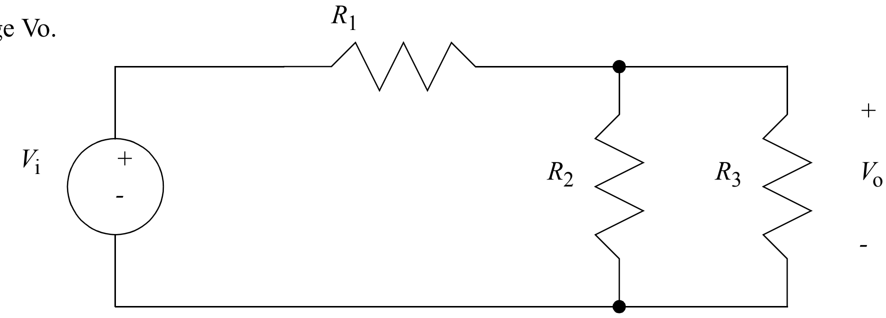


Figure 7.5 Voltage and current sources

A circuit containing a voltage source and resistors is shown in Figure 7.6. The circuit is analyzed using the node voltage method.

Find the output voltage Vo.



Examining the circuit there are two loops, but only one node, so the node current methods is the most suitable for calculations. The currents into the upper right node, Vo, will be solved.

$$\sum I = \frac{V_o - V_i}{R_1} + \frac{V_o}{R_2} + \frac{V_o}{R_3} = 0$$

$$V_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o\left(\frac{R_2 R_3 + R_1 R_3 + R_1 R_2}{R_1 R_2 R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o = V_i\left(\frac{R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}\right)$$

Aside: when doing node voltage methods, select currents out of a node as positive, and in as negative. This will reduce the chances of careless mistakes.

Figure 7.6 A circuit calculation

Dependent (variable) current and voltage sources are shown in Figure 7.7. The voltage and current values of these supplies are determined by their relationship to some other circuit voltage or current. The dependent voltage source will be accompanied by a '+' and '-' symbol, while the current source has an arrow inside.

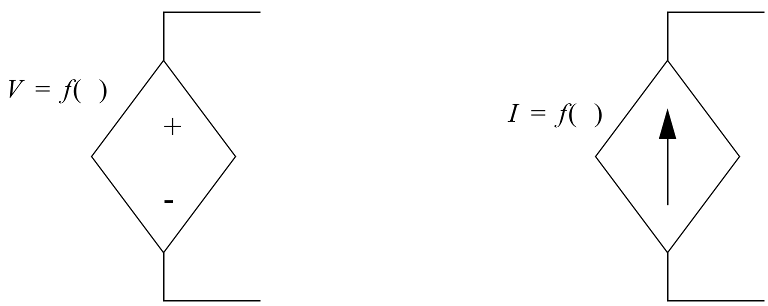
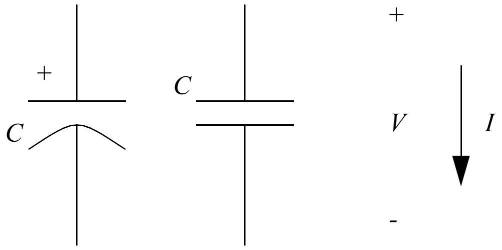


Figure 7.7 Dependent voltage sources

Capacitors

Capacitors are composed of two isolated metal plates very close together. When a voltage is applied across the capacitor, electrons will be forced into one plate, and forced out of the other plate. Temporarily this creates a small current flow until the plates reach equilibrium. So, any voltage change will result in some current flow. In practical terms this means that the capacitor will block any DC voltages, except for transient effects. But, high frequency AC currents will pass through the device. The equation for a capacitor and schematic symbols are given in Figure 7.8.



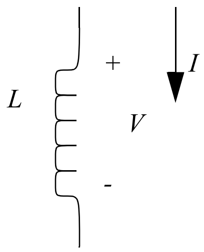
$$I = C\left(\frac{d}{dt}\right)V = CDV$$

Figure 7.8 Capacitors

The symbol on the left is for an electrolytic capacitor. These contain a special fluid that increases the effective capacitance of the device but requires that the positive and negative sides must be observed in the circuit. (Warning: reversing the polarity on an electrolytic capacitor can make them leak, fail and possibly explode.) The other capacitor symbol is for a regular capacitor, normally with values under a microfarad.

Inductors

While a capacitor will block a DC current, an inductor will pass it. Inductors are basically coils of wire. When a current flows through the coils, a magnetic field is generated. If the current through the inductor changes then the magnetic field must change, otherwise the field is maintained without effort (i.e., no voltage). Therefore the inductor resists changes in the current. The schematic symbol and relationship for an inductor are shown in Figure 7.9.



$$V = L\left(\frac{d}{dt}\right)I = LDI$$

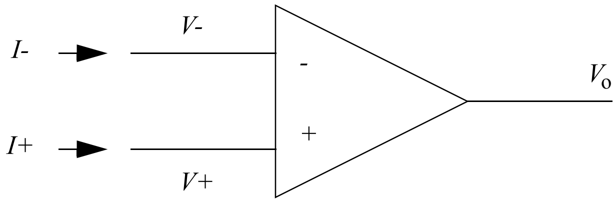
Figure 7.9 An inductor

An inductor is normally constructed by wrapping wire in loops about a core. The core can be hollow, or be made of ferrite to increase the inductance. Inductors usually cost more than capacitors. In addition, inductors are susceptible to interference when metals or other objects disturb their magnetic fields. When possible, designers normally try to avoid using inductors in circuits.

Op-Amps

The ideal model of an op-amp is shown in Figure 7.10. On the left hand side are the inverting and non-inverting inputs. Both of these inputs are assumed to have infinite impedance, and so no current will flow. Op-amp application circuits are designed so that the inverting and non-inverting inputs are driven to the same voltage level. The output of the op-amp is shown on the right. In circuits op-amps are used with feedback to perform standard operations such as those listed below.

- Adders, subtracters, multipliers, and dividers - simple analog math operations.
- Amplifiers - increase the amplitude of a signal.
- Impedance isolators - hide the resistance of a circuit while passing a voltage.



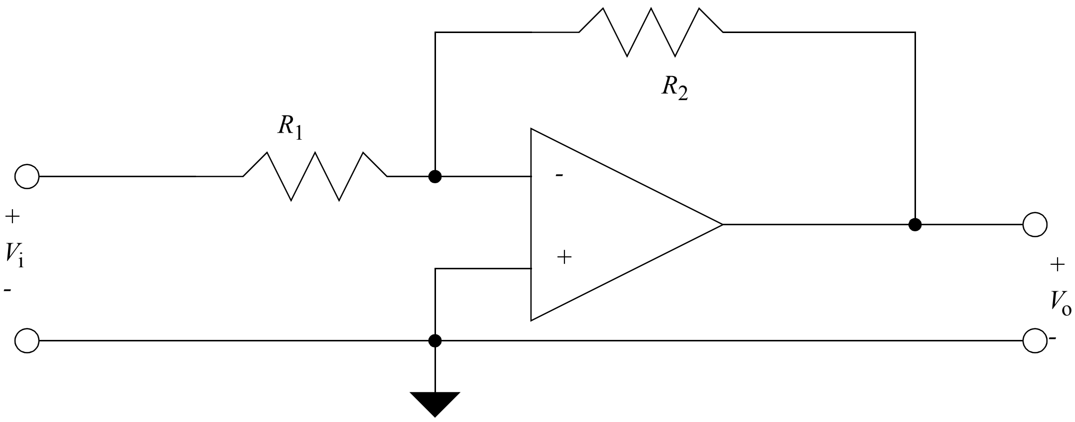
Note: for analysis use,

$$I_- = I_+ = 0$$

$$V_- = V_+$$

Figure 7.10 An ideal op-amp

A simple op-amp example is given in Figure 7.11. As expected both of the op-amp input voltages are the same. This is a function of the circuit design. (Note: most op-amp circuits are designed to force both inputs to have the same voltage, so it is normally reasonable to assume they are the same.) The non-inverting input is connected directly to ground, so it will force both of the inputs to 0V. When the currents are summed at the inverting input, an equation including the input and output voltages is obtained. The final equation shows the system is a simple multiplier, or amplifier. The gain of the amplifier is determined by the ratio of the input and feedback resistors.



The voltage at the non-inverting input will be 0V, by design the voltage at the inverting input will be the same.

$$V_+ = 0V$$

$$V_- = V_+ = 0V$$

The currents at the inverting input can be summed.

$$\sum I_{V-} = \frac{V_- - V_i}{R_1} + \frac{V_- - V_o}{R_2} = 0$$

$$\frac{0 - V_i}{R_1} + \frac{0 - V_o}{R_2} = 0$$

$$V_o = \frac{-R_2 V_i}{R_1}$$

$$V_o = \left(\frac{-R_2}{R_1}\right) V_i$$

Figure 7.11 A simple inverting operational amplifier configuration

An op-amp circuit that can subtract signals is shown in Figure 7.12.

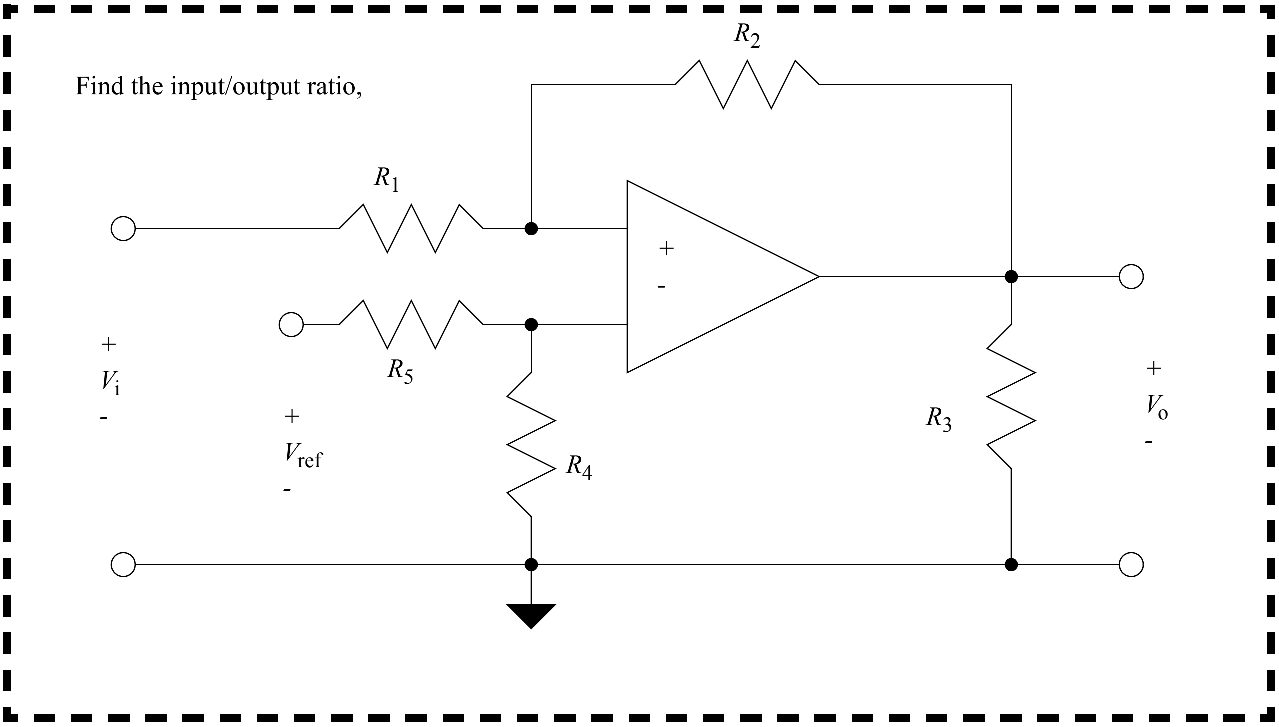


Figure 7.12 Op-amp example

For ideal op-amp problems the node voltage method is normally the best choice. The equations for the circuit in Figure 7.12 are derived in Figure 7.13. The general approach to this solution is to sum the currents into the inverting and non-inverting input nodes. Notice that the current into the op-amp is assumed to be zero. Both the inverting and non-inverting input voltages are then set to be equal. After that, algebraic manipulation results in a final expression for the op-amp. Notice that if all of the resistor values are the same then the circuit becomes a simple subtracter.

Note: normally node voltage methods work best with op-amp circuits, although others can be used if the non-ideal op-amp model is used.

First sum the currents at the inverting and non-inverting op-amp terminals.

$$\sum I_{V+} = \frac{V_+ - V_i}{R_1} + \frac{V_+ - V_o}{R_2} = 0$$

$$V_+\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+ = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right) \qquad \text{eqn 7.1}$$

$$\sum I_{V-} = \frac{V_- - V_{ref}}{R_5} + \frac{V_-}{R_4} = 0$$

$$V_-\left(\frac{1}{R_4} + \frac{1}{R_5}\right) = V_{ref}\left(\frac{1}{R_5}\right)$$

$$V_- = V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) \qquad \text{eqn 7.2}$$

Now the equations can be combined.

$$V_- = V_+$$

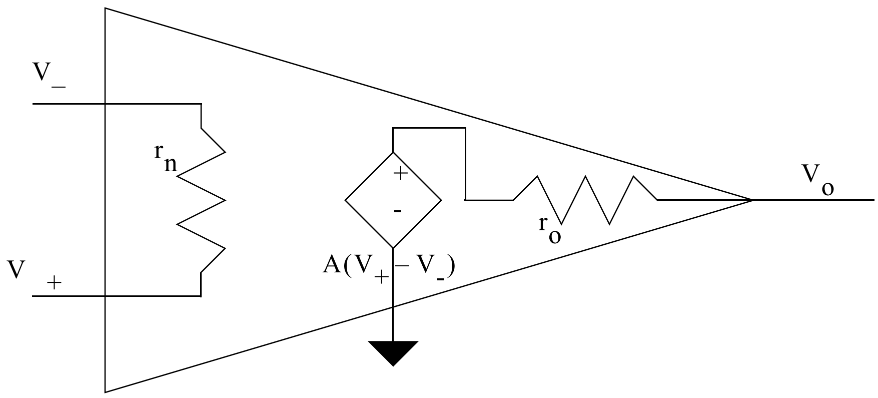
$$V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right)$$

$$V_o\left(\frac{R_1}{R_1 + R_2}\right) = -V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_{ref}\left(\frac{R_4}{R_4 + R_5}\right)$$

$$V_o = -V_i\left(\frac{R_2}{R_1}\right) + V_{ref}\left(\frac{R_4(R_1 + R_2)}{R_1(R_4 + R_5)}\right) \qquad \text{eqn 7.3}$$

Figure 7.13 Op-amp example (continued)

An op-amp (operational amplifier) has an extremely high gain, typically 100,000 times. The gain is multiplied by the difference between the inverting and non-inverting terminals to form an output. A typical op-amp will work for signals from DC up to about 100KHz. When the op-amp is being used for high frequencies or large gains, the model of the op-amp in Figure 7.14 should be used. This model includes a large resistance between the inverting and non-inverting inputs. The voltage difference drives a dependent voltage source with a large gain. The output resistance will limit the maximum current that the device can produce, normally less than 100mA.



typically,

$r_n > 1M\Omega$
$A > 10^5$
$r_o < 100\Omega$

Figure 7.14 A non-ideal op-amp model

7.1 Impedance

Circuit components can be represented in impedance form as shown in Figure 7.15. When represented this way the circuit solutions can focus on impedances, ‘Z’, instead of resistances, ‘R’. Notice that the primary difference is that the differential operator has been replaced. In this form we can use impedances as if they are resistances.

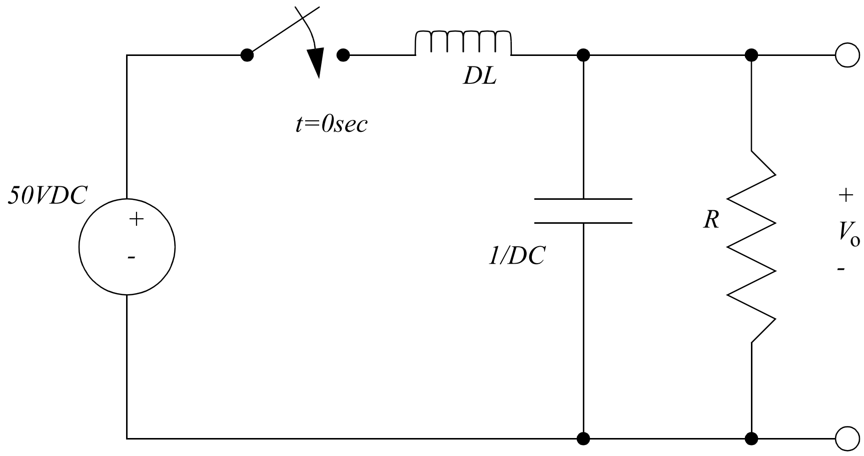
Device	*Time domain*	*Impedance*
Resistor	$V(t) = RI(t)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$Z = \frac{1}{DC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$Z = LD$

Note: Impedance is like resistance, except that it includes time variant features also.

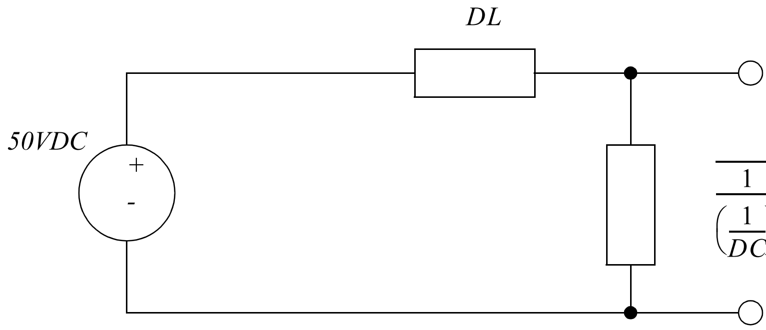
$$V = ZI$$

Figure 7.15 Impedances for electrical components

When representing component values with impedances the circuit solution is done as if all circuit components are resistors. An example of this is shown in Figure 7.16. Notice that the two impedances at the right (resistor and capacitor) are equivalent to two resistors in parallel, and the overall circuit is a voltage divider. The impedances are written beside the circuit elements.



Find the equivalent for the capacitor and resistor in parallel.



$$\frac{1}{\frac{1}{\left(\frac{1}{DC}\right)}+\frac{1}{R}} = \frac{1}{DC+\frac{1}{R}} = \frac{R}{RCD+1}$$

Treat the circuit as a voltage divider,

$$V_o = 50V\frac{\left(\frac{R}{1+DCR}\right)}{DL+\left(\frac{R}{1+DCR}\right)} = 50V\left(\frac{R}{D^2RLC+DL+R}\right)$$

Figure 7.16 A impedance example for a circuit

7.2 Example Systems

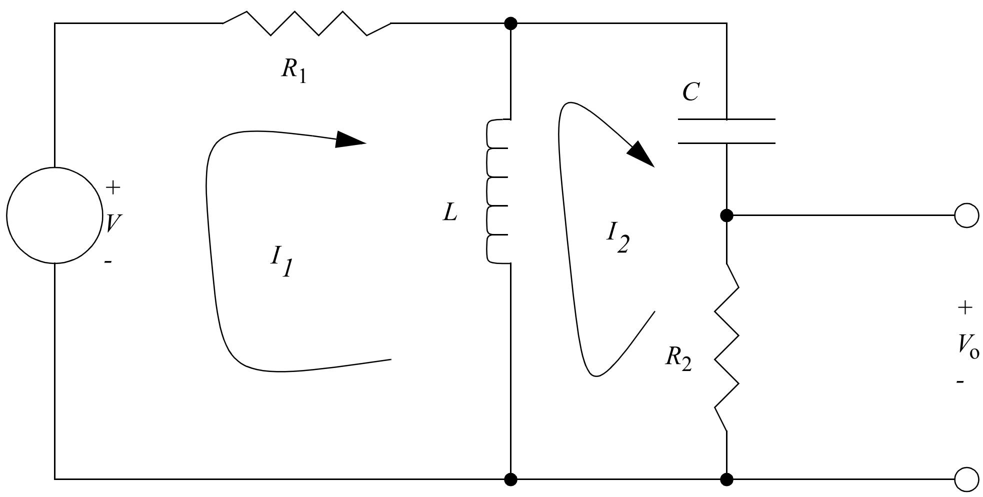
The list of instructions below can be useful when approaching a circuits problem. The most important concept to remember is that a minute of thinking about the solution approach will save ten minutes of backtracking and fixing mistakes.

1. Look at the circuit to determine if it is a standard circuit type such as a voltage divider, current divider or an op-amp inverting amplifier. If so, use the standard solution to solve the problem.
2. Otherwise, consider the nodes and loops in the circuit. If the circuit contains fewer loops, select the current loop method. If the circuit contains fewer nodes, select the node voltage method. Before continuing, verify that the select method can be used for the circuit.
3. For the node voltage method define node voltages and current directions. For the current loop method define current loops and indicate voltage rises or drops by adding ‘+’ or ‘-’ signs.
4. Write the equations for the loops or nodes.
5. Identify the desired value and eliminate unwanted values using algebra techniques.
6. Use numerical values to find a final answer.

Note: The units for various electrical quantities are listed to the right. They may be used to check equations by doing a unit balance.

coefficient	units
C	$\frac{As}{V}$
L	$\frac{Vs}{A}$
R	$\frac{V}{A}$

The circuit in Figure 7.17 could be solved with two loops, or two nodes. An arbitrary decision is made to use the current loop method. The voltages around each loop are summed to provide equations for each loop.



First, sum the voltages around the loops and then eliminate I1.

$$\sum V_{L1} = -V + R_1 I_1 + L(DI_1 - DI_2) = 0$$

$$(R_1 + LD)I_1 = V + (LD)I_2 \qquad \text{eqn 7.4}$$

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right)I_2 \qquad \text{eqn 7.5}$$

$$\sum V_{L2} = L(DI_2 - DI_1) + \frac{I_2}{CD} + R_2 I_2 = 0$$

$$(DL)I_1 = \left(LD + \frac{1}{CD} + R_2\right)I_2 = \left(\frac{LCD^2 + 1 + R_2 CD}{CD}\right)I_2 \qquad \text{eqn 7.6}$$

$$I_1 = \left(\frac{LCD^2 + 1 + R_2 CD}{CLD^2}\right)I_2 \qquad \text{eqn 7.7}$$

Note: when summing voltages in a loop remember to deal with sources that increase the voltage by flipping the sign.

Figure 7.17 Example problem

The equations in Figure 7.17 are manipulated further in Figure 7.18 to develop an input-output equation for the second current loop. This current can be used to find the current through the output resistor R_2. The output voltage can then be found by multiplying the R_2 and I_2.

First, sum the voltages around the loops and then eliminate I_1.

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right) I_2 = \left(\frac{LCD^2 + 1 + R_2 CD}{CLD^2}\right) I_2$$

$$V = \left(\frac{(LCD^2 + 1 + R_2 CD)(R_1 + LD)}{CLD^2} - LD\right) I_2$$

$$\left(\left(R_1 LCD^2 + R_1 + R_1 R_2 CD + L^2 CD^3 + LD + R_2 CLD^2\right) - CL^2 D^3\right) I_2 = CLD^2 V$$

$$I_2\left(D^3(L^2 C - CL^2) + D^2(R_1 LC + R_2 CL) + D(R_1 R_2 C + L) + (R_1)\right) = V(CLD^2)$$

$$V_o = R_2 I_2$$

$$\ddot{V}_o LC(R_1 + R_2) + \dot{V}_o(R_1 R_2 C + L) + V_o(R_1) = \ddot{V}(R_2 CL)$$

$$\ddot{V}_o + \dot{V}_o \frac{(R_1 R_2 C + L)}{LC(R_1 + R_2)} + V_o\left(\frac{R_1}{LC(R_1 + R_2)}\right) = \ddot{V}\left(\frac{R_2 CL}{LC(R_1 + R_2)}\right)$$

$$\ddot{V}_o + \dot{V}_o\left(\frac{R_1 R_2 C + L}{LC(R_1 + R_2)}\right) + V_o\left(\frac{R_1}{LC(R_1 + R_2)}\right) = \ddot{V}\left(\frac{R_2}{(R_1 + R_2)}\right)$$

Figure 7.18 Example problem (continued)

The equations can also be manipulated into state equations, as shown in Figure 7.19. In this case a dummy variable is required to replace the two first derivatives in the first equation. The dummy variable is used in place of I_1, which now becomes an output variable. In the remaining state equations I_1 is replaced by q_1. In the final matrix form the state equations are in one matrix, and the output variable must be calculated separately.

State equations can also be developed using equations (1) and (3).

equation 1 becomes $R_1 I_1 + L\dot{I}_1 = V + L\dot{I}_2$

$$L\dot{I}_1 - L\dot{I}_2 = V - R_1 I_1$$

$$\dot{I}_1 - \dot{I}_2 = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$q_1 = I_1 - I_2 \qquad \text{eqn 7.8}$$

$$\dot{q}_1 = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$I_1 = q_1 + I_2 \qquad \text{eqn 7.9}$$

$$\dot{q}_1 = \frac{V}{L} - \frac{R_1}{L}(q_1 + I_2)$$

$$\dot{q}_1 = q_1\left(-\frac{R_1}{L}\right) + I_2\left(-\frac{R_1}{L}\right) + \frac{V}{L} \qquad \text{eqn 7.10}$$

equation 3 becomes $L\dot{I}_2 - L\dot{I}_1 + \frac{I_2}{C} + R_2\dot{I}_2 = 0$

$$V - R_1 I_1 + \frac{I_2}{C} + R_2\dot{I}_2 = 0$$

$$V - R_1(q_1 + I_2) + \frac{I_2}{C} = -R_2\dot{I}_2$$

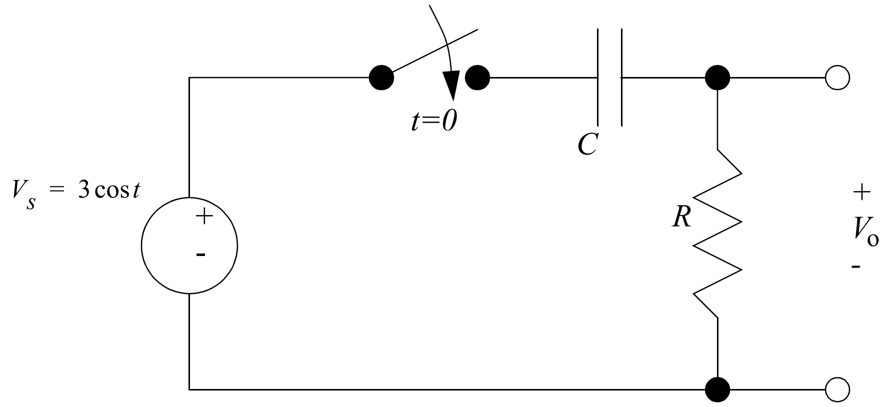
$$\dot{I}_2 = I_2\left(-R_1 + \frac{1}{C}\right) + q_1(-R_1) + V \qquad \text{eqn 7.11}$$

These can be put in matrix form,

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{R_1}{L} \\ -R_1 & -R_1 + \frac{1}{C} \end{bmatrix}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} \frac{V}{L} \\ V \end{bmatrix} \qquad \begin{bmatrix} I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix}$$

Figure 7.19 Example problem (continued)

The circuit in Figure 7.20 can be evaluated as a voltage divider when the capacitor is represented as an impedance. In this case the result is a first-order differential equation.



As normal we relate the source voltage to the output voltage. The we find the values for the various terms in the frequency domain.

$$V_o = V_s\left(\frac{Z_R}{Z_R + Z_C}\right) \qquad \text{where,} \qquad Z_R = R \qquad Z_C = \frac{1}{DC}$$

Next, we may combine the equations, and convert it to a differential equation.

$$V_o = V_s\left(\frac{R}{R + \frac{1}{DC}}\right)$$

$$V_o = V_s\left(\frac{CRD}{CRD + 1}\right)$$

$$V_o(CRD + 1) = V_s(CRD)$$

$$\dot{V}_o(CR) + V_o = \dot{V}_s(CR)$$

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = \dot{V}_s$$

Figure 7.20 Circuit solution using impedances

The first-order differential equation in Figure 7.20 is continued in Figure 7.21 where the equation is integrated. The solution is left in variable form, except for the supply voltage.

First, write the homogeneous solution using the known relationship.

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = 0 \qquad \text{yields} \qquad V_h = C_1 e^{-\frac{t}{CR}}$$

Next, the particular solution can be determined, starting with a guess.

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = \left(\frac{d}{dt}\right)(3\cos t) = -3\sin t$$

$$V_p = A\sin t + B\cos t$$

$$V_p' = A\cos t - B\sin t$$

$$(A\cos t - B\sin t) + (A\sin t + B\cos t)\left(\frac{1}{CR}\right) = -3\sin t$$

$$A + B\left(\frac{1}{CR}\right) = 0$$

$$A = B\left(\frac{-1}{CR}\right)$$

$$-B + A\left(\frac{1}{CR}\right) = -3$$

$$-B + B\left(\frac{-1}{CR}\right)\left(\frac{1}{CR}\right) = -3$$

$$B\left(\frac{1}{C^2R^2} + 1\right) = 3$$

$$B = \frac{3C^2R^2}{1 + C^2R^2} \qquad A = \left(\frac{3C^2R^2}{1 + C^2R^2}\right)\left(\frac{-1}{CR}\right) = \frac{-3CR}{1 + C^2R^2}$$

$$V_p = \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

The homogeneous and particular solutions can now be combined. The system will be assumed to be at rest initially.

$$V_0 = V_h + V_p = C_1 e^{-\frac{t}{CR}} + \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$0 = C_1 e^0 + \sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$C_1 = -\sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

Figure 7.21 Circuit solution using impedances (continued)

7.1 Electromechanical Systems - Motors

Permanent Magnet DC Motors

DC motors apply a torque between the rotor and stator that is related to the applied voltage/current. When a voltage is applied the torque will cause the rotor to accelerate. For any voltage and load on the motor there will tend to be a final angular velocity due to friction and drag in the motor. And, for a given voltage the ratio between steady-state torque and speed will be a straight line, as shown in Figure 7.22.

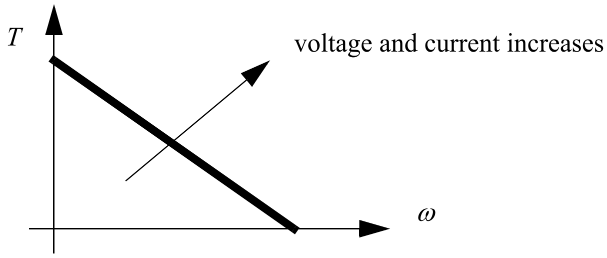
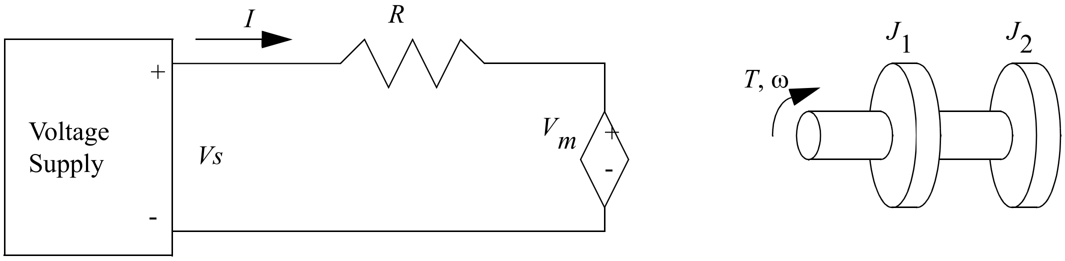


Figure 7.22 Torque speed curve for a permanent magnet DC motor

The basic equivalent circuit model is shown in Figure 7.23, includes the rotational inertia of the rotor and any attached loads. On the left hand side is the resistance of the motor and the 'back emf' dependent voltage source. On the right hand side the inertia components are shown. The rotational inertia J1 is the motor rotor, and the second inertia is an attached disk.



Because a motor is basically wires in a magnetic field, the electron flow (current) in the wire will push against the magnetic field. And, the torque (force) generated will be proportional to the current.

$$T_m = KI \qquad \backslash I = \frac{T_m}{K}$$

Next, consider the power in the motor,

$$P = V_m I = T\omega = KI\omega \qquad \backslash V_m = K\omega$$

Consider the dynamics of the rotating masses by summing moments.

$$\sum M = T_m - T_{load} = J\left(\frac{d}{dt}\right)\omega \qquad \backslash T_m = J\left(\frac{d}{dt}\right)\omega + T_{load}$$

Figure 7.23 The torque and inertia in a basic motor model

These basic equations can be manipulated into the first-order differential equation in Figure 7.24.

The current-voltage relationship for the left hand side of the equation can be written and manipulated to relate voltage and angular velocity.

$$I = \frac{V_s - V_m}{R}$$

$$\therefore \frac{T_m}{K} = \frac{V_s - K\omega}{R}$$

$$\therefore \frac{J\left(\frac{d}{dt}\right)\omega + T_{load}}{K} = \frac{V_s - K\omega}{R}$$

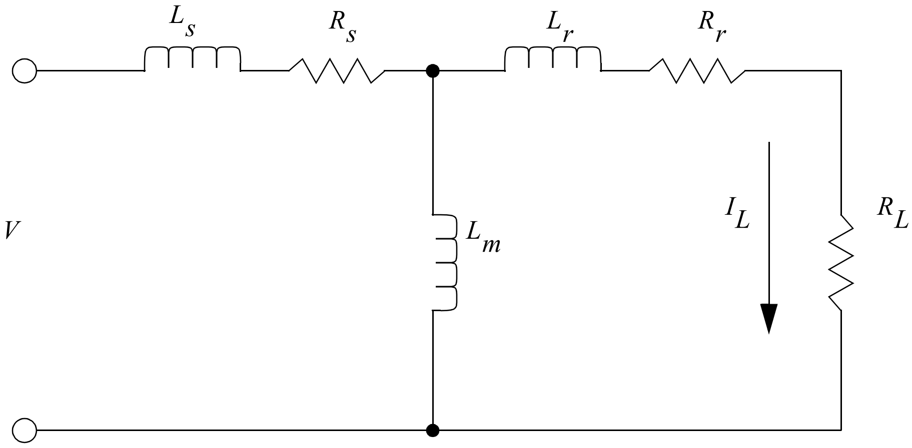
$$\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J}$$

Figure 7.24 The first-order model of a motor

Induction Motors

AC induction motors are extremely common because of the low cost of construction, and compatibility with the power distribution system. The motors are constructed with windings in the stator (outside of the motor). The rotor normally has windings, or a squirrel cage. The motor does not have bushes to the rotor. The motor speed is close to, but always less than the rotating AC fields. The rotating fields generate currents, and hence opposing magnetic fields in the stator. The maximum motor speed is a function of the frequency of the AC power, and the number of pole of the machine. For example, an induction motor with three poles being used with a 60Hz AC supply would have a maximum speed of 2*(60Hz/3) = 40Hz = 2400RPM.

The equivalent circuit for an AC motor is given in Figure 7.25. The slip of the motor determines the load current, I_L. It is a function of the fraction, f, of full speed.



$$R_L = \frac{1-f}{f} R_r$$

Figure 7.25 Basic model of an induction motor

The torque relationship for AC motors is given in Figure 7.26. These can be combined with the equivalent circuit model to determine the response of the motor to a load.

First the torques on the motor are summed,

$$\sum M = T_{rotor} - T_{load} = J \frac{d}{dt} \omega$$

[[Note: example must continue to state variable solution]]

Brushless Servo Motors

Brushless servo motors are becoming very popular because of their low maintenance requirements. The motors eliminate the need for brushes by using permanent magnets on the rotor, with windings on the stator, as shown in Figure 7.26. The windings on the stator are switched at a given frequency to produce a desired rotational speed, or held static to provide a holding torque.

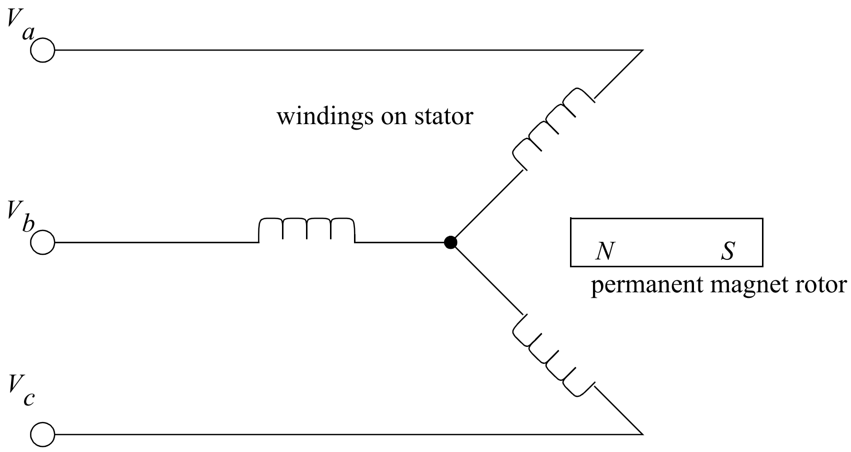


Figure 7.26 The construction of a brushless servo motor

The basic relationships for brushless DC motors are given in Figure 7.27.

$$V_t = \left(R_m + \frac{d}{dt}L\right)I_m + E$$

$$E = K_e\omega$$

$$T = K_t I_m$$

where,

V_t = *terminal voltage across motor windings*
R_m = *resistance of a motor winding*
L = *phase to phase inductance*
I_m = *current in winding*
E = *back e.m.f. of motor*
K_e = *motor speed constant*
ω = *motor speed*
K_t = *motor torque constant*
T = *motor torque*

Figure 7.27 Basic relationships for a brushless motor

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{T}{K_t} + K_e\omega$$

$$\sum M = T - T_{load} = J\frac{d}{dt}\omega$$

$$T = J\frac{d}{dt}\omega + T_{load}$$

where,

J = *combined moments of inertia for the rotor and external loads*

T_{load} = *the applied torque in the system*

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{J\frac{d}{dt}\omega + T_{load}}{K_t} + K_e\omega$$

$$V_t = \frac{JR_m}{K_t}\frac{d}{dt}\omega + \frac{LJ}{K_t}\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{K_t}T_{load} + \frac{L}{K_t}\frac{d}{dt}T_{load} + K_e\omega$$

$$(LJ)\left(\frac{d}{dt}\right)^2\omega + (JR_m)\frac{d}{dt}\omega + K_eK_t\omega = K_tV_t - L\frac{d}{dt}T_{load} - R_mT_{load}$$

$$\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{L}\frac{d}{dt}\omega + \frac{K_eK_t}{LJ}\omega = \frac{K_tV_t}{LJ} - \frac{1}{J}\frac{d}{dt}T_{load} - \frac{R_mT_{load}}{LJ}$$

Figure 7.28 An advanced model of a brushless servo motor

To rotate the motor at a constant velocity the waveform in Figure 7.29 would be applied to each phase. Although each phase would be 120 degrees apart for a three pole motor. A more sophisticated motor controller design would smooth the waves more to approach a sinusoidal shape.

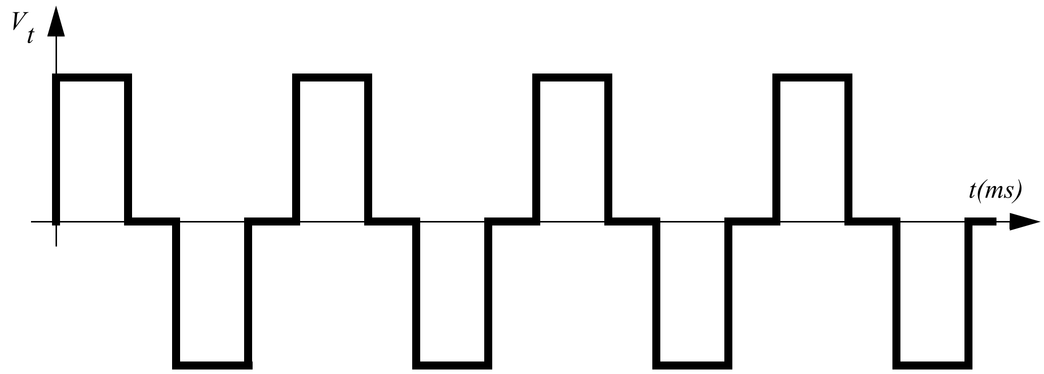


Figure 7.29 Typical supply voltages

7.2 Filters

Filters are useful when processing data signals. Low pass often used to eliminate noise, high pass filters eliminate static signals and leave dynamic signals. Band pass filters reject all frequencies outside a desired frequency band. A low pass filter is shown in Figure 7.30. At high frequencies the capacitor, C, has a very low impedance, and grounds the input signal. At low frequencies the capacitor impedance is high, increasing the gain of the op-amp circuit. This is easier to conceptualize if the R_1-C pair are viewed as a voltage divider.

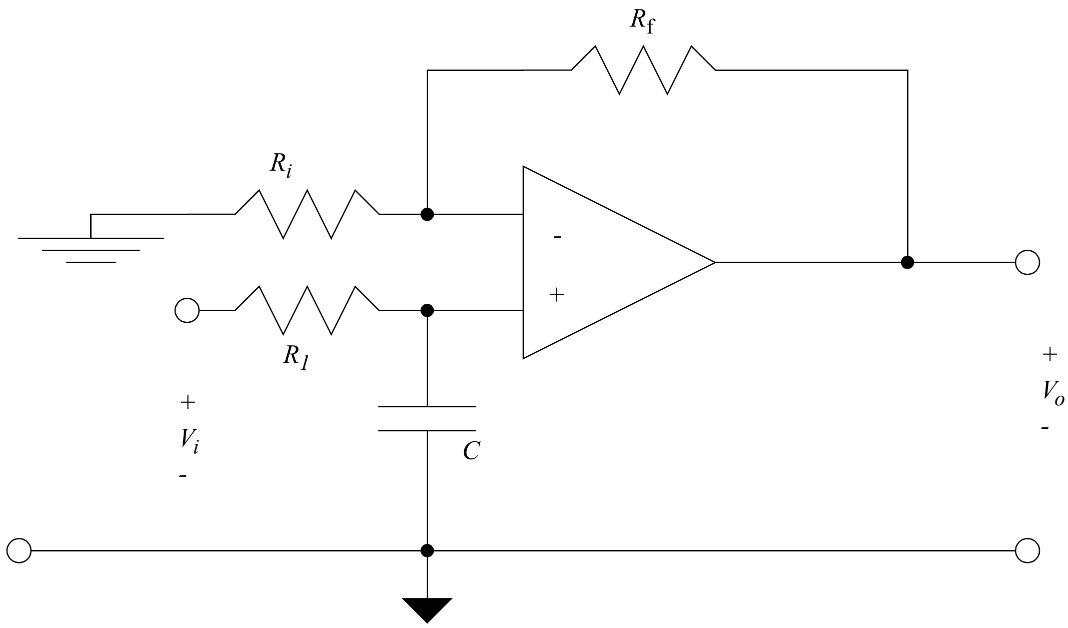


Figure 7.30 Low-Pass Filter

A high pass filter is shown in Figure 7.31. In this case the voltage divider in the previous circuit is reversed. In this circuit the gain will increase for signals with higher frequencies.

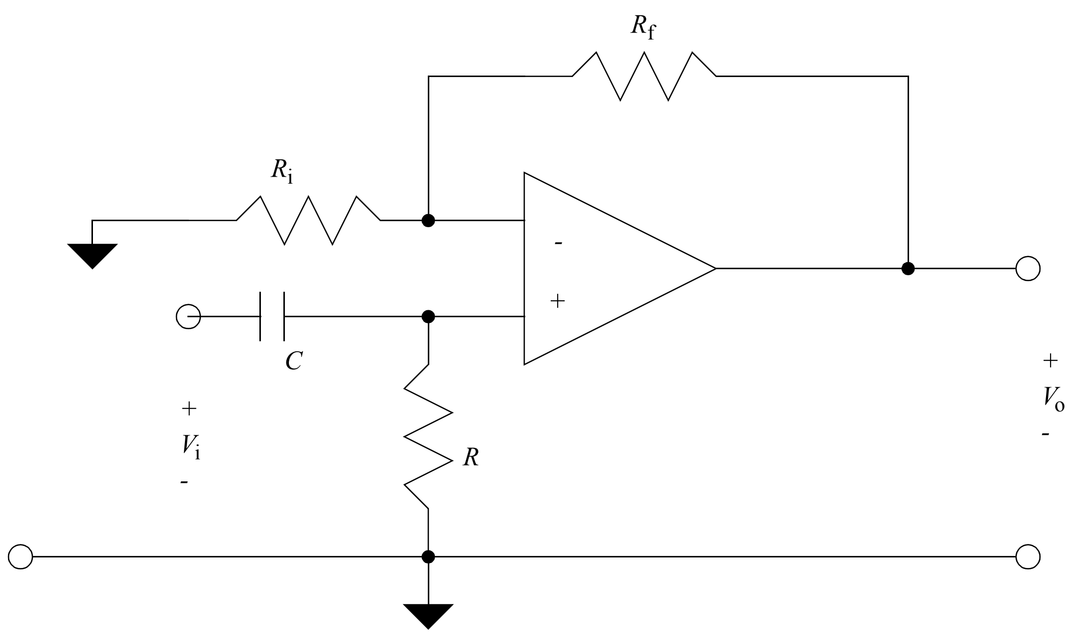


Figure 7.31 High-Pass Filter

7.3 Other Topics

The relationships in Figure 7.32 can be used to calculate the power and energy in a system. Notice that the power calculations focus on resistance, as resistances will dissipate power in the form of heat. Other devices, such as inductors and capacitors, store energy, but don't dissipate it.

$$P = IV = I^2R = \frac{V^2}{R} \qquad E = Pt$$

Figure 7.32 Electrical power and energy

7.4 Summary

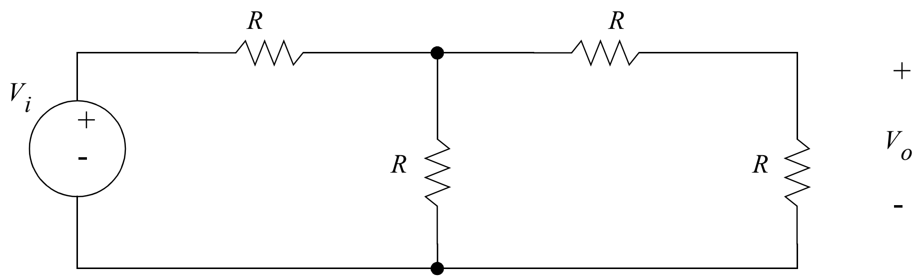
- Basic circuit components are resistors, capacitors, inductors op-amps.
- Node and loop methods can be used to analyze circuits.
- Capacitor and inductor impedances can be used as resistors in calculations.

7.5 Problems With Solutions

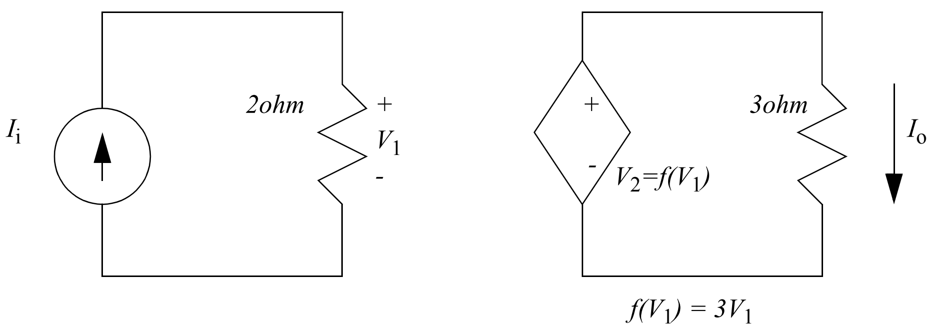
Problem 7.1 Derive the equations for combined values for resistors, capacitors and inductors in series and parallel.

Problem 7.2 Evaluate the circuit in Figure 7.6 using the current loop method.

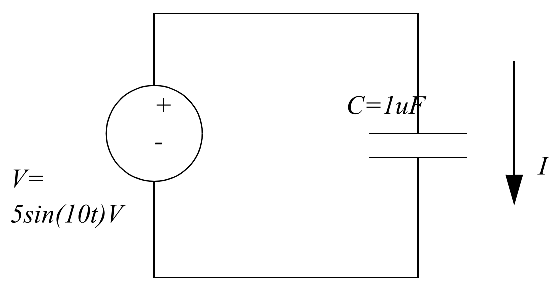
Problem 7.3 Find the output voltage as a function of input voltage.



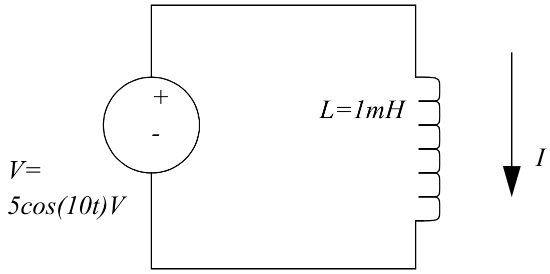
Problem 7.4 *Find the output current, Io, if Ii = 1A. What if the input current is Ii = 1sin(2t)A?*



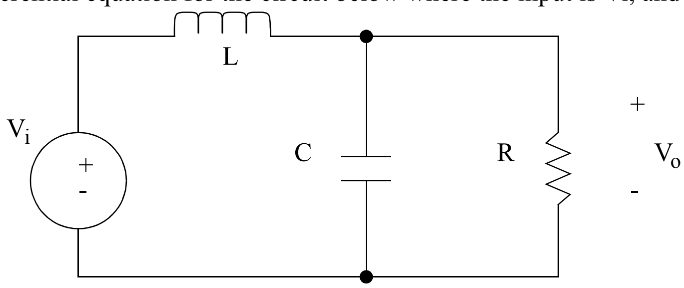
Problem 7.5 *Find the current as a function of time.*



Problem 7.6 *Find the current as a function of time.*



Problem 7.7 a) Find the differential equation for the circuit below where the input is Vi, and the output is Vo.



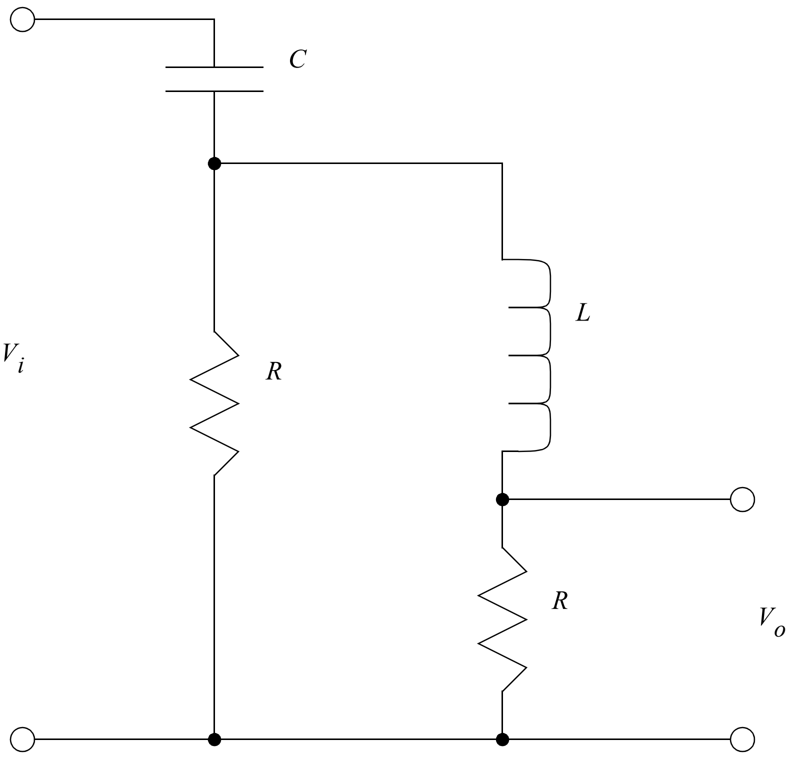
b) Convert the equation to an input-output equation.
c) Solve the differential equation found in part b) using the numerical values given below. Assume at time t=0, the circuit has the voltage Vo and the first derivative shown below.

$L = 10mH$ $C = 1\mu F$ $R = 1K\Omega$ $V_i = 10V$

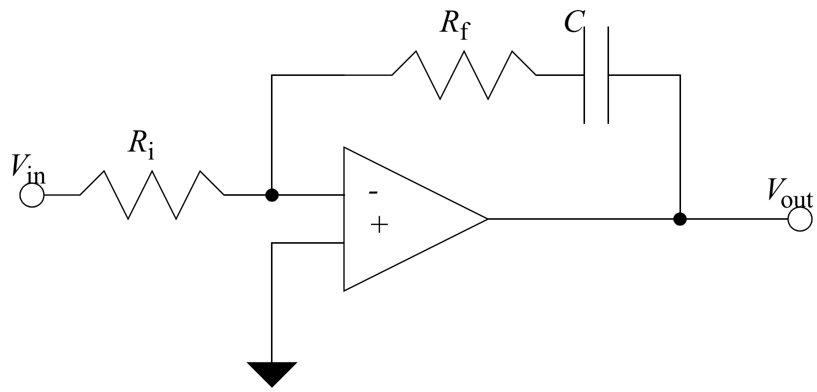
at t = 0s $V_o = 2V$ $V_o' = 3\frac{V}{s}$

Problem 7.8 Evaluate the circuit in Figure 7.17 using the node voltage method.

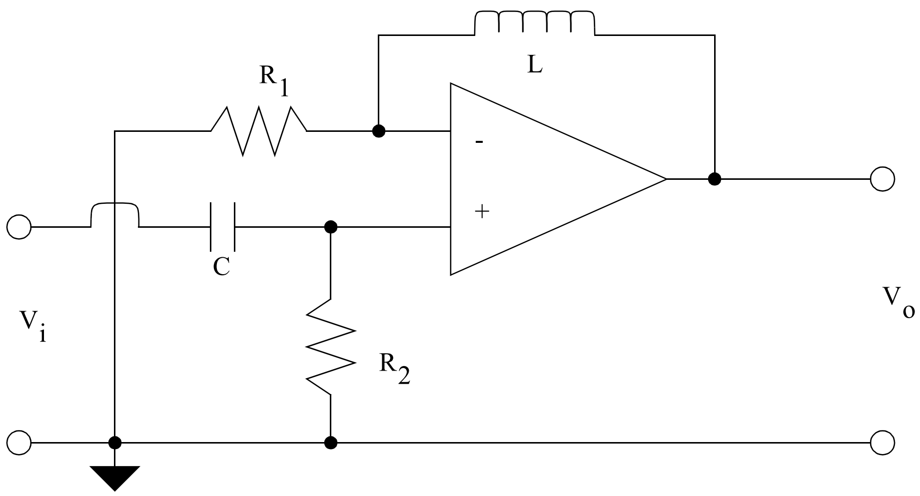
Problem 7.9 Develop a transfer function for the circuit.



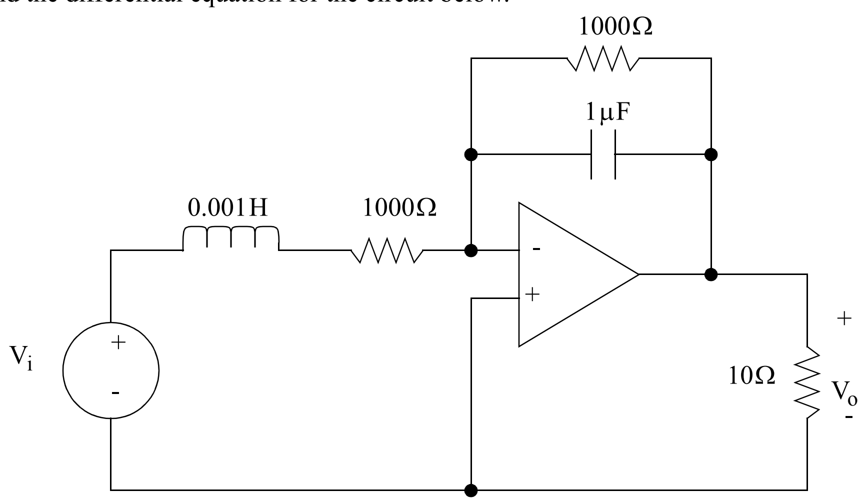
Problem 7.10 *Find the equation relating the output and input voltages,*



Problem 7.11 Find the transfer function for the system below.

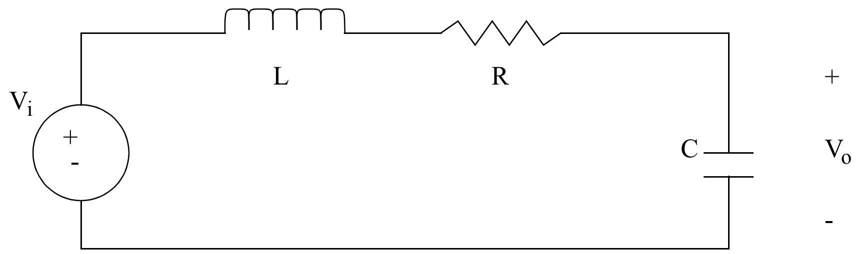


Problem 7.12 a) Find the differential equation for the circuit below.

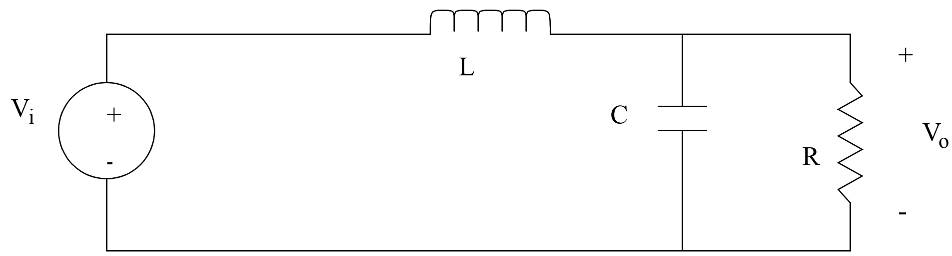


b) Put the differential equation in state variable form and a numerical method to produce a graph of the output voltage Vo using a computer. Assume the system starts at rest, and the input is Vi=5V.

Problem 7.13 Write the differential equation for the following circuit.

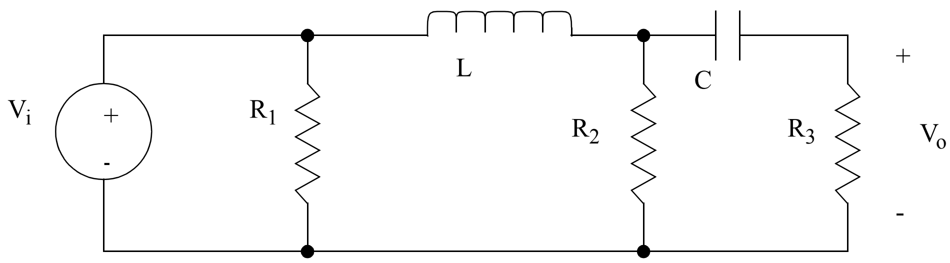


Problem 7.14 Consider the following circuit.

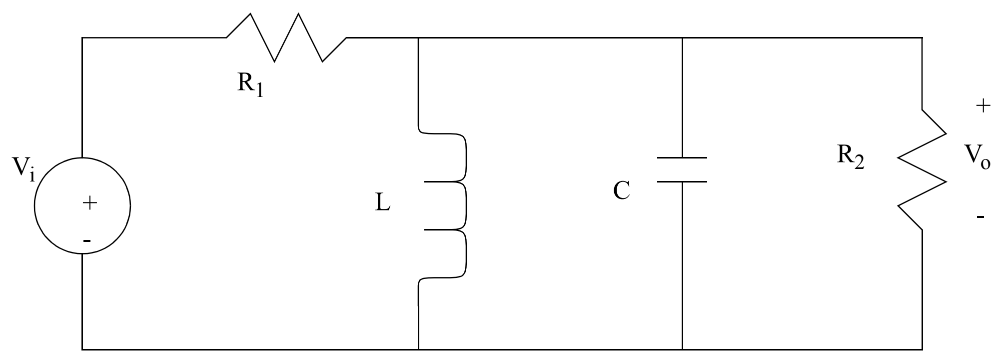


a) Develop a differential equation for the circuit.
b) Put the equation in state variable matrix form.

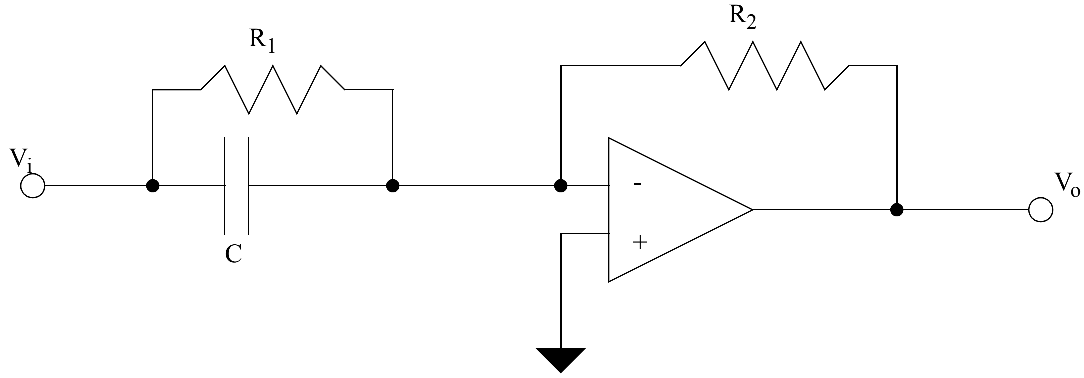
Problem 7.15 Consider the following circuit. Develop a differential equation for the circuit.



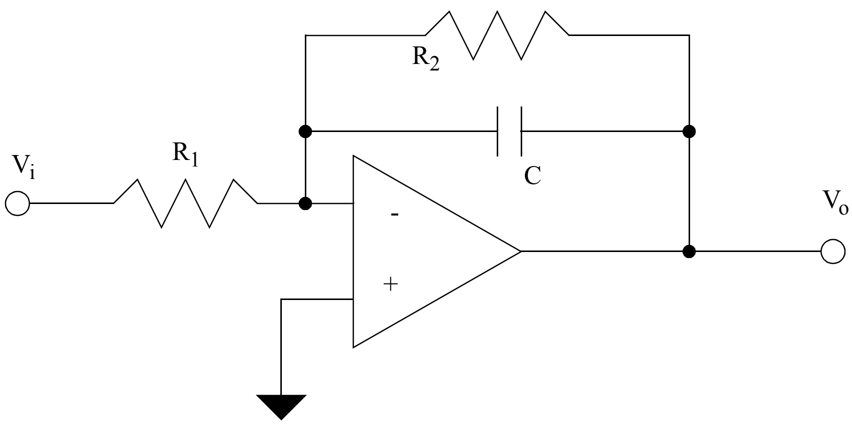
Problem 7.16 Find the input-output equation for the circuit below, and then find the natural frequency and damping factor.



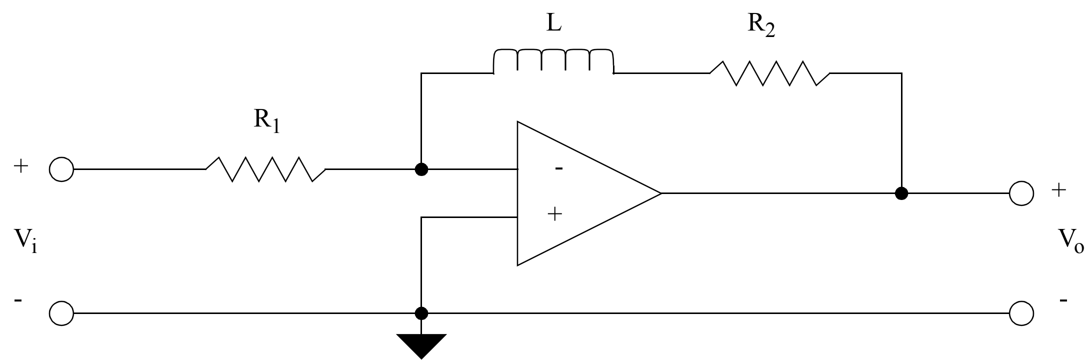
Problem 7.17 a) Write the differential equations for the system pictured below. b) Put the equations in input-output form.



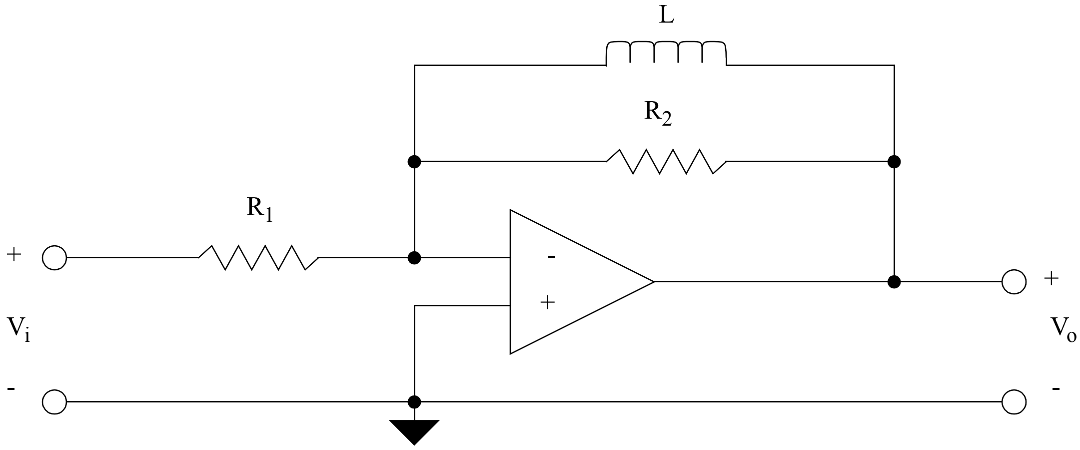
Problem 7.18 Given the circuit below, find the ratio of the output over the input (this is also known as a transfer function). Simplify the results.



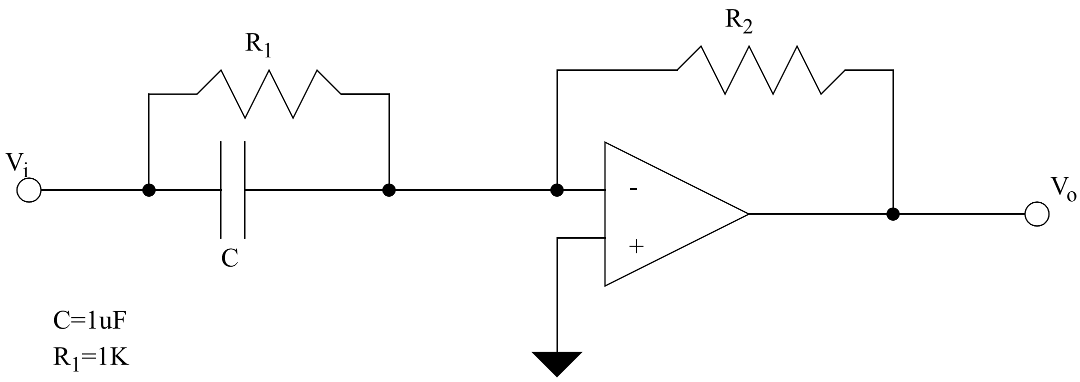
Problem 7.19 Examine the following circuit and then derive the differential equation.



Problem 7.20 Examine the following circuit and then derive the differential equation.



Problem 7.21 a) Write the differential equations for the system pictured below.
b) Put the equations in state variable form.
c) Use numerical methods to find the ratio between input and output voltages for a range of frequencies. The general method is put in a voltage such as Vi=1sin(___t), and see what the magnitude of the output is. Divide the magnitude of the output sine wave by the input magnitude. Note: This should act as a high pass or low pass filter.
d) Plot a graph of gain against the frequency of the input.



C=1uF
R_1=1K
R_2=1K

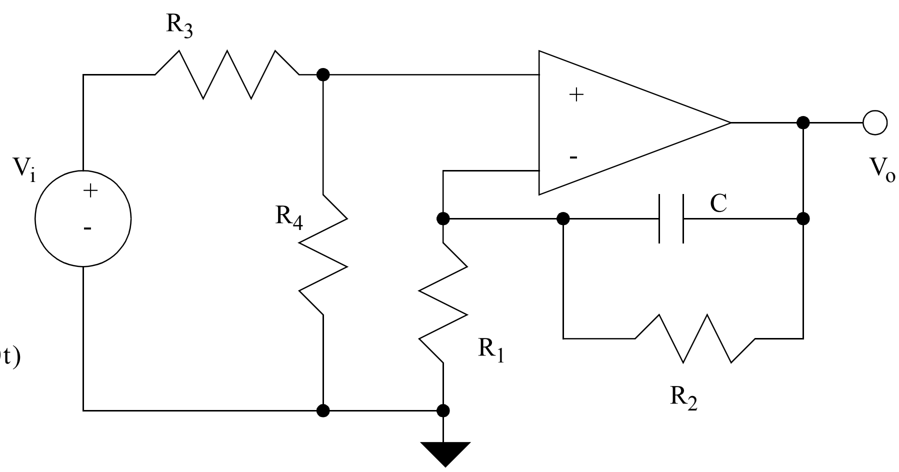
Problem 7.22 Develop the differential equation(s) for the system below, and use them to find the response to the following inputs. Assume that the circuit is off initially.

$R_1=R_2=R_3=R_4=1$Kohm
C=1uF

a) $V_i = 5\sin(100t)$

b) $V_i = 5\sin(1000000t)$

c) $V_i = 5$



Problem 7.23 Use the following differential equation to construct an op-amp circuit.

$$V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right)$$

7.6 Problem Solutions

Answer 7.1

$$R_{parallel} = \frac{R_1 R_2}{R_1 + R_2} \qquad R_{series} = R_1 + R_2$$

$$C_{series} = \frac{C_1 C_2}{C_1 + C_2} \qquad C_{parallel} = C_1 + C_2$$

$$L_{parallel} = \frac{L_1 L_2}{L_1 + L_2} \qquad L_{series} = L_1 + L_2$$

Answer 7.2

$$V_o = V_i\left(\frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}\right)$$

Answer 7.3

$$\frac{V_o}{V_i} = \frac{1}{5}$$

Answer 7.4

$$I_o = 2A$$

Answer 7.5

$$I(t) = 50 \times 10^{-6}\cos(10t)$$

Answer 7.6

$$I(t) = 5000\sin(10t)$$

Answer 7.7

a) Sum currents at node V0.

$$\sum I_{V_o} = \frac{(V_o - V_i)}{DL} + (V_o)DC + \frac{(V_o)}{R} = 0$$

$$V_o - V_i + V_o D^2 LC + \frac{V_o DL}{R} = 0$$

$$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

b) $$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

c)

$$\ddot{V}_o(LC) + \dot{V}_o\left(\frac{L}{R}\right) + V_o = V_i$$

$$\ddot{V}_o(10^{-2}10^{-6}) + \dot{V}_o\left(\frac{10^{-2}}{10^{3}}\right) + V_o = 10$$

$$\ddot{V}_o(10^{-8}) + \dot{V}_o(10^{-5}) + V_o = 10$$

$$\ddot{V}_o + \dot{V}_o(10^{3}) + V_o(10^{8}) = (10^{9})$$

homogeneous:

$$A^2e^{At} + Ae^{At}(10^3) + e^{At}(10^8) = 0$$

$$A^2 + A(10^3) + (10^8) = 0 \qquad A = \frac{-10^3 \pm \sqrt{(10^3)^2 - 4(10^8)}}{2} = -500 \pm 9987j$$

$$V_h = C_1e^{-500t}\cos(9987t + C_2)$$

particular:

$$V_p = A \qquad (0) + (0)(10^3) + (A)(10^8) = (10^9) \qquad A = 10$$

$$V_p = 10$$

initial conditions

$$V_o = C_1e^{-500t}\cos(9987t + C_2) + 10$$

for t=0, V0=2V

$$2 = C_1e^{-500(0)}\cos(9987(0) + C_2) + 10$$

$$-8 = C_1\cos(C_2) \qquad \text{eqn 7.1}$$

$$\dot{V}_o = -500C_1e^{-500t}\cos(9987t + C_2) - 9987C_1e^{-500t}\sin(9987t + C_2)$$

for t=0, d/dt V0=3V

$$3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$$

$$3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$$

$$3 = -4000 - 9987\left(\frac{-8}{\cos(C_2)}\right)\sin(C_2)$$

$$\frac{-3997}{8(9987)} = \frac{\sin(C_2)}{\cos(C_2)} = \tan(C_2) \qquad C_2 = -0.050$$

$$C_1 = \frac{-8}{\cos(C_2)} = -8.01$$

$$V_o = -8.01e^{-500t}\cos(9987t - 0.050) + 10$$

Answer 7.8

$$CL(R_1+R_2)I_2''' + L(CR_1^2 + 2CR_1R_2 + L)I_2'' + R_1(CR_1R_2 + 2L)I_2' + (R_1^2)I_2 = CLV''$$

$$\sum I_{V1} = \frac{V - V_1}{R_1} + \frac{0 - V_1}{LD} + \frac{V_o - V_1}{\left(\frac{1}{CD}\right)} = 0$$

$$V_1\left(\frac{1}{R_1} + \frac{1}{LD} + CD\right) = \frac{V}{R_1} + V_oCD$$

$$V_1 = V\left(\frac{LD}{LD + R_1 + R_1LCD^2}\right) + V_oR_1CD\left(\frac{LD}{LD + R_1 + R_1LCD^2}\right)$$

$$\sum I_{V2} = \frac{V_1 - V_o}{\left(\frac{1}{CD}\right)} + \frac{0 - V_o}{R_2} = 0$$

$$CDV_1 = V_o\left(CD + \frac{1}{R_2}\right)$$

$$V_1 = V_o\left(\frac{1 + CR_2D}{CDR_2}\right)$$

$$V_o\left(\frac{1 + CR_2D}{CDR_2}\right) = V\left(\frac{LD}{LD + R_1 + R_1LCD^2}\right) + V_oR_1CD\left(\frac{LD}{LD + R_1 + R_1LCD^2}\right)$$

$$V_o((1 + CR_2D)(LD + R_1 + R_1LCD^2)) = VLD^2CR_2 + V_oR_1C^2D^3LR_2$$

$$V_o\left(LD + R_1 + R_1LCD^2 + CR_2LD^2 + CR_2R_1D + CR_2R_1LCD^3\right) = VLD^2CR_2 + V_oR_1C^2D^3LR_2$$

$$V_o\left(D^2(R_1L(C) + CR_2L) + D(L + CR_2R_1) + (R_1)\right) = VLD^2CR_2$$

$$\ddot{V}_o(R_1L(C) + CR_2L) + \dot{V}_o(L + CR_2R_1) + V_o(R_1) = \ddot{V}(LCR_2)$$

$$\ddot{V}_o + \dot{V}_o\left(\frac{L + CR_2R_1}{LC(R_1 + R_2)}\right) + V_o\left(\frac{R_1}{LC(R_1 + R_2)}\right) = \ddot{V}\left(\frac{R_2}{R_1 + R_2}\right)$$

Answer 7.9

Method 1: Node Voltage

$$\frac{V_o - 0}{R} + \frac{V_o - V_{CR}}{LD} = 0$$

$$V_{CR} = V_o\left(\frac{LD}{R} + 1\right) = V_o\left(\frac{LD + R}{R}\right)$$

$$\frac{V_{CR} - V_i}{\left(\frac{1}{CD}\right)} + \frac{V_{CR} - 0}{R} + \frac{V_{CR} - V_o}{LD} = 0$$

$$(CDV_{CR} - CDV_i) + \frac{V_{CR}}{R} + \frac{V_{CR}}{LD} = V_o\left(\frac{1}{LD}\right)$$

$$V_o\left(\frac{1}{LD}\right) = V_{CR}\left(CD + \frac{1}{R} + \frac{1}{LD}\right) + V_i(-CD)$$

$$V_o\left(\frac{1}{LD}\right) = V_o\left(\frac{LD + R}{R}\right)\left(\frac{RCLD^2 + LD + R}{RLD}\right) + V_i(-CD)$$

$$V_o(R^2) = V_o(LD + R)(RCLD^2 + LD + R) + V_i\left(-CLR^2D^2\right)$$

$$V_o(RCL^2D^3 + L^2D^2 + RLD + R^2CLD^2 + RLD + R^2 - R^2) = V_i\left(CLR^2D^2\right)$$

$$\frac{V_o}{V_i} = \frac{D(CR^2)}{D^2(RCL) + D(L + R^2C) + (2R)}$$

Method 2: Simplification

$$Z_o = R + LD$$

$$Z_c = \frac{RZ_o}{R + Z_o} = \frac{R^2 + RLD}{2R + LD}$$

$$V_{CR} = V_i\left(\frac{Z_c}{\left(\frac{1}{CD}\right) + Z_c}\right) = V_i\left(\frac{CDZ_c}{1 + CDZ_c}\right) = V_i\left(\frac{CD(R^2 + RLD)}{2R + LD + CD(R^2 + RLD)}\right)$$

$$V_o = V_{CR}\left(\frac{R}{R + LD}\right) = V_i\left(\frac{CD(R^2 + RLD)}{2R + LD + CD(R^2 + RLD)}\right)\left(\frac{R}{R + LD}\right)$$

$$\frac{V_o}{V_i} = \frac{D(R^2C)}{D^2(CLR) + D(CR^2 + L) + (2R)}$$

Answer 7.10

$$\dot{V}_o = \frac{V_i}{CR_i} - \frac{R_f}{R_i}\dot{V}_i$$

Answer 7.11

For Vi use a voltage divider,

$$V_+ = V_i\left(\frac{R_2}{R_2 + \frac{1}{CD}}\right) = V_i\left(\frac{R_2CD}{R_2CD+1}\right)$$

Sum currents at V-

$$\frac{(V_o - V_-)}{LD} + \left(\frac{0 - V_-}{R_1}\right) = 0$$

$$V_-\left(\frac{1}{R_1} + \frac{1}{LD}\right) = V_o\left(\frac{1}{LD}\right)$$

$$V_- = V_o\left(\frac{1}{LD}\right)\left(\frac{R_1LD}{R_1+LD}\right) = V_o\left(\frac{R_1}{R_1+LD}\right)$$

Combine

$$V_+ = V_- = V_i\left(\frac{R_2CD}{R_2CD+1}\right) = V_o\left(\frac{R_1}{R_1+LD}\right)$$

$$\frac{V_o}{V_i} = \left(\frac{R_2CD}{R_2CD+1}\right)\left(\frac{R_1+LD}{R_1}\right)$$

Answer 7.12 a)

Create a node between the inductor and resistor Va, and use the node voltage method

$$\sum I_{V_A} = \frac{(V_A - V_i)}{0.001D} + \frac{(V_A - V_-)}{1000} = 0 \qquad V_- = V_+ = 0V$$

$$1000000(V_A - V_i) + V_AD = 0$$

$$V_A = V_i\left(\frac{1000000}{1000000+D}\right)$$

$$\sum I_{V_-} = \frac{(V_- - V_A)}{1000} + \frac{(V_- - V_o)}{1000} + (V_- - V_o)(0.000001D) = 0$$

$$\frac{(-1)}{1000}V_i\left(\frac{1000000}{1000000+D}\right) + \frac{(-V_o)}{1000} + (-V_o)(0.000001D) = 0$$

$$V_o(-1 - 0.001D) = V_i\left(\frac{1000000}{1000000+D}\right)$$

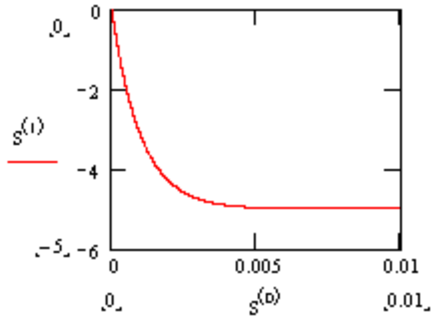
$$V_o(-1000000 - D - 1000D - 0.001D^2) = 1000000V_i$$

$$V_o''(-10^{-9}) + V_o'(-1.001(10^{-3})) + V_o(-1) = V_i$$

b)

$$\frac{d}{dt}V_o = \dot{Y}_o$$

$$\frac{d}{dt}\dot{Y}_o = -1000000000V_i - 1001000Y_o - 1000000000V_o$$



Answer 7.13

$$\ddot{V}_o + \dot{V}_o\left(\frac{R}{L}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

Answer 7.14

a) $$\ddot{V}_o + \dot{V}_o\left(\frac{1}{RC}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

b) $$\dot{V}_o = X_o$$

$$\dot{X}_o = X_o\left(\frac{-1}{RC}\right) + V_o\left(\frac{-1}{LC}\right) + V_i\left(\frac{1}{LC}\right)$$

Answer 7.15

$$\ddot{V}_o + \dot{V}_o\left(\frac{R_2R_3C + L}{LC(R_2 + R_3)}\right) + V_o\left(\frac{R_2}{LC(R_2 + R_3)}\right) = \dot{V}_i\left(\frac{R_2R_3}{L(R_2 + R_3)}\right)$$

Answer 7.16

a) $$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR_1} + \frac{1}{CR_2}\right) + V_o\left(\frac{1}{LC}\right) = \dot{V}_i\left(\frac{1}{CR_1}\right)$$

b) $$\omega_n = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{\sqrt{L}(R_1 + R_2)}{2\sqrt{C}R_1R_2}$$

Answer 7.17

a) $$V_i + \dot{V}_i(R_1C) + V_o\left(\frac{R_1}{R_2}\right) = 0$$

b) $$V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right)$$

Answer 7.18

$$\frac{V_o}{V_i} = \frac{-R_2}{R_1 + DR_1R_2C}$$

Answer 7.19

$$V_o = \dot{V}_i\left(\frac{-L}{R_1}\right) + V_i\left(\frac{-R_2}{R_1}\right)$$

Answer 7.20

$$\dot{V}_o + V_o\left(\frac{R_2}{L}\right) = \dot{V}_i\left(\frac{-R_2}{R_1}\right)$$

Answer 7.21

a) $V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right)$

b) Not a state equation

c) assume Vi=1sin(wt)

$$V_o = \sqrt{(-1)^2 + (-10^{-3}\omega)^2}\sin(\omega t + \operatorname{atan}(-1, -10^{-3}\omega))$$

d) $GAIN = \sqrt{1^2 + (10^{-3}\omega)^2}$

Answer 7.22

$$\dot{V}_o + V_o(10^3) = \dot{V}_i(0.5) + V_i(10^3)$$

a) $V_o(t) = 0.2475e^{-10^3 t} + 4.9813\sin(100t - 0.0497)$

b) $V_o(t) = 2500e^{-10^3 t} + 2500\sin(10^6 t - 1.569)$

c) $V_o(t) = -5e^{-10^3 t} + 5$

$$V_o(t) = -2.5e^{-10^3 t} + 5$$

Answer 7.23

$$V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right) = V_i(-CDR_2) + V_i\left(\frac{-R_2}{R_1}\right)$$

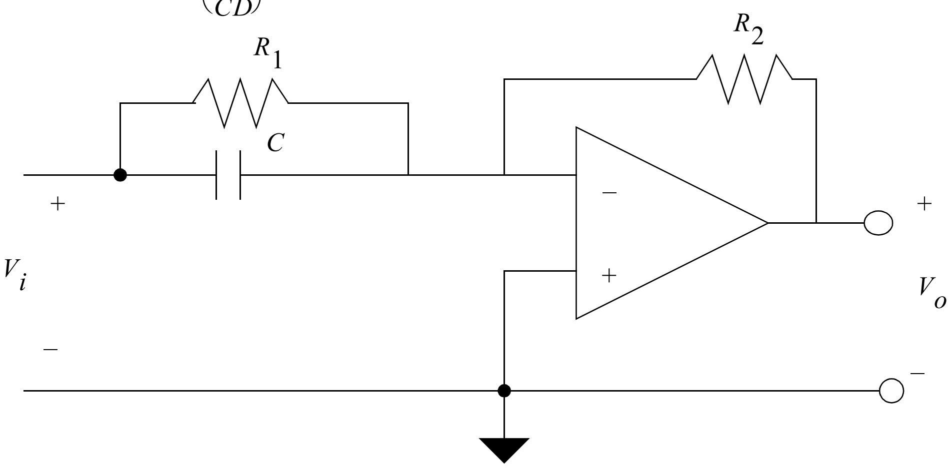
An arbitrary choice to begin, assume the node voltage method. Each term should be voltage over impedance.

$$\frac{V_o}{R_2} = \frac{-V_i}{\left(\frac{1}{CD}\right)} + V_i\left(\frac{-R_2}{R_1R_2}\right) = \frac{-V_i}{\left(\frac{1}{CD}\right)} + \frac{-V_i}{R_1}$$

$$\frac{V_o}{R_2} + \frac{V_i}{\left(\frac{1}{CD}\right)} + \frac{V_i}{R_1} = 0$$

It is worth noting that there are only two variables, and the node voltage is zero. We commonly see this with an op-amp circuit.

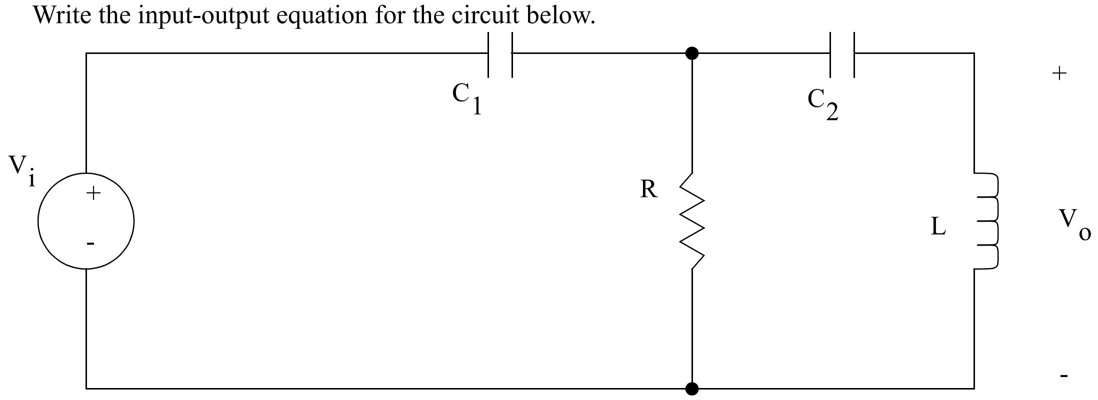
$$\frac{V_o - 0}{R_2} + \frac{V_i - 0}{\left(\frac{1}{CD}\right)} + \frac{V_i - 0}{R_1} = 0$$



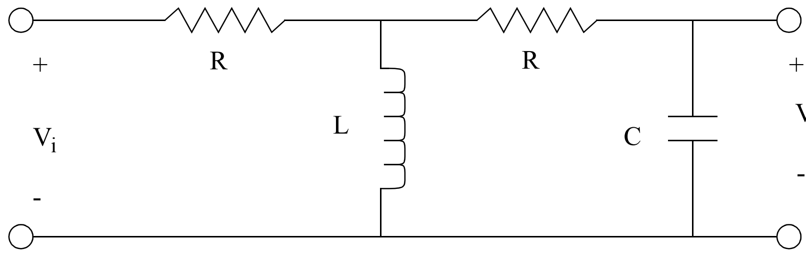
7.1 Problems Without Solutions

Problem 7.24 Evaluate the voltage divider in Figure 7.3 using the current loop method. Hint: Put a voltage supply on the left, and an output resistor on the right. Remember that the output resistance should be infinite.

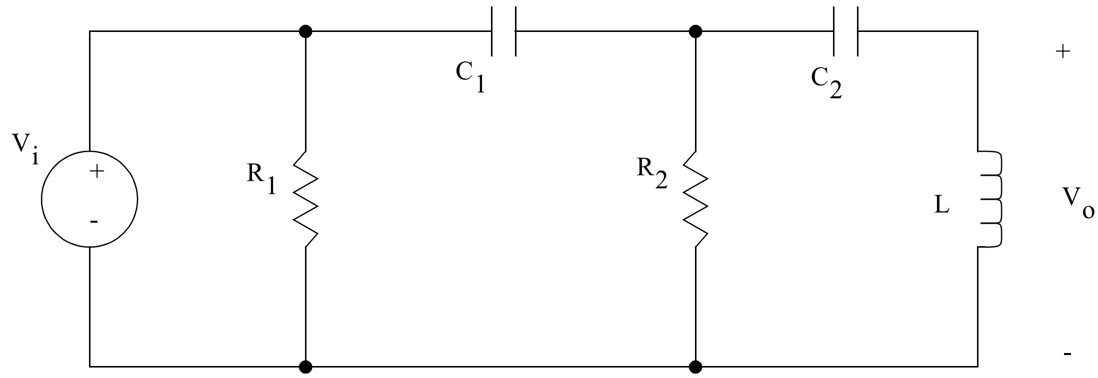
Problem 7.25 Write the input-output equation for the circuit below.



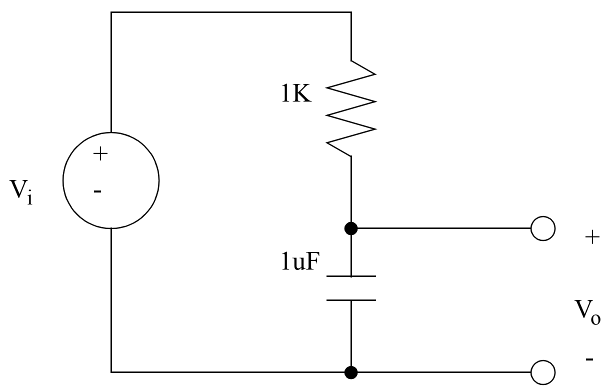
Problem 7.26 Write the differential equation for the following schematic.



Problem 7.27 Write the input-output equation for the circuit below.



Problem 7.28 Study the circuit below. Assume that for t<0s the circuit is discharged and off. Starting at t=0s an input of Vi=5sin(100,000t) is applied.



a) Write a differential equation for the circuit.

b) Find the output of the circuit using explicit integration (i.e., homogeneous and particular solutions).

Answer 7.28

$$V_o = V_i\left(\frac{\left(\frac{1}{CD}\right)}{R+\left(\frac{1}{CD}\right)}\right) = V_i\left(\frac{1}{CDR+1}\right) = V_i\left(\frac{1}{1+D10^{-3}}\right)$$

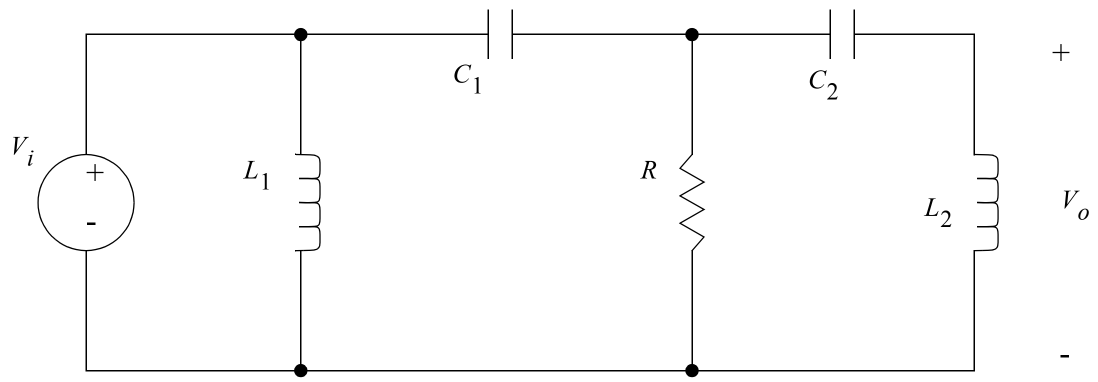
$$V_o(1+D10^{-3}) = V_i$$

$$\dot{V}_o + 1000V_o = 1000V_i$$

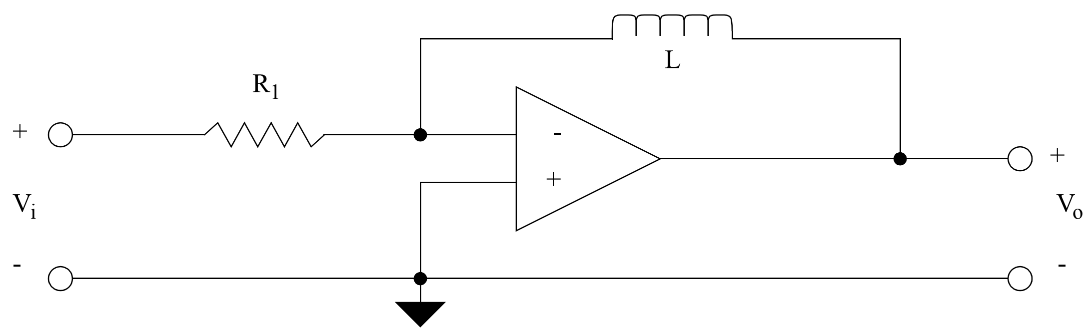
$$\dot{V}_o + 1000V_o = 1000(5\sin(100000t))$$

$$V_o(t) = \frac{5000}{10001}e^{-1000t} + \frac{\sqrt{1^2+100^2}}{10001}\sin\left(100000t + \operatorname{atan}\left(\frac{-100}{1}\right)\right)$$

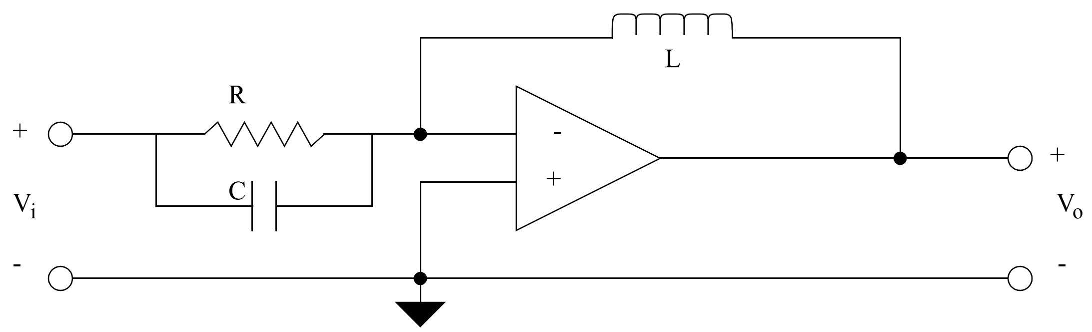
Problem 7.29 Write the input-output equation for the circuit below.



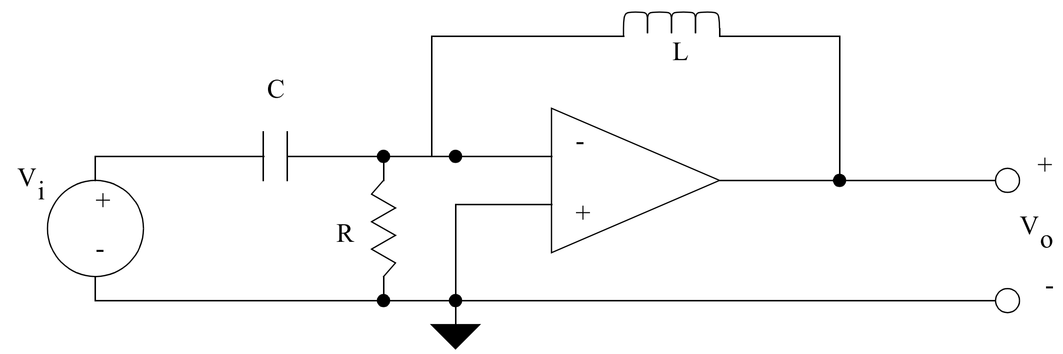
Problem 7.30 Write the differential equation for the following schematic.



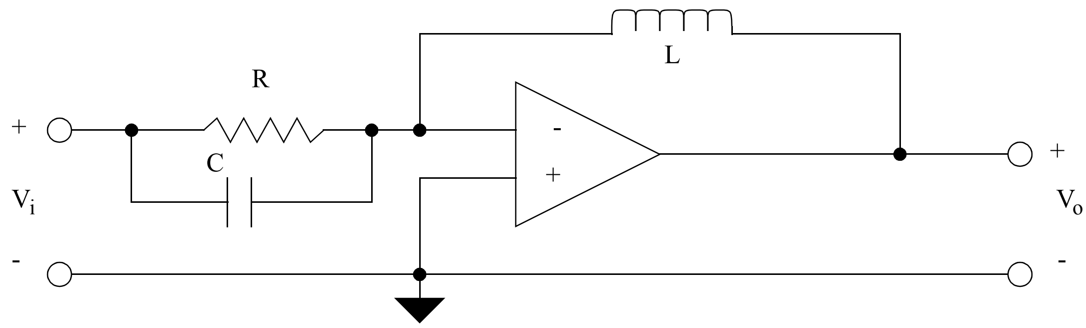
Problem 7.31 Write the differential equation for the following schematic.



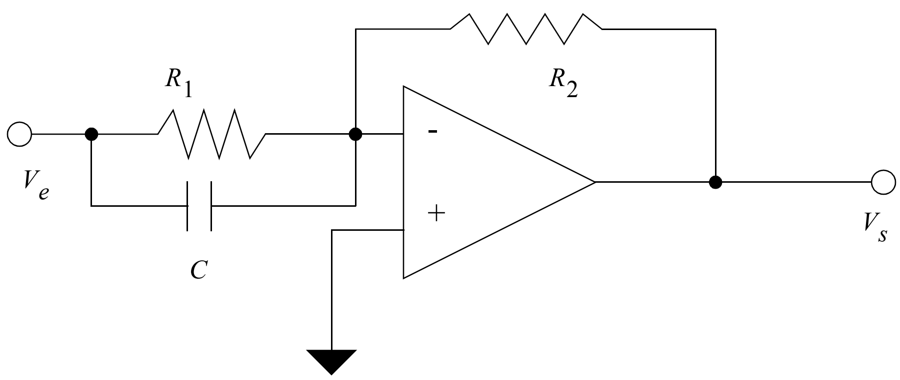
Problem 7.32 Write the input-output equation for the circuit below.



Problem 7.33 Write the input-output equation for the circuit below.



Problem 7.34 Write the transfer function for the circuit.



8. System Block Diagrams

Topic 8.1 *Transfer functions, block diagrams and simplification.*
Objective 8.1 *To be able to select controller parameters to meet design objectives.*

Every engineered component has some function. A function can be described as a transformation of inputs to outputs. For example it could be an amplifier that accepts a signal from a sensor and amplifies it. Or, consider a mechanical gear box with an input and output shaft. A manual transmission has an input shaft from the motor and from the shifter. When analyzing systems we will often use transfer functions that describe a system as a ratio of output to input.

8.1 Transfer Functions

Transfer functions are used for equations with one input and one output variable. An example of a transfer function is shown below in Figure 8.1. The general form calls for output over input on the left hand side. The right hand side is comprised of constants and the 'D' operator. In the example 'x' is the output, while 'F' is the input.

The general form

$$\frac{output}{input} = f(D)$$

An example

$$\frac{x}{F} = \frac{4 + D}{D^2 + 4D + 16}$$

Figure 8.1 A transfer function example

If both sides of the example were inverted then the output would become 'F', and the input 'x'. This ability to invert a transfer function is called reversibility. In reality many systems are not reversible.

There is a direct relationship between transfer functions and differential equations. This is shown for the second-order differential equation in Figure 8.2. The homogeneous equation (the left hand side) ends up as the denominator of the transfer function. The non-homogeneous solution ends up as the numerator of the expression.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f}{M}$$

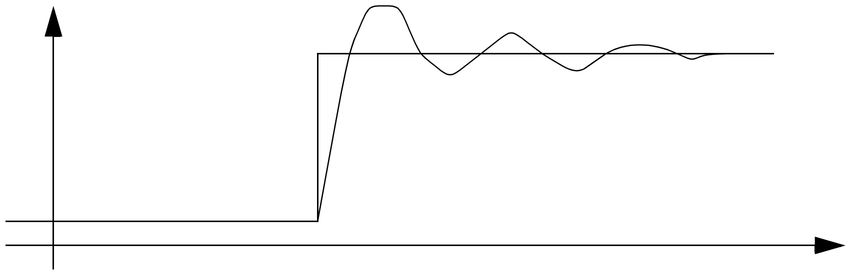
$$xD^2 + 2\zeta\omega_n xD + \omega_n^2 x = \frac{f}{M}$$

$$x(D^2 + 2\zeta\omega_n D + \omega_n^2) = \frac{f}{M}$$

$$\frac{x}{f} = \frac{\left(\frac{1}{M}\right)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

particular (numerator)

homogeneous (denominator)



ω_n	Natural frequency of system - Approximate frequency of control system oscillations.
ξ	Damping factor of system - If < 1 then underdamped, and the system will oscillate. If =1 critically damped. If > 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.

Figure 8.2 The relationship between transfer functions and differential equations for a mass-spring-damper example

The transfer function for a first-order differential equation is shown in Figure 8.3. As before the homogeneous and non-homogeneous parts of the equation becomes the denominator and the numerator of the transfer function.

$$\dot{x} + \frac{1}{\tau}x = f$$

$$xD + \frac{1}{\tau}x = f$$

$$x\left(D + \frac{1}{\tau}\right) = f$$

$$\frac{x}{f} = \frac{1}{D + \frac{1}{\tau}}$$

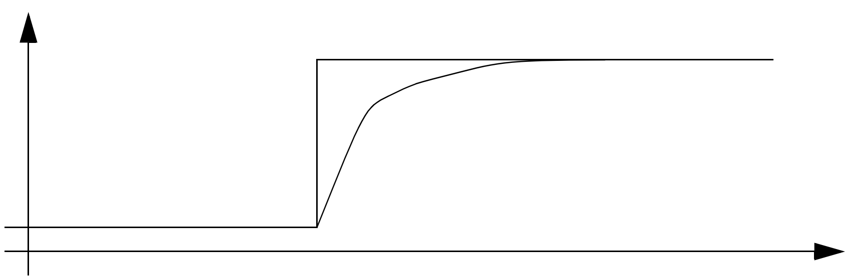


Figure 8.3 A first-order system response

8.2 Control Systems

Figure 8.4 shows a transfer function block for a car. The input, or control variable is the gas pedal angle. The system output, or result, is the velocity of the car. In standard operation the gas pedal angle is controlled by the driver. When a cruise control system is engaged the gas pedal must automatically be adjusted to maintain a desired velocity setpoint. To do this a control system is added, in this figure it is shown inside the dashed line. In this control system the output velocity is subtracted from the setpoint to get a system error. The subtraction occurs in the summation block (the circle on the left hand side). This error is used by the controller function to adjust the control variable in the system. Negative feedback is the term used for this type of controller.

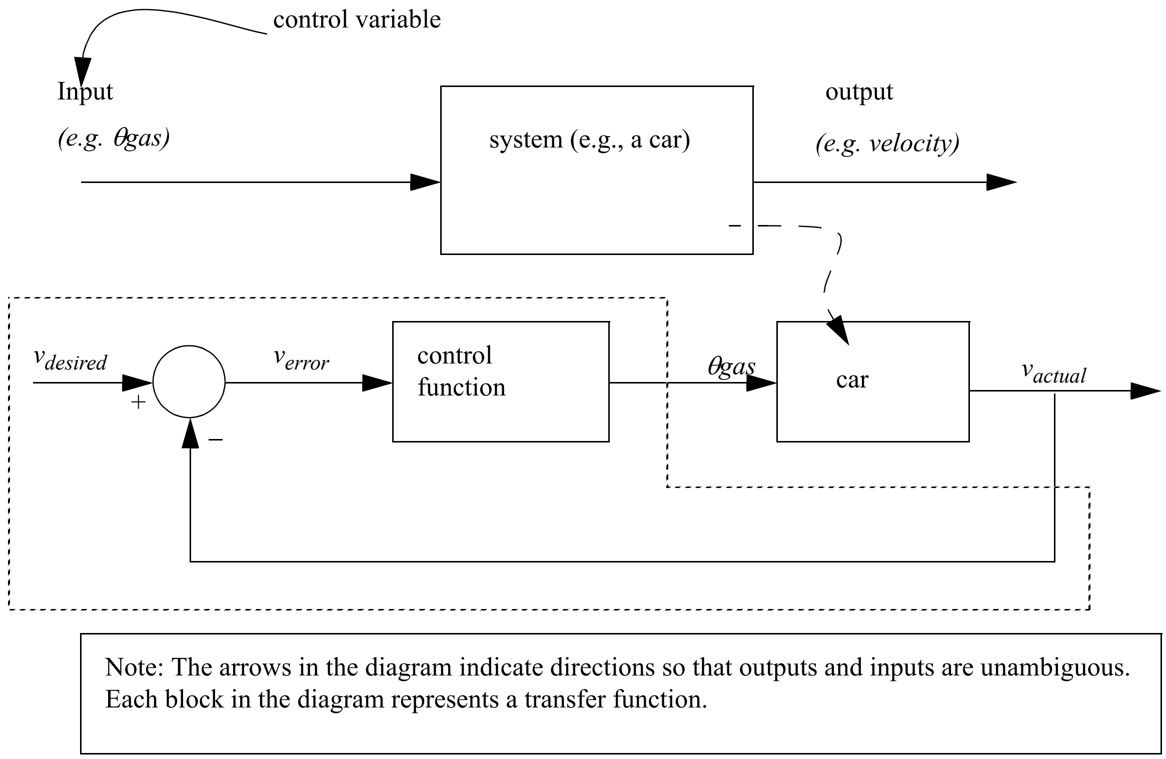


Figure 8.4 An automotive cruise control system

There are two main types of feedback control systems: negative feedback and positive feedback. In a positive feedback control system the setpoint and output values are added. In a negative feedback control the setpoint and output values are subtracted. As a rule negative feedback systems are more stable than positive feedback systems. Negative feedback also makes systems more immune to random variations in component values and inputs.

The control function in Figure 8.4 can be defined many ways. A possible set of rules for controlling the system is given in Figure 8.5. Recall that the system error is the difference between the setpoint and actual output. When the system output matches the setpoint the error is zero. Larger differences between the setpoint and output will result in larger errors. For example if the desired velocity is 50mph and the actual velocity 60mph, the error is -10mph, and the car should be slowed down. The rules in the figure give a general idea of how a control function might work for a cruise control system.

Human rules to control car (also like expert system/fuzzy logic):

1. *If* v_{error} *is not zero, and has been positive/negative for a while, increase/decrease* θ_{gas}
2. *If* v_{error} *is very big/small increase/decrease* θ_{gas}
3. *If* v_{error} *is near zero, keep* θ_{gas} *the same*
4. *If* v_{error} *suddenly becomes bigger/smaller, then increase/decrease* θ_{gas}.
5. *etc.*

Figure 8.5 Example control rules

In following sections we will examine mathematical control functions that are easy to implement in actual control systems.

PID Control Systems

The Proportional Integral Derivative (PID) control function shown in Figure 8.6 is the most popular choice in industry. In the equation given the 'e' is the system error, and there are three separate gain constants for the three terms. The result is a control variable value.

$$u = K_p e + K_i \int e dt + K_d\left(\frac{de}{dt}\right)$$

Figure 8.6 A PID controller equation

Figure 8.7 shows a basic PID controller in block diagram form. In this case the potentiometer on the left is used as a voltage divider, providing a setpoint voltage. At the output the motor shaft drives a potentiometer, also used as a voltage divider. The voltages from the setpoint and output are subtracted at the summation block to calculate the feedback error. The resulting error is used in the PID function. In the proportional branch the error is multiplied by a constant, to provide a long term output for the motor (a ballpark guess). If an error is largely positive or negative for a while the integral branch value will become large and push the system towards zero. When there is a sudden change in the error value the differential branch will give a quick response. The results of all three branches are added together in the second summation block. This result is then amplified to drive the motor. The overall performance of the system can be changed by adjusting the gains in the three branches of the PID function.

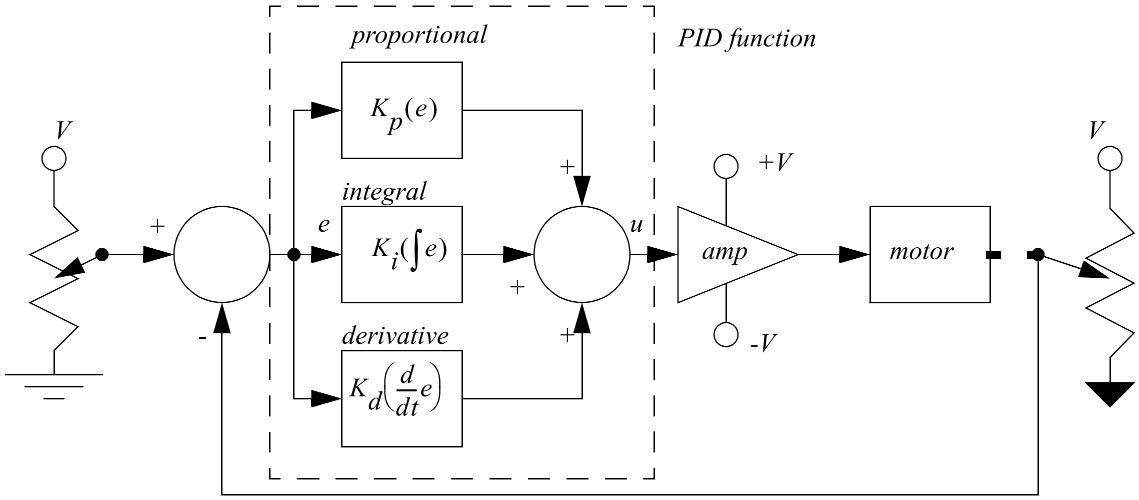


Figure 8.7 A PID control system

There are other variations on the basic PID controller shown in Figure 8.8. A PI controller results when the derivative gain is set to zero. (Recall the second order response.) This controller is generally good for eliminating long term errors, but it is prone to overshoot. In a P controller only the proportional gain in non-zero. This controller will generally work, but often cannot eliminate errors. The PD controller does not deal with long term errors, but is very responsive to system changes.

For a PI Controller

$$\theta_{gas} = K_p v_{error} + K_i \int v_{error} dt$$

For a P Controller

$$\theta_{gas} = K_p v_{error}$$

For a PD Controller

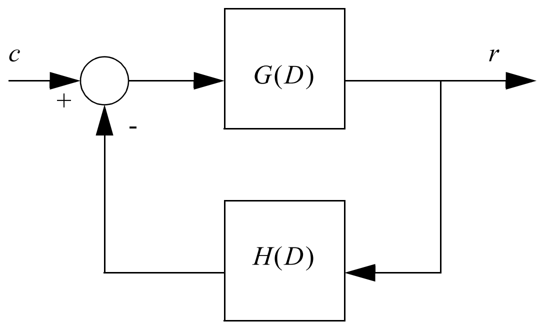
$$\theta_{gas} = K_p v_{error} + K_d\left(\frac{dv_{error}}{dt}\right)$$

Figure 8.8 Some other control equations

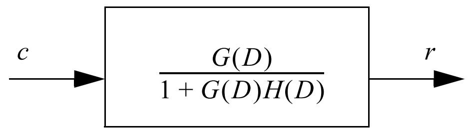
Aside: The manual process for tuning a PID controlled is to set all gains to zero. The proportional gain is then adjusted until the system is responding to input changes without excessive overshoot. After that the integral gain is increased until the long term errors disappear. The differential gain will be increased last to make the system respond faster.

Manipulating Block Diagrams

A block diagram for a system is not unique, meaning that it may be manipulated into new forms. Typically a block diagram will be developed for a system. The diagram will then be simplified through a process that is both graphical and algebraic. For example, equivalent blocks for a negative feedback loop are shown in Figure 8.9, along with an algebraic proof.



is equal to



$$r = G(D)e$$

$$r = G(D)(c - H(D)r)$$

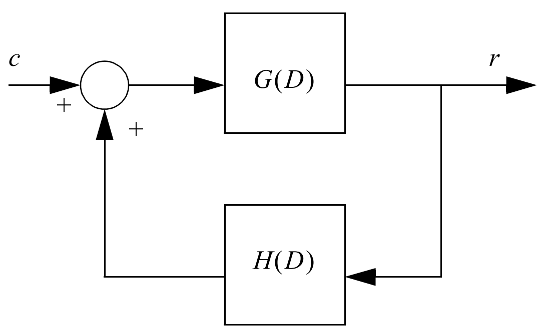
$$r(1 + G(D)H(D)) = G(D)c$$

$$r = \left(\frac{G(D)}{1 + G(D)H(D)}\right)c$$

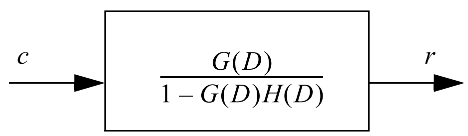
Note: Units can be used to verify the feedback loop calculations.

Figure 8.9 A negative feedback block reduction

Other block diagram equivalences are shown in Figure 8.10 to Figure 8.16. In all cases these operations are reversible. Proofs are provided, except for the cases where the equivalence is obvious.



is equal to



$$r = G(D)e$$

$$r = G(D)(c + H(D)r)$$

$$r(1 - G(D)H(D)) = G(D)c$$

$$r = \left(\frac{G(D)}{1 - G(D)H(D)}\right)c$$

Figure 8.10 A positive feedback block reduction

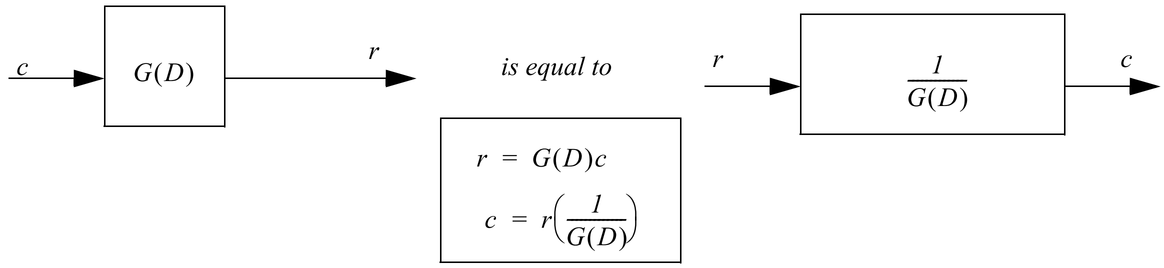


Figure 8.11 Reversal of function blocks

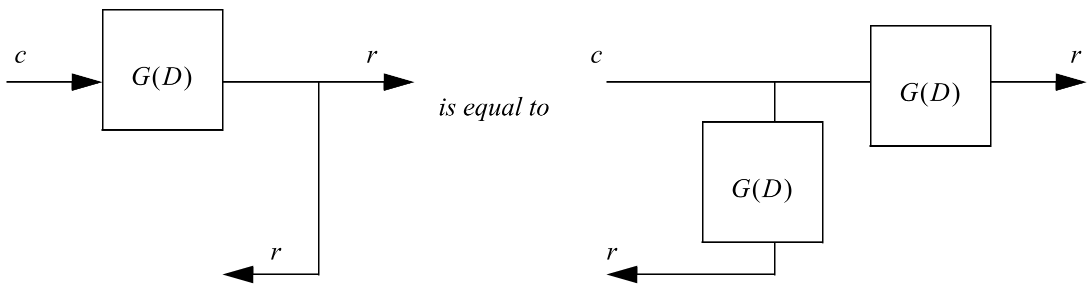


Figure 8.12 Moving branches before blocks

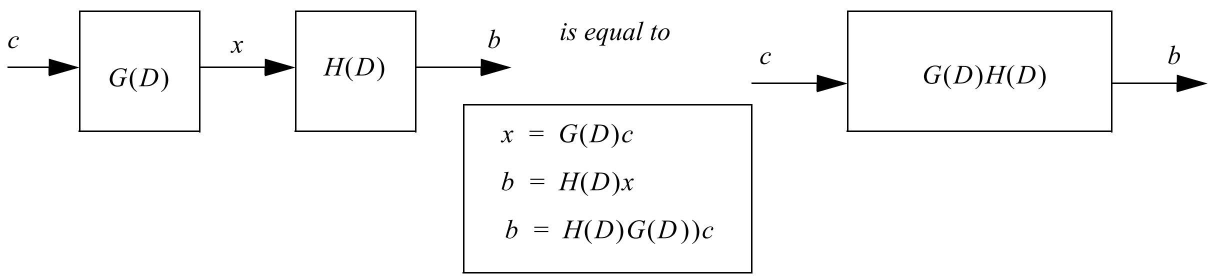


Figure 8.13 Combining sequential function blocks

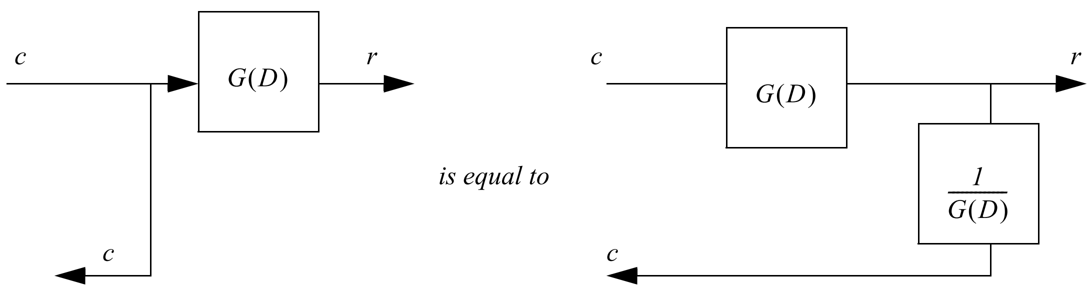


Figure 8.14 Moving branches after blocks

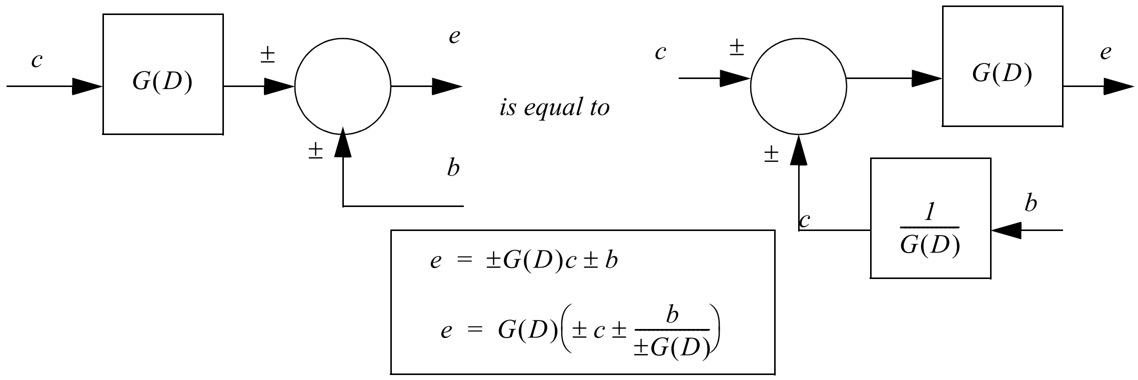


Figure 8.15 Moving summation functions before blocks

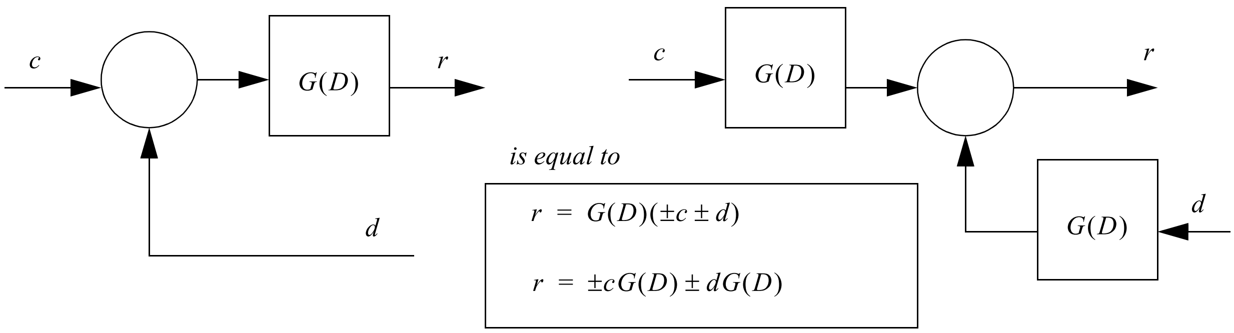


Figure 8.16 Moving summation function past blocks

Recall the example of a cruise control system for an automobile presented in Figure 8.4. This example is extended in Figure 8.17 to include mathematical models for each of the function blocks. This block diagram is first simplified by multiplying the blocks in sequence. The feedback loop is then reduced to a single block. Notice that the feedback line doesn't have a function block on it, so by default the function is '1' - everything that goes in, comes out.

e.g. The block diagram of the car speed control system

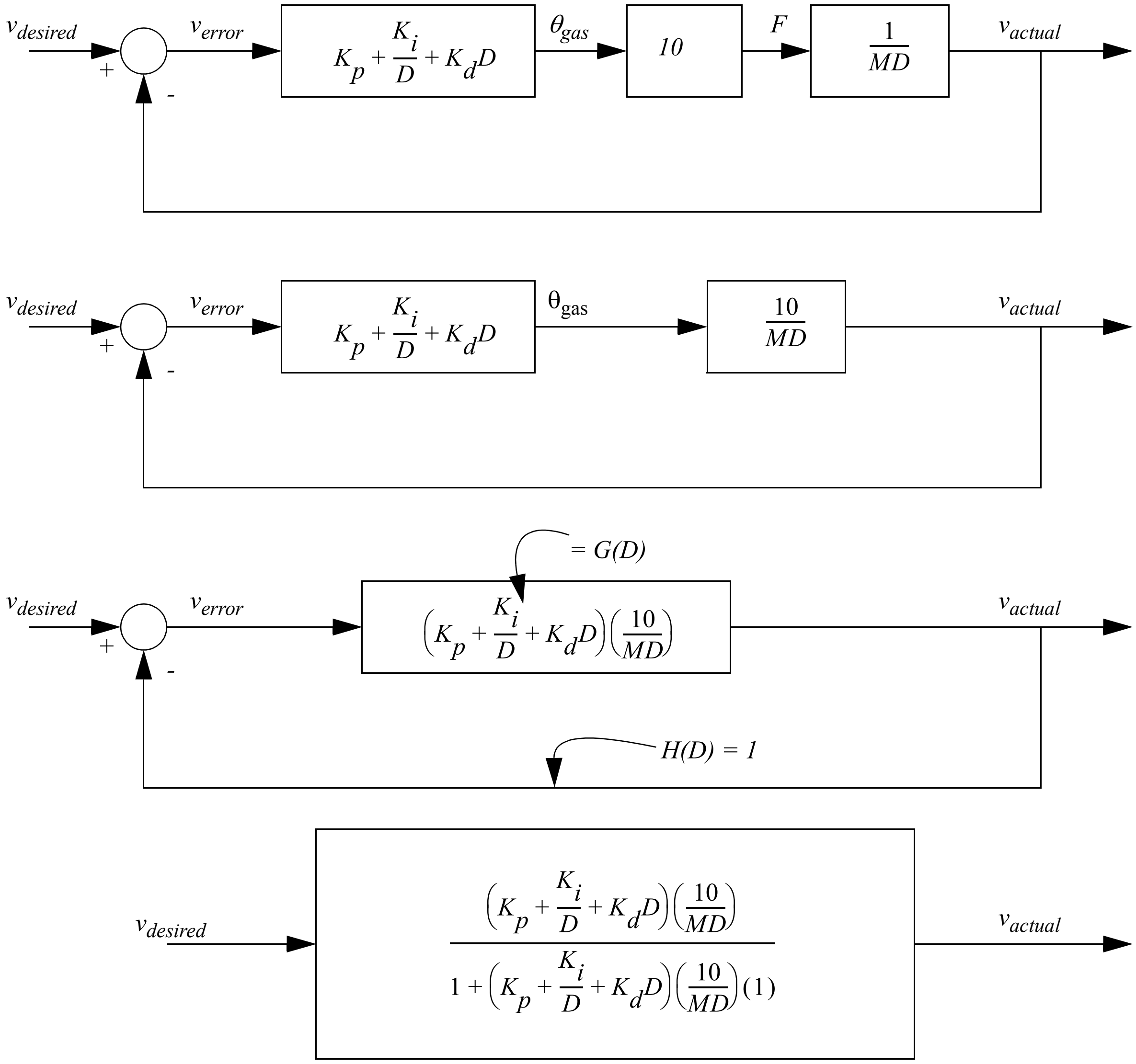


Figure 8.17 An example of simplifying a block diagram

The function block is further simplified in Figure 8.18 to a final transfer function for the whole system.

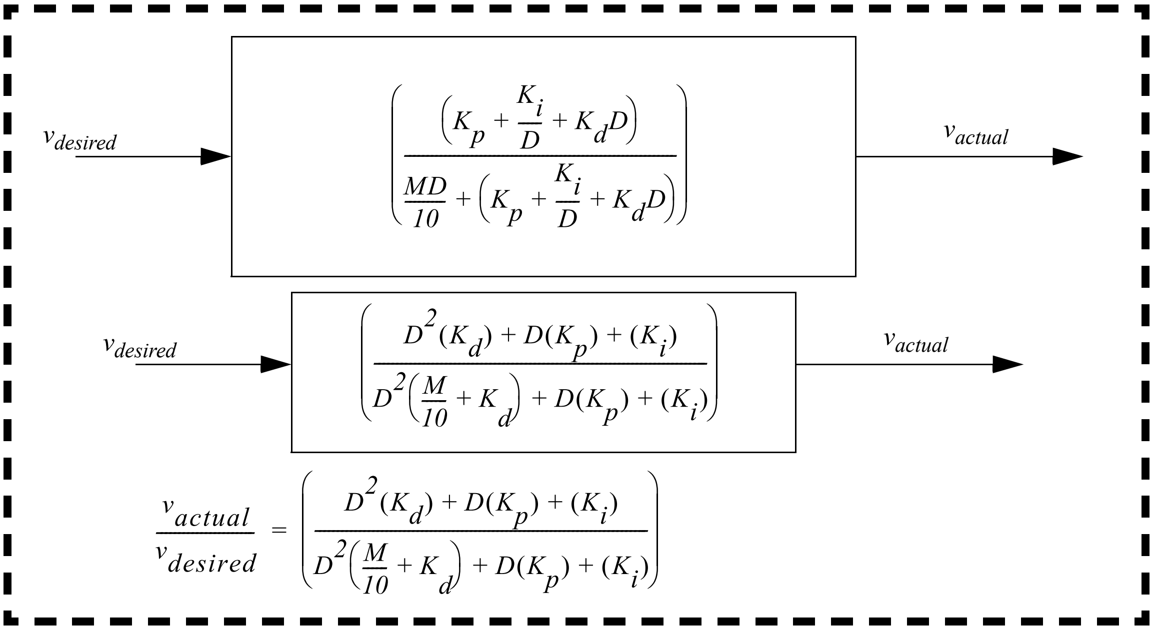


Figure 8.18 An example of simplifying a block diagram (continued)

A Motor Control System Example

Consider the example of a DC servo motor controlled by a computer. The purpose of the controller is to position the motor. The system in Figure 8.19 shows a reasonable control system arrangement. Some elements such as power supplies and commons for voltages are omitted for clarity.

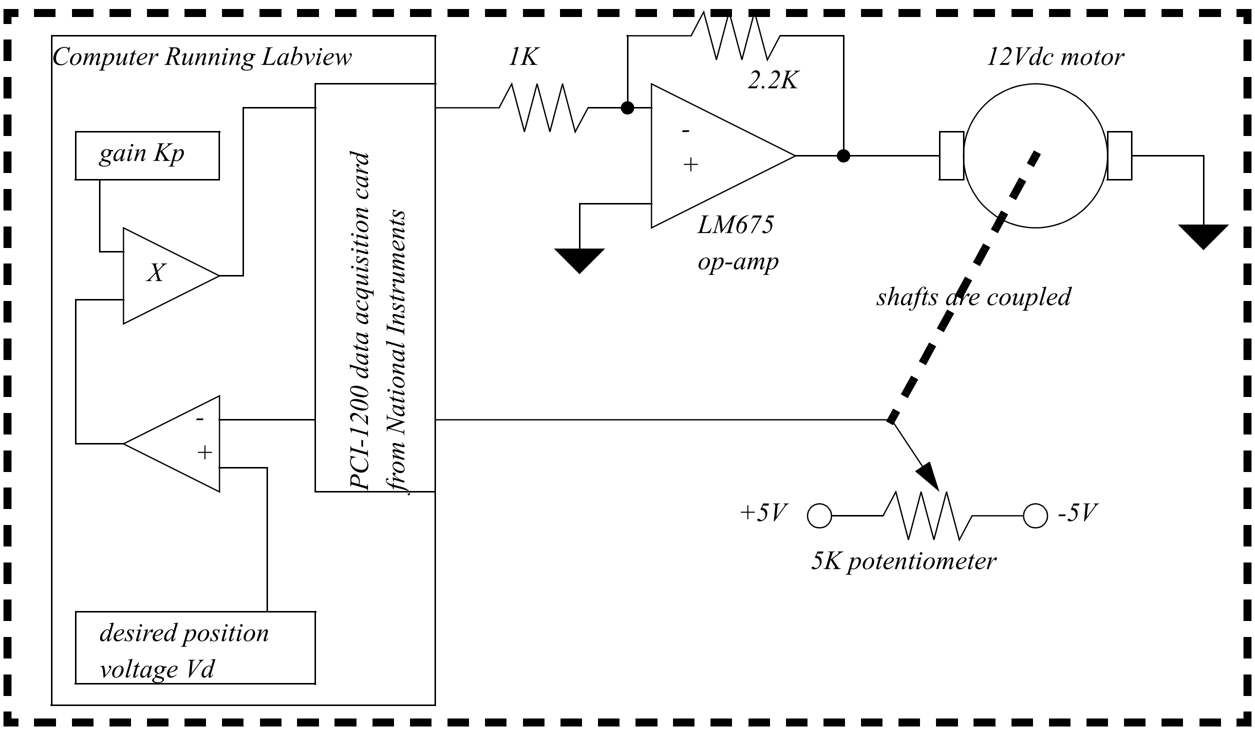


Figure 8.19 A motor feedback control system

The feedback controller can be represented with the block diagram in Figure 8.20.

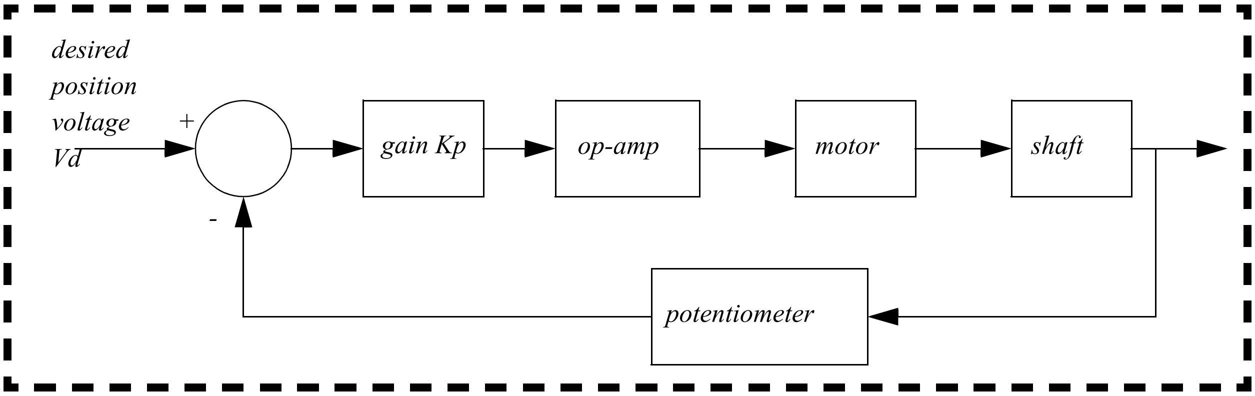


Figure 8.20 A block diagram for the feedback controller

The transfer functions for each of the blocks are developed in Figure 8.21. Two of the values must be provided by the system user. The op-amp is basically an inverting amplifier with a fixed gain of -2.2 times. The potentiometer is connected as a voltage divider and the equation relates angle to voltage. Finally the velocity of the shaft is integrated to give position.

Given or selected values:

- *desired potentiometer voltage Vd*
- *gain K*

For the op-amp:

$$\sum I_{V+} = \frac{V_+ - V_i}{1K} + \frac{V_+ - V_o}{2.2K} = 0 \qquad V_+ = V_- = 0V$$

$$\frac{-V_i}{1K} + \frac{-V_o}{2.2K} = 0$$

$$\frac{V_o}{V_i} = -2.2$$

For the potentiometer assume that the potentiometer has a range of 10 turns and 0 degrees is in the center of motion. So there are 5 turns in the negative and positive direction.

$$V_o = 5V\left(\frac{\theta}{5(2\pi)}\right)$$

$$\frac{V_o}{\theta} = 0.159 Vrad^{-1}$$

For the shaft, it integrates the angular velocity into position:

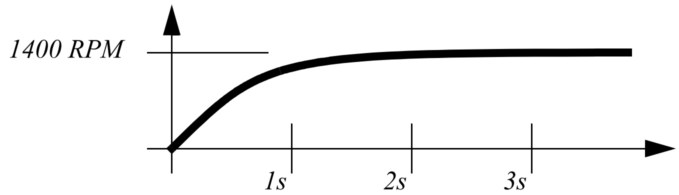
$$\omega = \frac{d}{dt}\theta$$

$$\frac{\theta}{\omega} = \frac{1}{D}$$

Figure 8.21 Transfer functions for the power amplifier, potentiometer and motor shaft

The basic equation for the motor is derived in Figure 8.22 using experimental data. In this case the motor was tested with the full inertia on the shaft, so there is no need to calculate ‘J’.

For the motor use the differential equation and the speed curve when Vs=10V is applied:



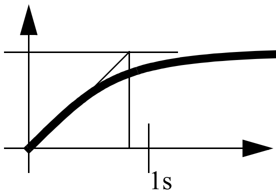
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\tau \approx 0.8s$$

$$\left(\frac{K^2}{JR}\right) = \frac{1}{\tau}$$

$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

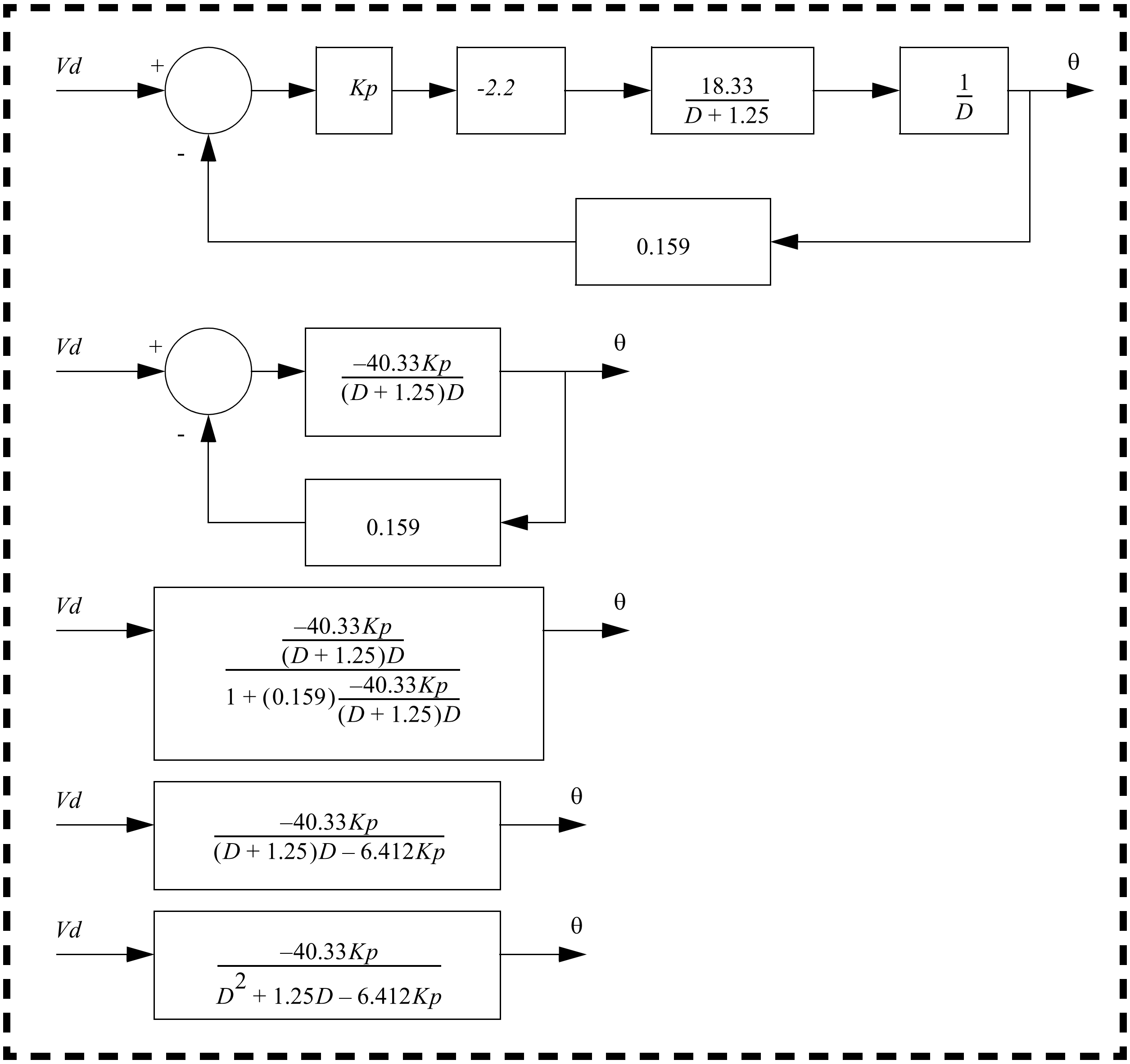
$$\frac{K}{JR} = 18.328$$

$$D\omega + \frac{1}{0.8}\omega = 18.33V_s$$

$$\frac{\omega}{V_s} = \frac{18.33}{D + 1.25}$$

Figure 8.22 Transfer function for the motor

The individual transfer functions for the system are put into the system block diagram in Figure 8.23. The block diagram is then simplified for the entire system to a single transfer function relating the desired voltage (setpoint) to the angular position (output). The transfer function contains the unknown gain value ‘Kp’.



$$\frac{\frac{-40.33Kp}{(D+1.25)D}}{1+(0.159)\frac{-40.33Kp}{(D+1.25)D}}$$

$$\frac{-40.33Kp}{(D+1.25)D-6.412Kp}$$

$$\frac{-40.33Kp}{D^2+1.25D-6.412Kp}$$

Figure 8.23 The system block diagram, and simplification

The value of 'Kp' can be selected to 'tune' the system performance. In Figure 8.24 the gain value is calculated to give the system an overall damping factor of 1.0, or critically damped. This is done by recognizing that the bottom (homogeneous) part of the transfer function is second-order and then extracting the damping factor and natural frequency. The final result of 'Kp' is negative, but this makes sense when the negative gain on the op-amp is considered.

We have specified, or been given the damping factor as a design objective.

$$\zeta = 1.0$$

The denominator of the system transfer function can be compared to the standard second-order response.

$$D^2 + 1.25D - 6.412K_p = x'' + 2\xi\omega_n x' + \omega_n^2 x$$

$$1.25 = 2\xi\omega_n$$

$$1.25 = 2(1.0)\omega_n$$

$$\omega_n = 0.625$$

$$-6.412K_p = \omega_n^2$$

$$-6.412K_p = 0.625^2$$

$$K_p = -0.0609$$

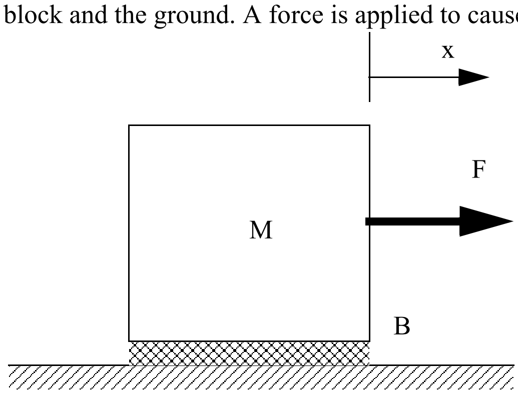
Figure 8.24 Calculating a gain Kp

8.3 Summary

- Transfer functions can be used to model the ratio of input to output.
- Block diagrams can be used to describe and simplify systems.

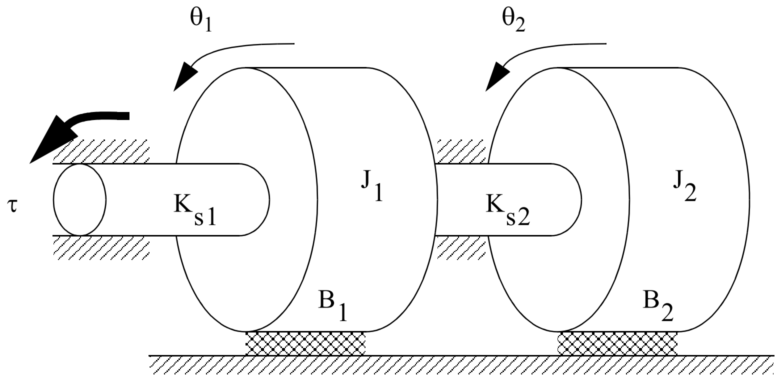
8.4 Problems With Solutions

Problem 8.1 Develop differential equations and then transfer functions for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the mass the accelerate.

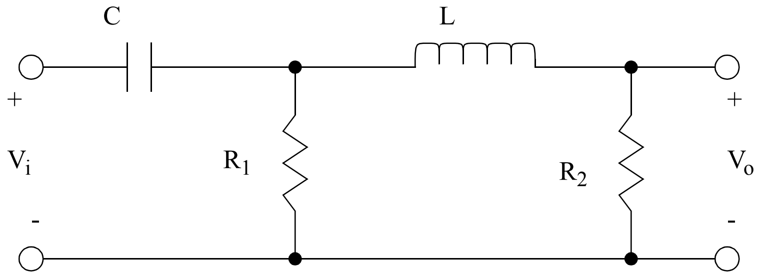


Problem 8.2 Find the transfer function for the systems below. Here the input is a torque, and the output is the angle of the sec-

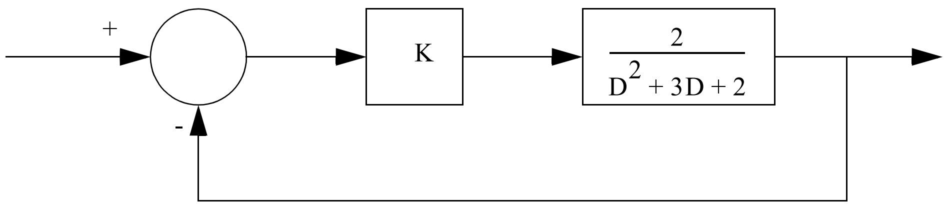
ond mass.



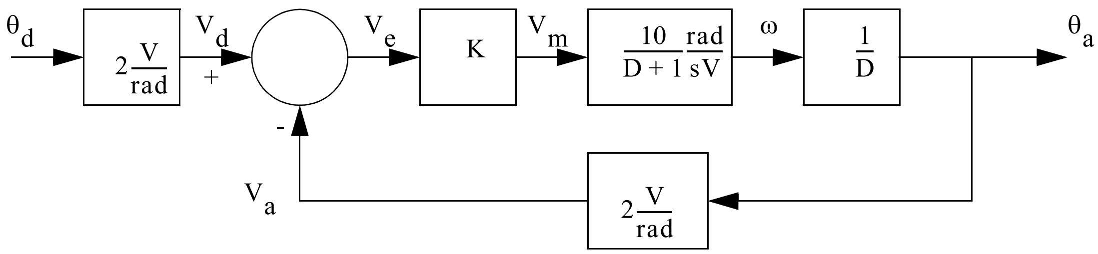
Problem 8.3 Find the transfer functions for the system below where Vi is the input and Vo is the output.



Problem 8.4 Given the block diagram below, select a system gain K that will give the overall system a damping ratio of 0.7 (for a step input). What is the resulting damped frequency of the system?

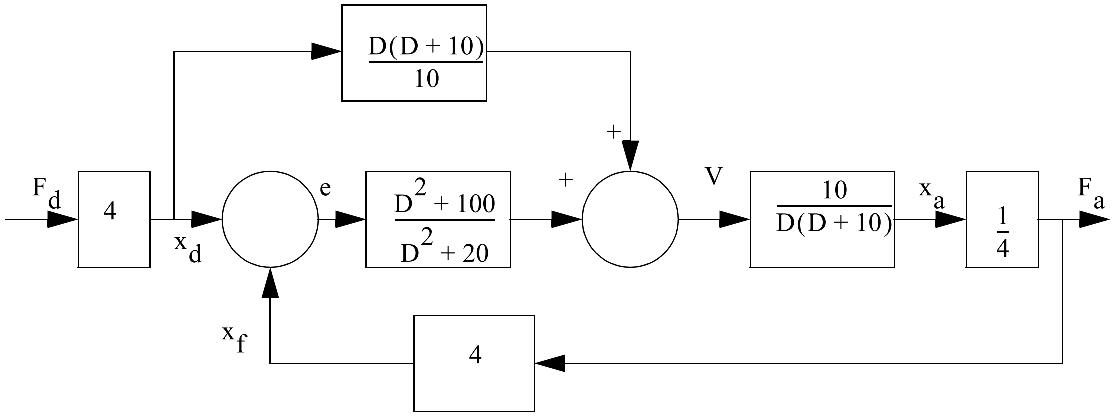


Problem 8.5 The block diagram below is for a servo motor position control system. The system uses a proportional controller.
a) Draw a sketch of what the actual system might look like. Identify components.

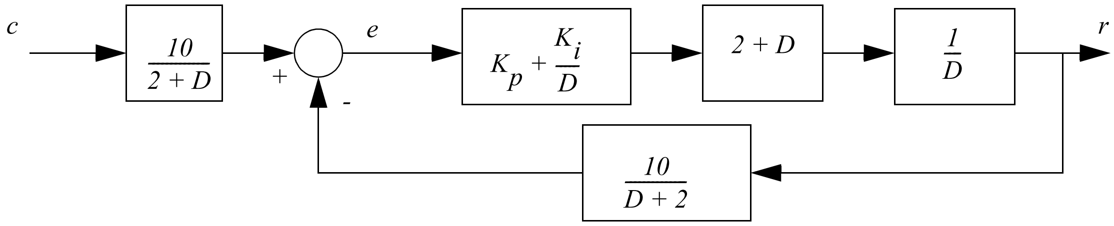


b) Convert the system to a transfer function.

Problem 8.6 Simplify the block diagram below and find the overall transfer function for the system.



Problem 8.7 The block diagram describes a system with a PI controller.

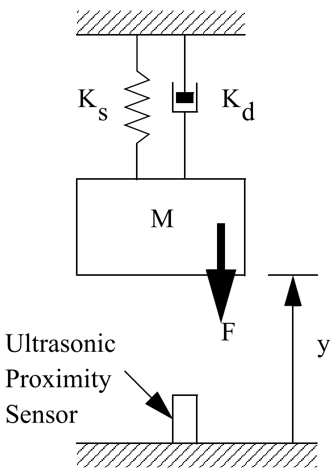


a) Find a transfer function for the system.
b) Select Kp and Ki values to make the system critically damped and the natural frequency is 1 rad/s.

Hint:

$$D^3 + D^2(P) + D(Q) + (R) = (D + A)(D^2 + 2\zeta\omega_n D + \omega_n^2)$$

Problem 8.8 Develop a transfer function for the system below. The input is the force ‘F’ and the output is the voltage ‘Vo’. The mass is suspended by a spring and a damper. When the spring is undeflected y=0. The height is measured with an ultrasonic proximity sensor. When y = 0, the output Vo=0V. If y=20cm then Vo=2V and if y=-20cm then Vo=-2V. Neglect gravity.



$K_s = 10\frac{N}{m}$ $K_d = 5N\frac{s}{m}$ $M = 0.5Kg$

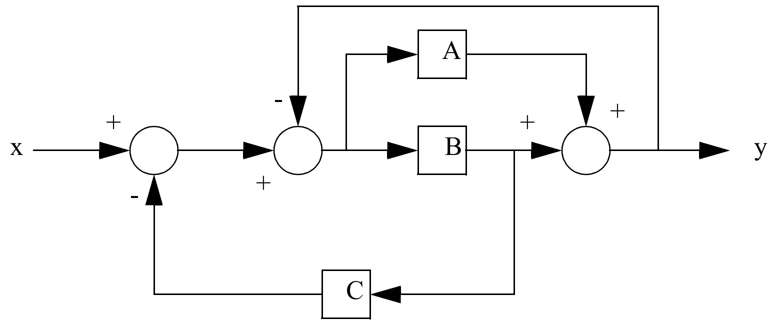
Problem 8.9 Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G = \frac{4}{D+2} = \frac{Y}{X}$$ $X(t) = 20$ When t >= 0

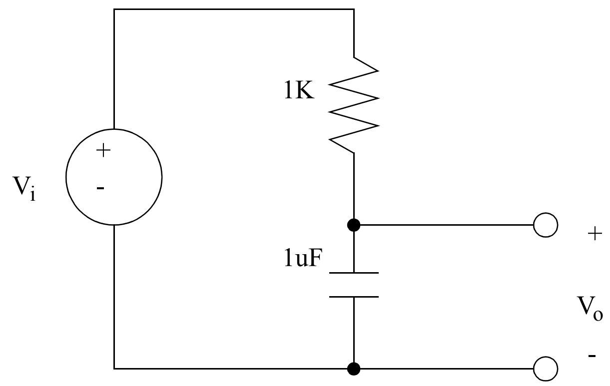
Problem 8.10 Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.
a) develop a transfer function for y/F.
b) find the input function F.
c) solve the input output equation to find an explicit equation of the position as a function of time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.
d) solve part c) numerically.

Problem 8.11 a) What is a Setpoint, and what is it used for? b) What does feedback do in control systems?

Problem 8.12 Simplify the block diagram below to a transfer function.

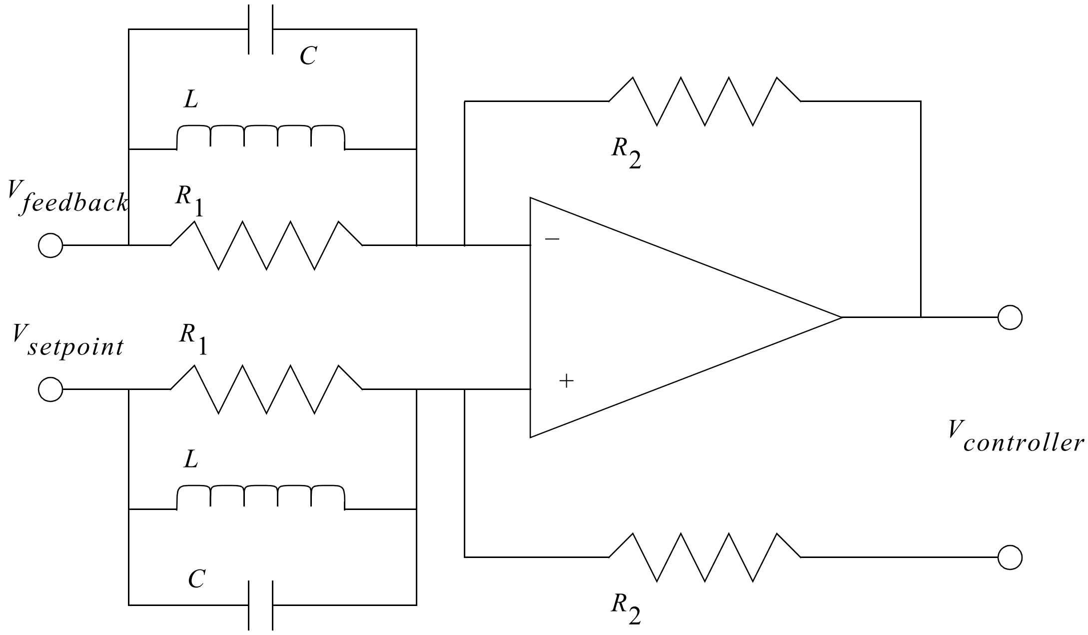


Problem 8.13 Study the circuit below. Assume that for t<0s the circuit is discharged and off. Starting at t=0s an input of Vi=5sin(100,000t) is applied.

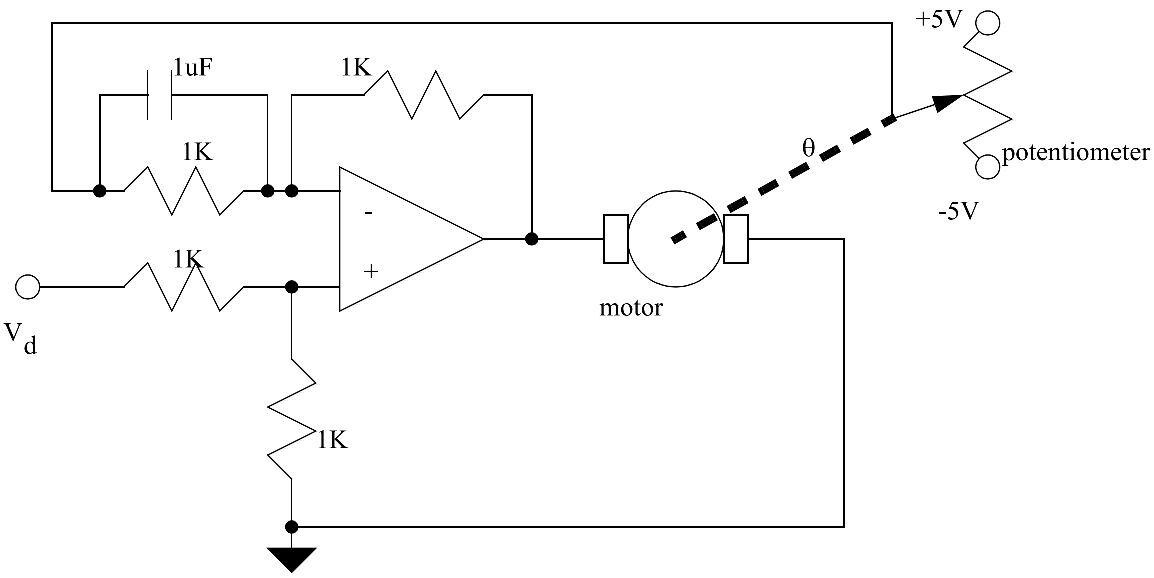


a) Write a differential equation and then a transfer function describing the circuit.
b) Find the output of the circuit using explicit integration (i.e., homogeneous and particular solutions).

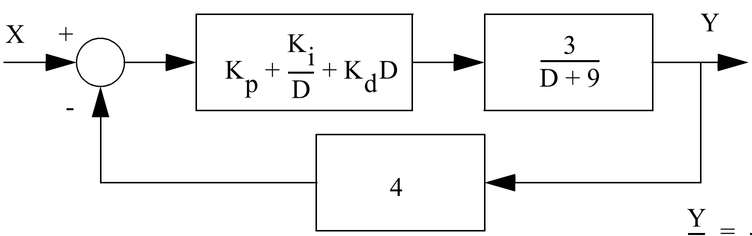
Problem 8.14 Write the differential equation for the control system below.



Problem 8.15 Develop a transfer function for the following system. The input is Vd and the output is the motor shaft position. Assume all components are ideal. The motor has a resistance of 10 ohms. With an input voltage of 4V the motor spins at 4000RPM (steady state), and has a time constant of 0.1s. When the potentiometer is rotated +180 degrees the output is 5V, a rotation of -180 degrees results in an output of -5V.



Problem 8.16 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer function is given.

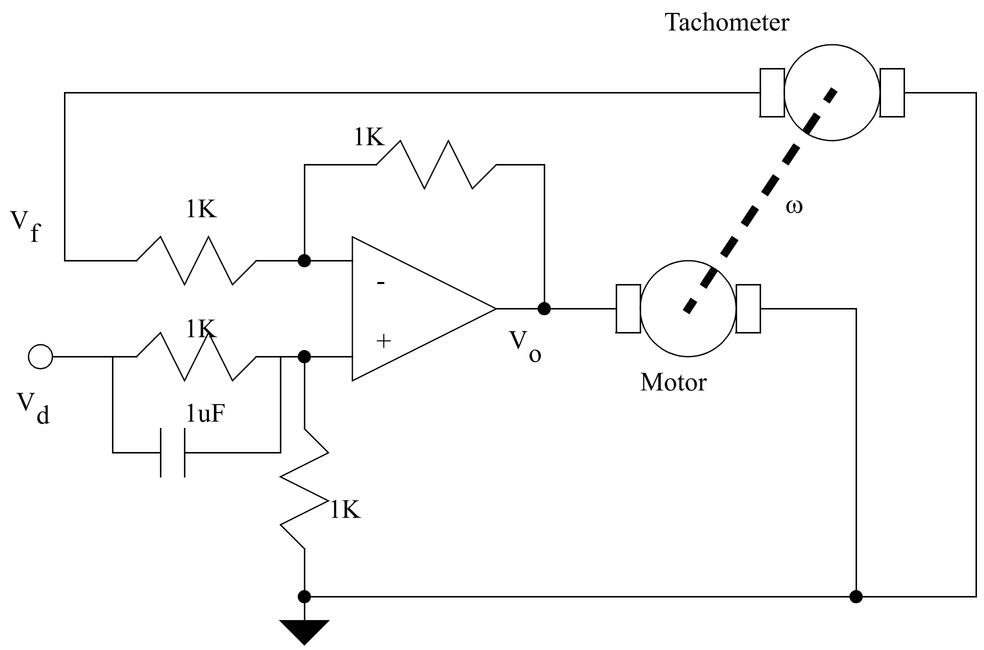


$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1 + 12K_d) + D(9 + 12K_p) + (12K_i)}$$

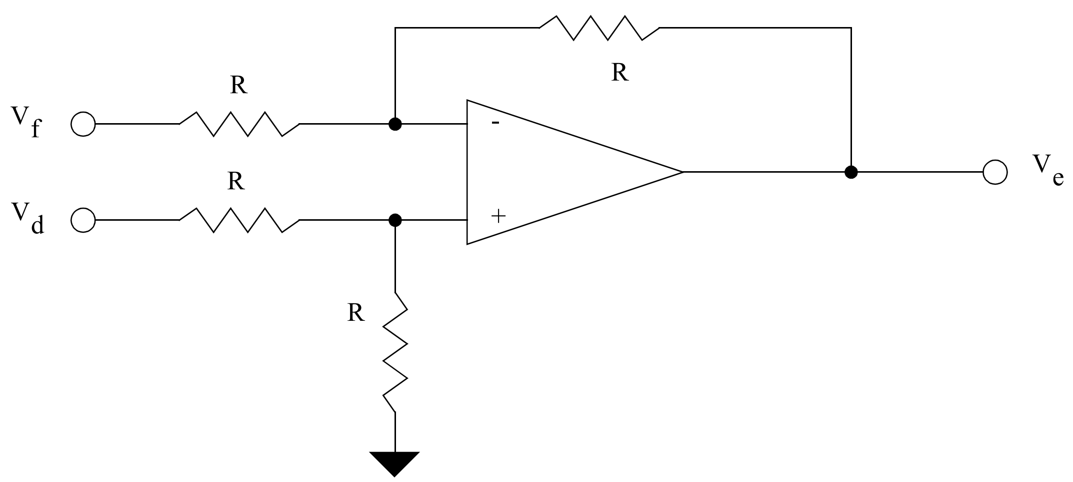
a) Verify the closed loop controller function given.

b) For the given transfer function select controller values that will result in a natural frequency of 3 rad/sec and damping factor of 0.5. (Hint: assume Kd=0.)
c) For the given transfer function, if the values are Kp=5 and Ki=1, and Kd=0, find the response equation to a unit ramp input (i.e., X=t) as a function of time by solving the differential equation to obtain an explicit solution.
d) For the given transfer function, if the values of Kp=1 and Ki=Kd=0, find the response to a unit ramp input (i.e., X=t) as a function of time using state equations and a numerical method. (email the program and graphical output to the instructor).

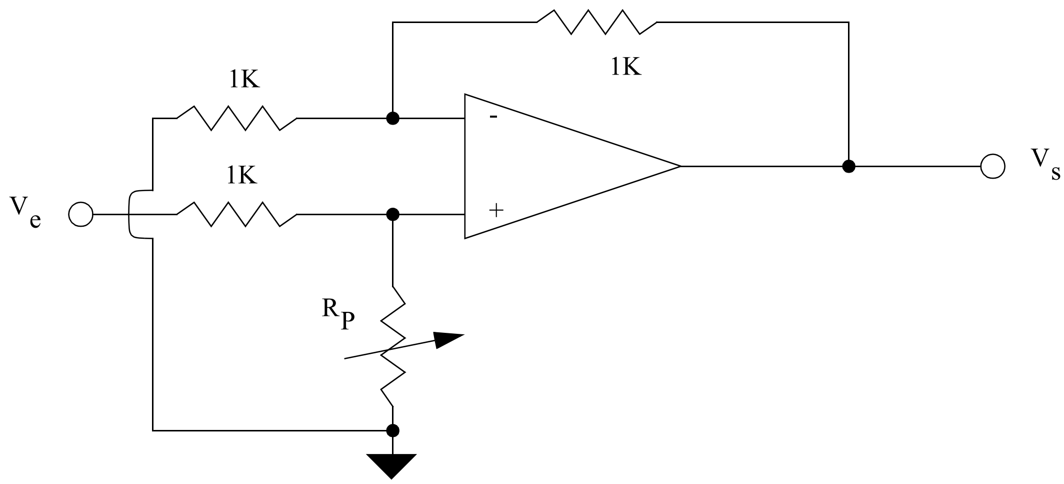
Problem 8.17 The system in the figure uses a high power op-amp to drive a motor. An identical motor is used for measuring motor speed. The input to the system is Vd, and the output is the motor shaft speed.
a) Develop an equation for the op-amp portion using Vf and Vd as inputs, and Vo as the output.
b) Find differential equations for the permanent magnet DC motors. Assume all components are ideal. The motors are identical with a resistance of 20 ohms. With an input voltage of 5V the connected motors spin at 2000RPM, and have a time constant of 0.25s.
c) Draw a block diagram for the system shown in the figure. Put values in each of the blocks.



Problem 8.18 a) Develop an equation for the system below relating the two inputs to the output. Put it in block diagram format. (Hint: think of a summation block.)

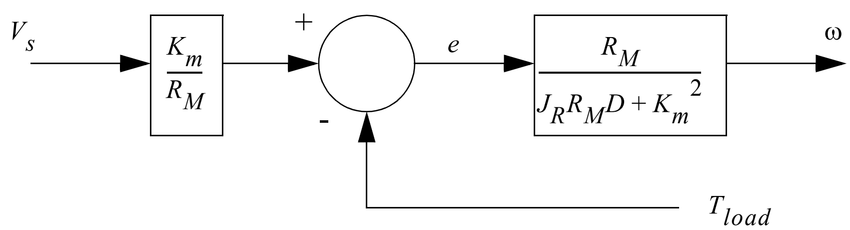


b) Develop an equation for the system below relating the input to the output. Put the result in block diagram format.

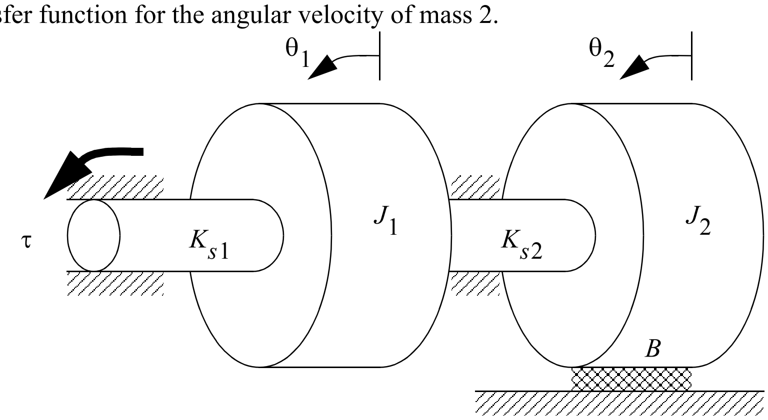


c) The equation below can be used to model a permanent magnet DC motor with an applied torque. An equivalent block diagram is given. Prove that the block diagram is equivalent to the equation.

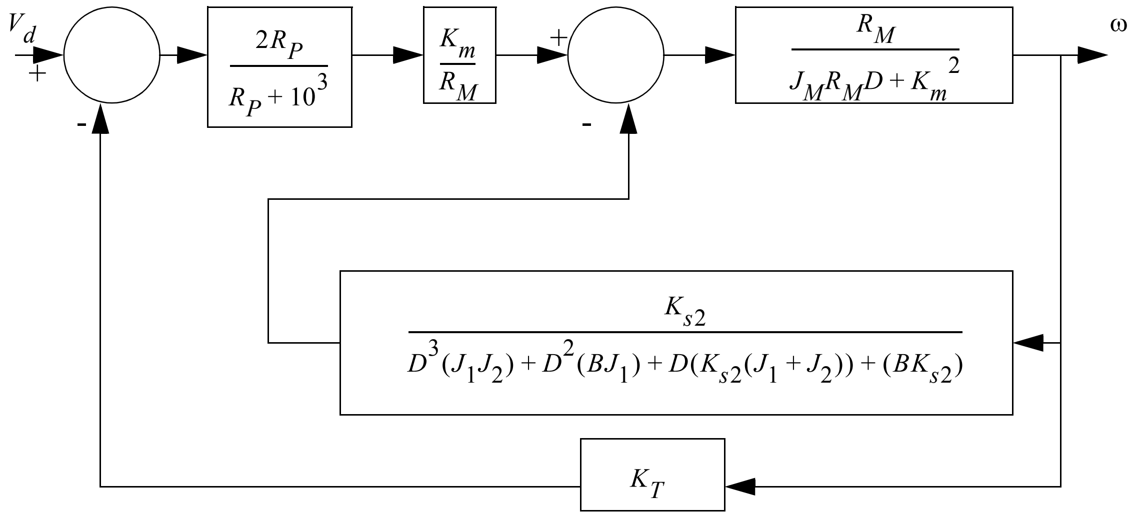
$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K_m^2}{J_R R_M}\right) = V_s\left(\frac{K_m}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$



d) Write the transfer function for the system below relating the input torque to the output angle theta2. Then write the transfer function for the angular velocity of mass 2.



e) The system below is a combination of previous components, and a tachometer for velocity feedback. Simplify the block diagram.



Problem 8.19 Given the transfer function below, develop a mechanical system that it could represent. (Hint: Differential Equations).

$$\frac{x(D)}{F(D)} = \frac{1}{10 + 20D}$$

where: x = displacement
F = force

Problem 8.20 Develop an op-amp circuit to implement the controller with values found in Problem 3.

8.5 Problem Solutions

Answer 8.1

$$\frac{x}{F} = \frac{1}{D(B - DM)}$$

Answer 8.2

$$\frac{\theta_2}{\tau} = \frac{K_{s2}}{D^4(J_1 J_2) + D^3(B_1 J_2 + B_2 J_1) + D^2(K_{s2}(J_1 + J_2) + B_1 B_2) + D(K_{s2}(B_1 + B_2))}$$

Answer 8.3

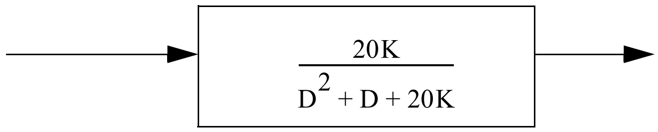
$$\frac{V_o}{V_i} = \frac{D(R_1 R_2 C)}{D^2(R_1 LC) + D(L + CR_1 R_2) + (R_1 + R_2)}$$

Answer 8.4

$$\omega_d = 1.530\frac{rad}{s} \qquad K = 1.30$$

Answer 8.5

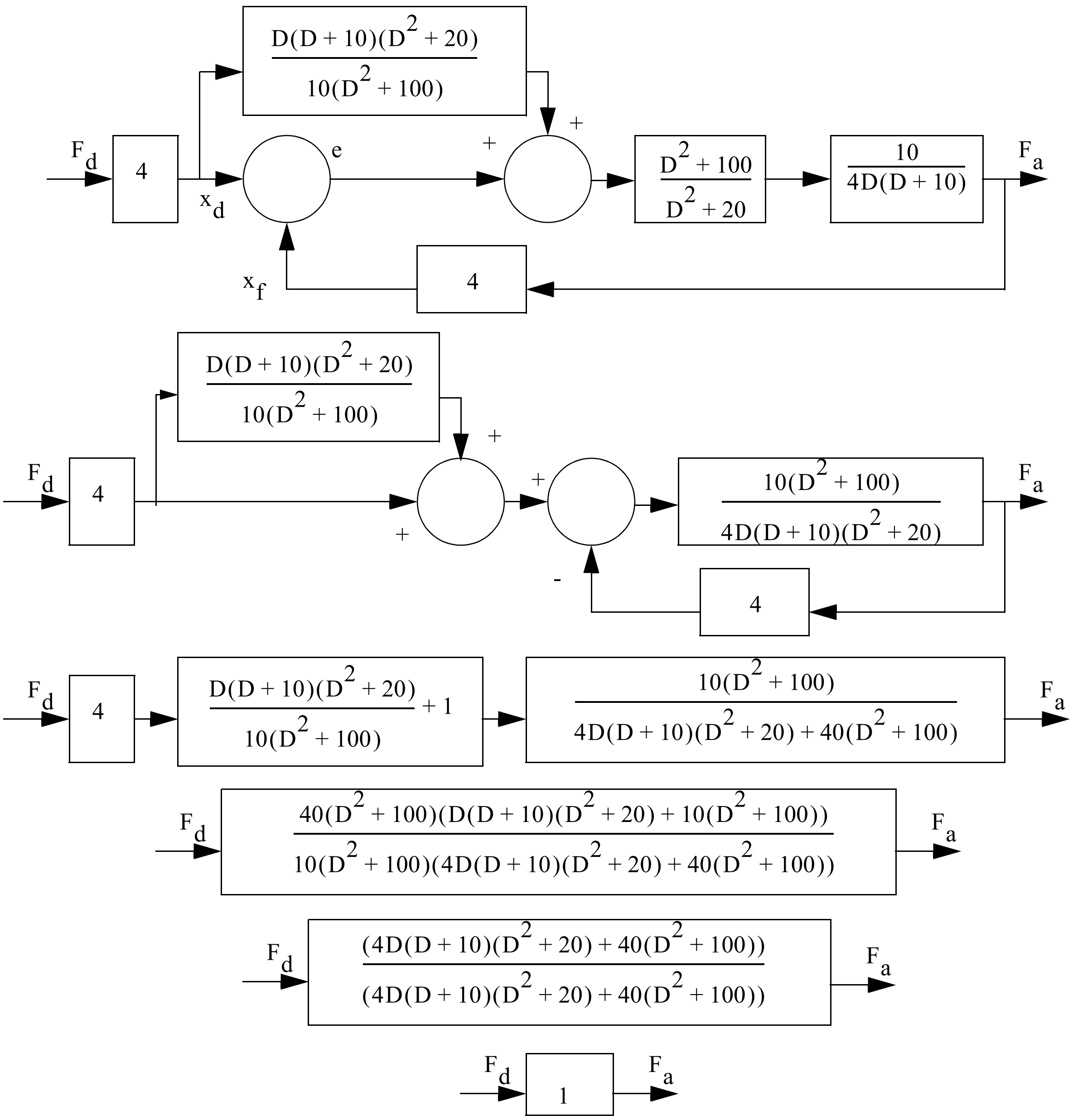
a)



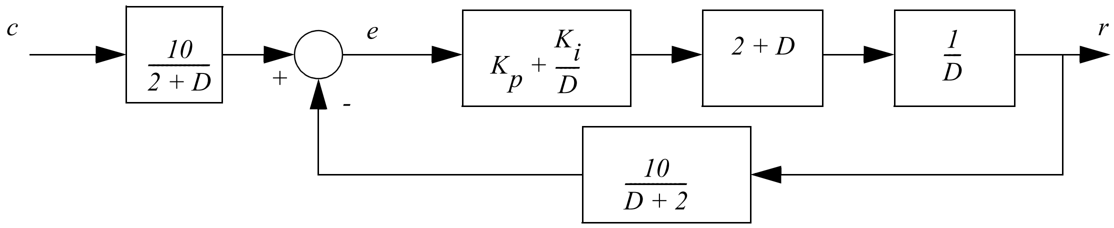
b)

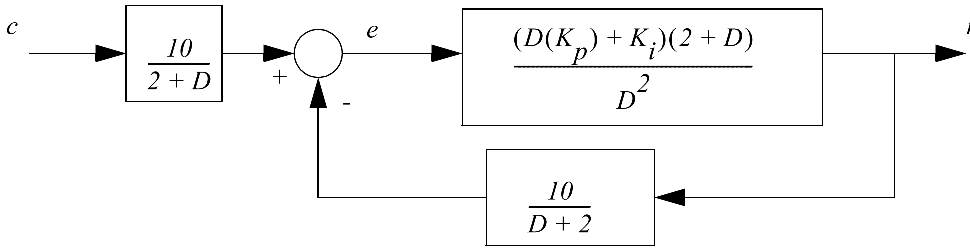
$$\frac{\theta_a}{\theta_d} = \frac{20K}{D^2 + D + 20K}$$

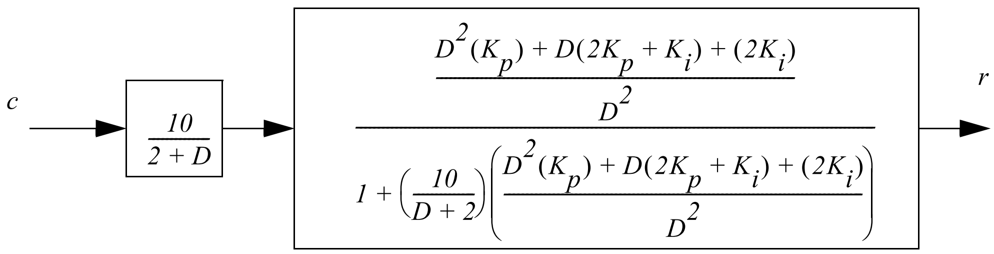
Answer 8.6

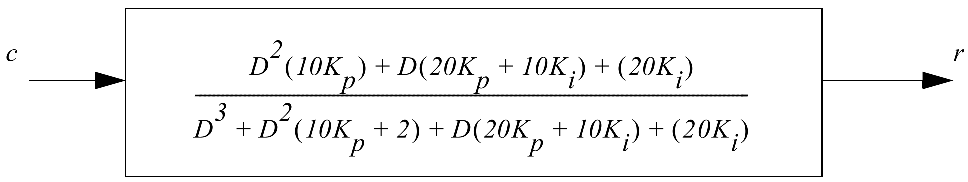


Answer 8.7 a)









$$\frac{r}{c} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{D^3 + D^2(10K_p + 2) + D(20K_p + 10K_i) + (20K_i)}$$

b)

$$\frac{r}{c} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{D^3 + D^2(10K_p + 2) + D(20K_p + 10K_i) + (20K_i)} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{(D + A)(D^2 + 2\zeta\omega_n D + \omega_n^2)}$$

$$D^3 + D^2(10K_p + 2) + D(20K_p + 10K_i) + (20K_i) = (D + A)(D^2 + 2\zeta\omega_n D + \omega_n^2)$$

$$= (D + A)(D^2 + 2D + 1)$$

$$= (D^3(1) + D^2(2 + A) + D(2A + 1) + A)$$

$$10K_p + 2 = 2 + A$$

$$20K_p + 10K_i = 2A + 1$$

$$20K_i = A$$

$$10K_p + 2 = 2 + A = 2 + 20K_i$$
$$K_p = 2K_i$$

$$20K_p + 10K_i = 2A + 1$$
$$20(2K_i) + 10K_i = 40K_i + 1$$

$K_i = 0.1$ $\quad K_p = 0.2$ $\quad A = 2.0$

Answer 8.8

$$\frac{V_o}{F} = \frac{V_o}{y}\frac{y}{F} = \frac{10\left(\frac{V}{m}\right)}{-MD^2 - K_d D - K_s} = \frac{10\left(\frac{V}{m}\right)}{-0.5kgD^2 - 5\frac{Ns}{m}D - 10\frac{N}{m}}$$

Answer 8.9

$$y(t) = -40e^{-2t} + 40$$

Answer 8.10

a) $$\frac{y}{F} = \frac{1}{D^2 M + DK_d + K_s}$$

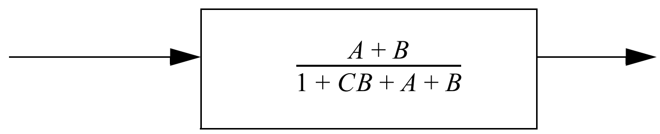
b) $$F = Mg$$

c) $$y(t) = -10.13e^{-0.25t}\cos(0.968t - 0.253) + 9.81$$

d) $$\begin{bmatrix} y(t+h) \\ v(t+h) \end{bmatrix} = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + h\left[\begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 9.81 \end{bmatrix}\right]$$

Answer 8.11 a) A setpoint is a system input that correlates to a desired output for a system with negative feedback control.
b) Feedback in a control system allows an out to be compared to an input. The difference, the error, is used to adjust the control variable and increase the accuracy.

Answer 8.12



Answer 8.13

a) $$\frac{V_o}{V_i} = \frac{1000}{D + 1000}$$

b) $$V_o = 0.050e^{-1000t} + 0.050\sin(10^5 t + 1.561)$$

Answer 8.14

$$\dot{V}_{controller} = \ddot{V}_s(CR_2) + \dot{V}_s\left(\frac{R_2}{R_1}\right) + V_s\left(\frac{R_2}{L}\right) + \ddot{V}_f(-CR_2) + \dot{V}_f\left(-\frac{R_2}{R_1}\right) + V_f\left(-\frac{R_2}{L}\right)$$

or

$$V_{controller} = e\left(D(CR_2) + \left(\frac{R_2}{R_1}\right) + \frac{1}{D}\left(\frac{R_2}{L}\right)\right)$$

Kd Kp Ki

Answer 8.15

For the op-amp:

$$V_+ = \frac{1}{2}V_d$$

$$\sum I_{V-} = \frac{(V_p - V_-)}{10^3} + \frac{(V_p - V_-)}{\left(\frac{1}{10^{-6}D}\right)} + \frac{(V_m - V_-)}{10^3} = 0$$

$$(V_p - V_-) + 10^{-3}D(V_p - V_-) + (V_m - V_-) = 0$$

$$V_m = V_-(2 + 10^{-3}D) + V_p(-1 - 10^{-3}D) = V_d\left(1 + \frac{10^{-3}}{2}D\right) + V_p(-1 - 10^{-3}D)$$

For the potentiometer:

$$V_p = \frac{5}{\pi}\theta$$

For the motor:

$$\dot{\omega} + \omega\left(\frac{1}{\tau}\right) = V_m(A) \qquad \dot{\omega} + \omega(10) = V_m(A)$$

$$(0) + \left(4000\left(\frac{2\pi}{60}\right)\right)(10) = 4(A) \qquad A = \left(\frac{4000}{4}\left(\frac{2\pi}{60}\right)\right)(10) = \frac{1000}{3}\pi$$

$$\dot{\omega} + 10\omega = \frac{1000}{3}\pi V_m \qquad \theta = \frac{\omega}{D}$$

$$D^2\theta + 10D\theta = \frac{1000}{3}\pi V_m \qquad V_m = \theta\left(\frac{3D^2 + 30D}{1000\pi}\right)$$

Combined:

$$V_m = V_d\left(1 + \frac{10^{-3}}{2}D\right) + \frac{5}{\pi}\theta(-1 - 10^{-3}D)$$

$$\theta\left(\frac{3D^2 + 30D}{1000\pi}\right) - \frac{5}{\pi}\theta(-1 - 10^{-3}D) = V_d\left(1 + \frac{10^{-3}}{2}D\right)$$

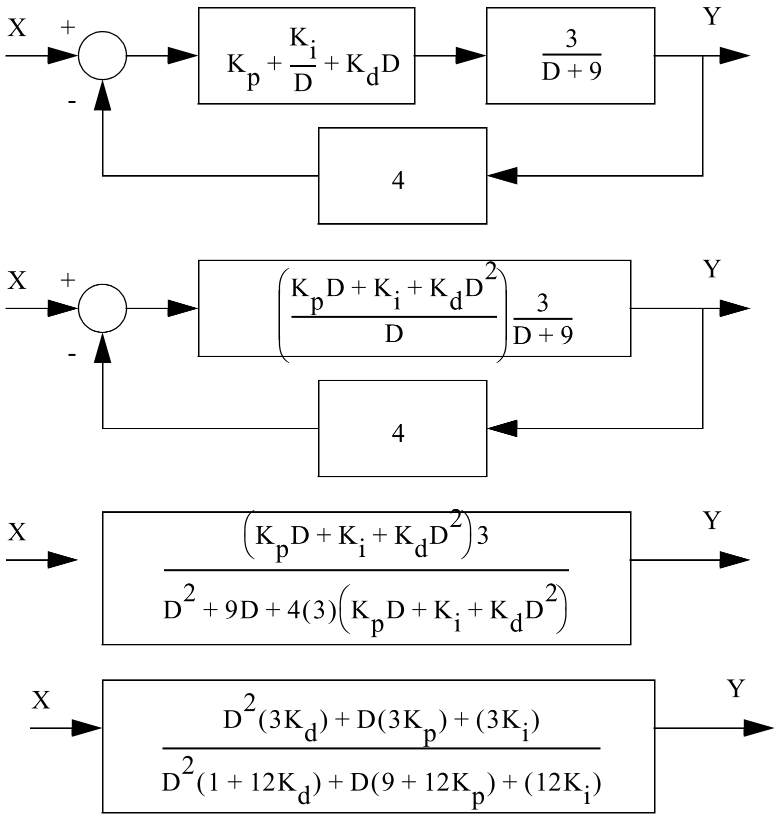
$$\theta\left(\frac{3D^2 + 30D}{1000\pi} + \frac{5}{\pi} + \frac{5}{\pi}10^{-3}D\right) = V_d\left(1 + \frac{10^{-3}}{2}D\right)$$

$$\theta((3D^2 + 30D) + 5000 + 5D) = V_d\left(1000\pi + \frac{\pi}{2}D\right)$$

$$\theta\left(D^2 + \frac{35}{3}D + \frac{5000}{3}\right) = V_d\left(\frac{1000}{3}\pi + \frac{\pi}{6}D\right)$$

$$\frac{\theta}{V_d} = \frac{D\left(\frac{\pi}{6}\right) + \left(\frac{1000}{3}\pi\right)}{D^2 + D\left(\frac{35}{3}\right) + \left(\frac{5000}{3}\right)}$$

Answer 8.16 a)



b)

$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1 + 12K_d) + D(9 + 12K_p) + (12K_i)} = \frac{\dots}{D^2 + D2\zeta\omega_n + \omega_n^2}$$

$$\omega_n^2 = \frac{12K_i}{1 + 12K_d} \qquad 3^2 = \frac{12K_i}{1 + 12K_d} \qquad 9 + 108K_d = 12K_i$$

$$2\zeta\omega_n = \frac{9 + 12K_p}{1 + 12K_d} \qquad 2(0.5)3 = \frac{9 + 12K_p}{1 + 12K_d} \qquad 3 + 36K_d = 9 + 12K_p$$

In simple terms there are two equations and three unknowns. The system can only be solved by adding one equation, or removing one unknown. Therefore I have made an arbitrary decision to set Kd = 0. However, any of the gains could be set to any value to move ahead. Note: We would prefer to avoid arbitrary solutions and under-constrained problems suggest ill-posed designs.

$$K_d = 0$$

$$9 + 108(0) = 12K_i \qquad K_i = \frac{3}{4}$$

$$3 + 36(0) = 9 + 12K_p \qquad K_p = -\frac{1}{2}$$

c)

$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1 + 12K_d) + D(9 + 12K_p) + (12K_i)} = \frac{15D + 3}{D^2 + 69D + 12}$$

$$\ddot{Y} + 69\dot{Y} + 12Y = 15\dot{X} + 3X$$

$$X = t$$

$$\ddot{Y} + 69\dot{Y} + 12Y = 15 + 3t \qquad\qquad X = t$$

Assume initial conditions are zero. Solution simplified from WolframAlpha using y'' + 69y' + 12y = 15 + 3t, y(0)=0, y'(0)=0

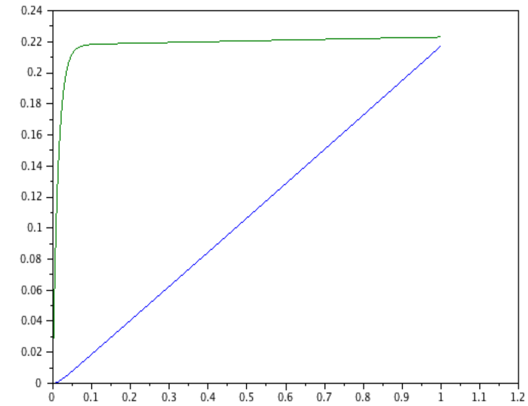
$$Y(t) = 62.500\times10^{-3}(4t - 3) + 3.1654\times10^{-3}e^{-68.8256t} + 184.33\times10^{-3}e^{-0.17435t}$$

d)

$$\frac{Y}{X} = \frac{D^2(3(0)) + D(3(1)) + (3(0))}{D^2(1 + 12(0)) + D(9 + 12(1)) + (12(0))} = \frac{(3)}{D(1) + (21)}$$

$$\dot{Y} + 21Y = 3t$$

$$\dot{Y} = Y(-21) + 3t$$




```
Y0 = 0 ;
ST = [ Y0 ];
t = [0];
h = 0.001;
n_steps = 1000;
function ST_der = state_eqns(ST, t)
    Y = ST($, 1) ;
    Y_der = - 21 * Y + 3 * t;
    ST_der = [ Y_der ];
endfunction
for i = 1 : n_steps
    ST = [ ST ; ST($, :) + h * state_eqns( ST($, :), t($) ) ]
    t = [ t ; t($) + h]
end
plot(t, ST);
```

Answer 8.17 a)

First, we will develop a differential equation for the op-amp.

$$\sum I_{V+} = \frac{V_d - V_+}{1000} + \frac{V_d - V_+}{\left(\frac{1}{10^{-6}D}\right)} + \frac{0 - V_+}{1000} = 0$$

$$V_+\left(\frac{2}{1000} + 10^{-6}D\right) = V_d\left(\frac{1}{1000} + 10^{-6}D\right) \qquad V_+ = V_d\left(\frac{1000 + D}{2000 + D}\right)$$

$$\sum I_{V-} = \frac{V_f - V_-}{1000} + \frac{V_o - V_-}{1000} = 0$$

$$V_f - 2V_- + V_o = 0$$

$$V_o = 2V_d\left(\frac{1000 + D}{2000 + D}\right) - V_f$$

b)

For the PM DC motor we begin with the basic differential equation. Using the given data the coefficients are calculated. Note: because the motors are connected the J will include the inertia for both motors.

$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_o\left(\frac{K}{JR}\right) \qquad R = 20\Omega$$

$$\omega_{ss}\left(\frac{K^2}{JR}\right) = V_o\left(\frac{K}{JR}\right) \qquad K = \frac{5V}{\left(\frac{2000 min(2\pi)}{60s}\right)} = \frac{3}{40\pi}\frac{Vs}{rad} = 0.02387\frac{Vs}{rad}$$

$$\frac{K^2}{JR} = \frac{1}{\tau} \qquad J = \frac{(0.25)\left(\frac{3}{40\pi}\right)^2}{20} = 7.124\times10^{-6}kgm^2$$

For the motor being used as a tachometer there is little load, so the voltage output Vf should be directly proportional to the shaft speed.

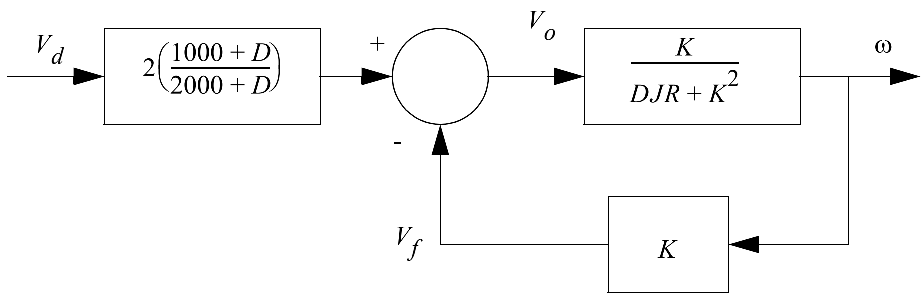
$$0 + \omega\left(\frac{K^2}{JR}\right) = V_f\left(\frac{K}{JR}\right) \qquad V_f = \omega K$$

c)

$$V_o = 2V_d\left(\frac{1000 + D}{2000 + D}\right) - V_f$$

$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_o\left(\frac{K}{JR}\right) \qquad \omega = V_o\left(\frac{K}{DJR + K^2}\right)$$

$$V_f = \omega K$$



Answer 8.18

a)

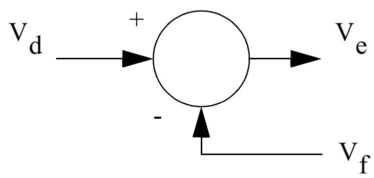
$$V_+ = V_d\left(\frac{R}{R + R}\right) = 0.5V_d$$

$$\sum I_{V_-} = \frac{V_- - V_f}{R} + \frac{V_- - V_e}{R} = 0$$

$$V_e = 2V_- - V_f$$

$$V_e = 2(0.5V_d) - V_f$$

$$V_e = V_d - V_f$$



b)

$$\sum I_{V_-} = \frac{V_- - 0}{10^3} + \frac{V_- - V_s}{10^3} = 0$$

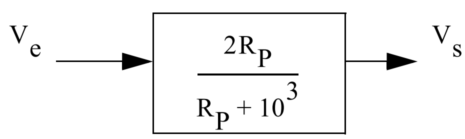
$$V_- = 0.5V_s$$ eqn 8.1

$$\sum I_{V_+} = \frac{V_+ - V_e}{10^3} + \frac{V_+ - 0}{R_P} = 0$$

$$V_+\left(\frac{1}{10^3} + \frac{1}{R_P}\right) = \frac{V_e}{10^3}$$

substitute in (1)

$$0.5V_s\left(\frac{R_P + 1K}{R_P(10^3)}\right) = \frac{V_e}{10^3}$$



$$\frac{V_s}{V_e} = \frac{2R_P}{R_P + 10^3}$$

note: a constant set by the variable resistor Rp

c)

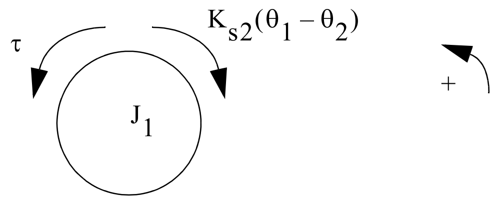
$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K_m^{\ 2}}{J_R R_M}\right) = V_s\left(\frac{K_m}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega\left(D + \frac{K_m^{\ 2}}{J_R R_M}\right) = V_s\left(\frac{K_m}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega = \left(\frac{JR_M}{J_R R_M D + K_m^{\ 2}}\right)\left(V_s\left(\frac{K_m}{J_R R_M}\right) - \frac{T_{load}}{J_R}\right)$$

$$\omega = \left(\frac{R_M}{J_R R_M D + K_m^{\ 2}}\right)\left(V_s\left(\frac{K_m}{R_M}\right) - T_{load}\right)$$

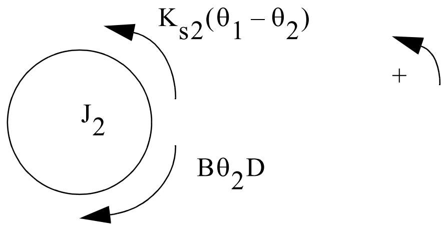
d)



$$\sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\theta_1\left(J_1 D^2 + K_{s2}\right) + \theta_2(-K_{s2}) = \tau \qquad \text{eqn 8.1}$$



$$\sum M = K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$\theta_2\left(J_2 D^2 + K_{s2} + BD\right) = \theta_1(K_{s2})$$

$$\theta_1 = \theta_2\left(\frac{J_2 D^2 + K_{s2} + BD}{K_{s2}}\right) \qquad \text{eqn 8.2}$$

substitute (2) into (1)

$$\theta_2\left(\frac{J_2 D^2 + BD + K_{s2}}{K_{s2}}\right)\left(J_1 D^2 + K_{s2}\right) + \theta_2(-K_{s2}) = \tau$$

$$\theta_2\left(\frac{D^4(J_1J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}{K_{s2}}\right) = \tau$$

$$\frac{\theta_2}{\tau} = \frac{K_{s2}}{D^4(J_1J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}$$

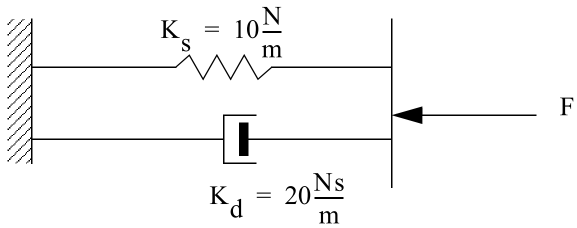
$$\omega_2 = D\theta_2$$

$$\frac{\omega_2}{\tau} = \frac{K_{s2}}{D^3(J_1J_2) + D^2(BJ_1) + D(K_{s2}(J_1 + J_2)) + (BK_{s2})}$$

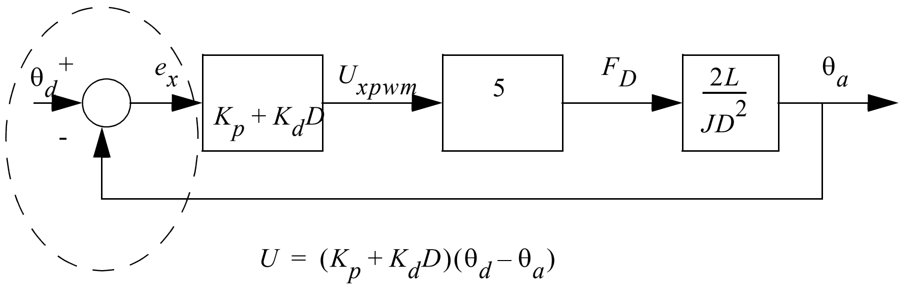
e)

$$\frac{\omega}{V_d} = \frac{D^3(\ ) + D^2(\ ) + D(\ ) + (\ )}{D^4(\ ) + D^3(\ ) + D^2(\ ) + D(\ ) + (\ )}$$

Answer 8.19



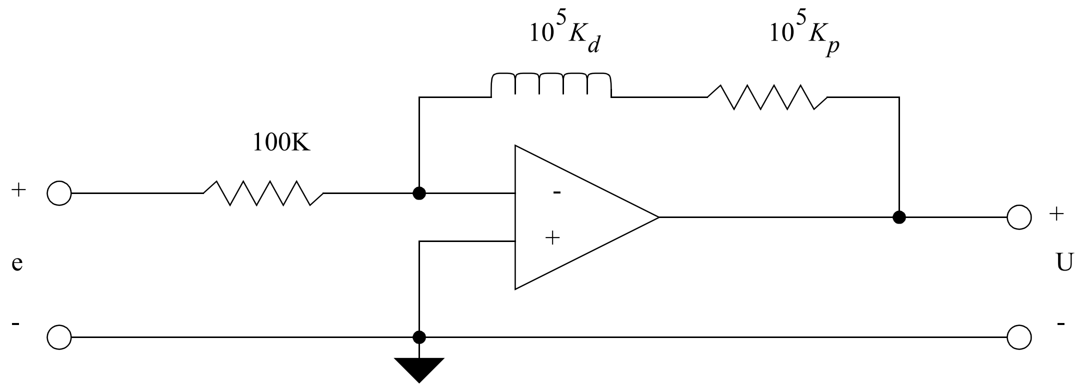
Answer 8.20



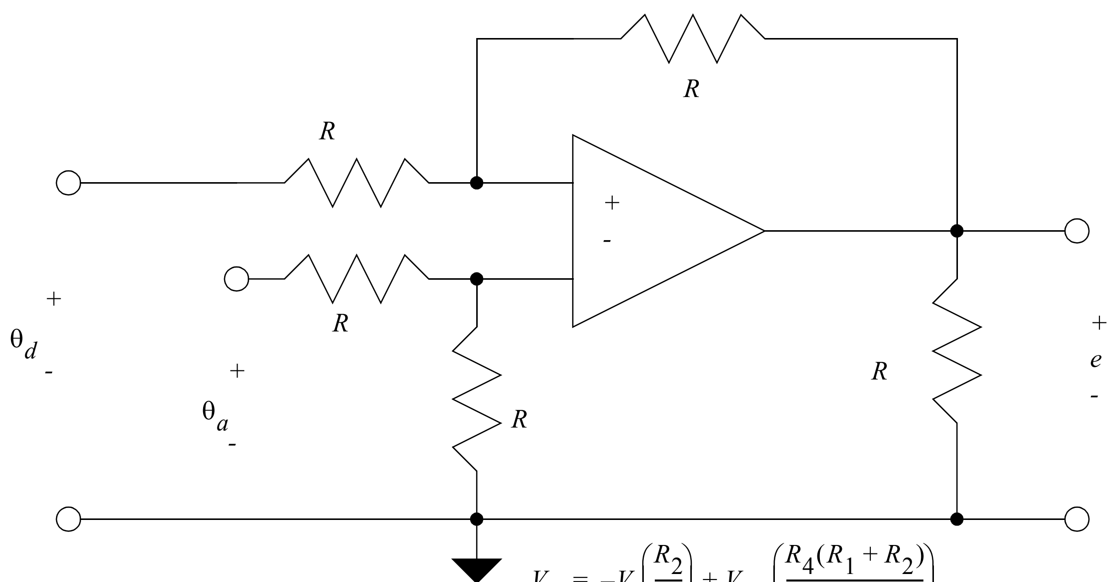
$$U = (K_p + K_d D)(\theta_d - \theta_a)$$

This type of function is a good application for an op-amp. The circuit pictured was presented earlier in the book using resistors. The dashed line circles represent two different op-amp circuits. The one on the left is a subtraction circuit. The second is a feedback amplifier with an inductor on the feedback loop. First we create the PD amplifier circuit by rearranging the negative feedback circuit. A value of 100K is chosen to provide a reasonable current draw.

$$U = (K_p + K_d D)e = \left(\frac{-Z_f}{Z_i}\right)e \qquad Z_i = 100K \qquad Z_f = 10^5 K_p + 10^5 K_d D$$



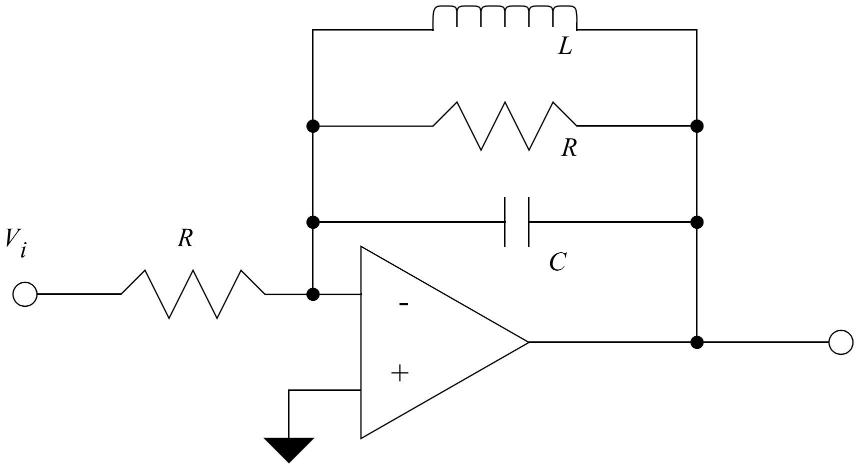
The subtraction circuit has been presented earlier. Making all of the resistor values makes the result a simple subtraction with no weights. The value of R is arbitrary but 10-100K would be reasonable. Not that the gains of both are negative, canceling out.



$$V_o = -V_i\left(\frac{R_2}{R_1}\right) + V_{ref}\left(\frac{R_4(R_1 + R_2)}{R_1(R_4 + R_5)}\right)$$

8.1 Problems Without Solutions

Problem 8.21 Develop a transfer function for the circuit below.



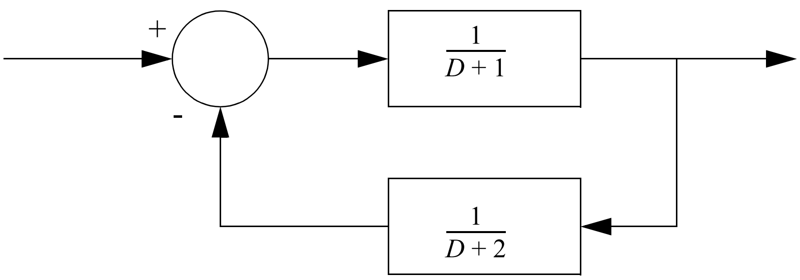
Problem 8.22 Given the transfer functions and input functions, F, use a numerical method to calculate the output of the system as a function of time for 0 to 0.5 seconds in 0.05 second intervals. Record the values in a table.

$$\frac{x}{F} = \frac{D^2}{(D+200\pi)^2} \qquad F = 5\sin(62.82t)$$

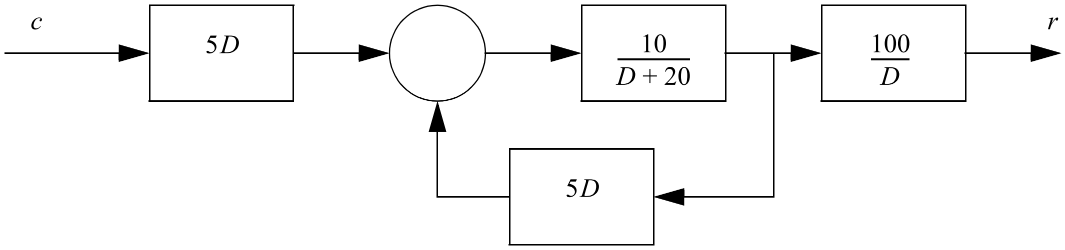
$$\frac{x}{F} = \frac{D(D+2\pi)}{(D+200\pi)^2} \qquad F = 5\sin(62.82t)$$

$$\frac{x}{F} = \frac{D^2(D+2\pi)}{(D+200\pi)^2} \qquad F = 5\sin(62.82t)$$

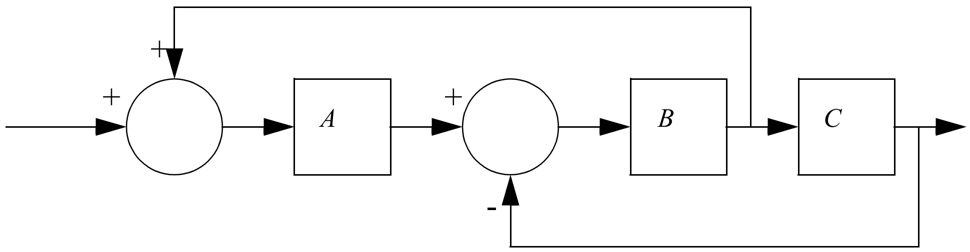
Problem 8.23 Simplify the following block diagram.



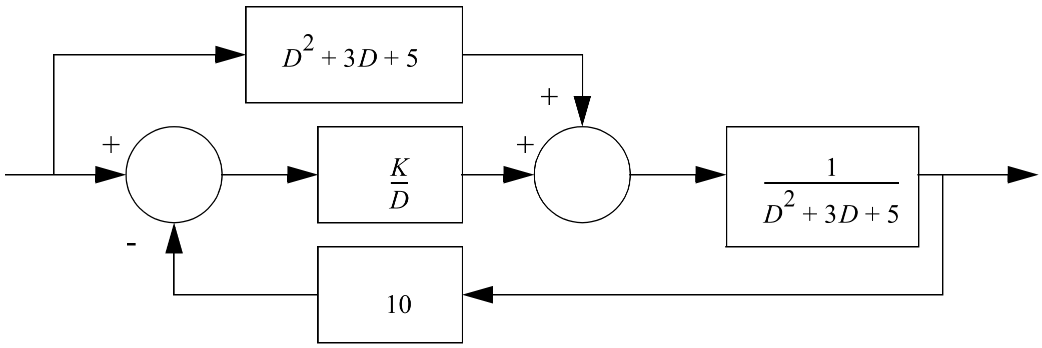
Problem 8.24 Simplify the block diagram.



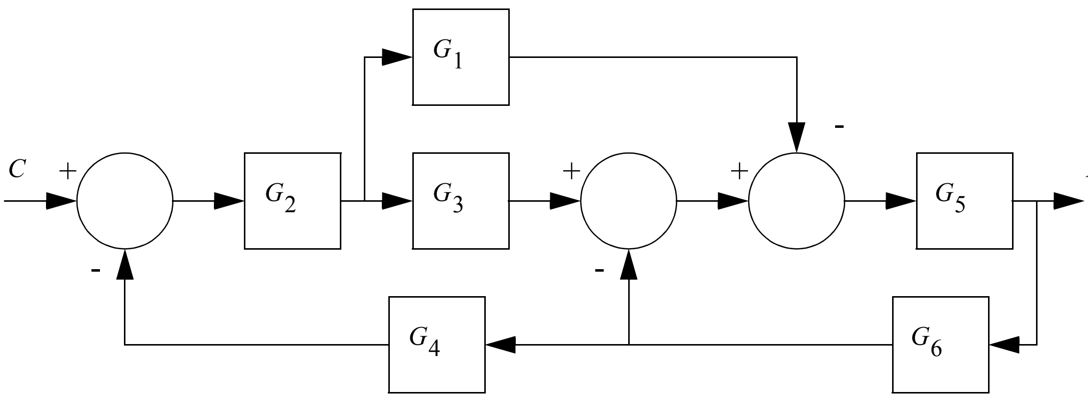
Problem 8.25 Simplify the following block diagram.



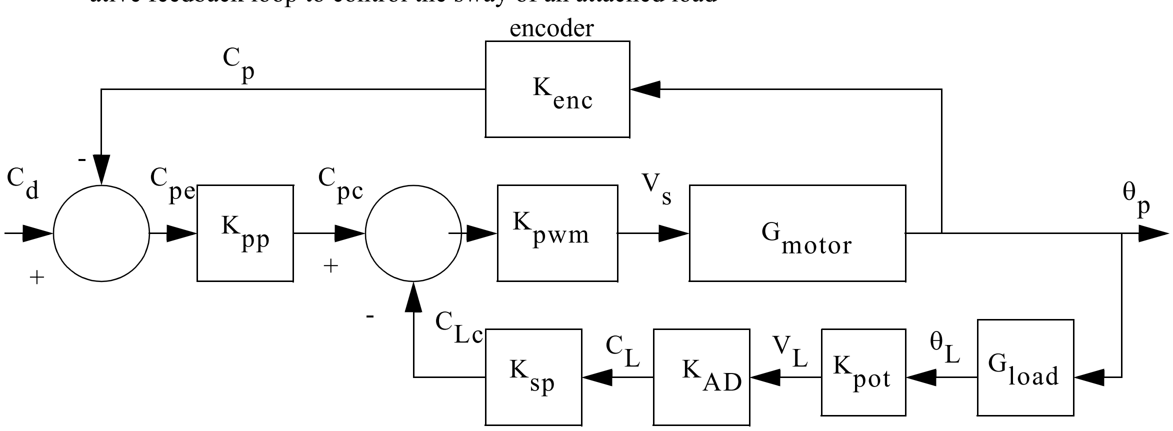
Problem 8.26 Simplify the following block diagram.



Problem 8.27 Simplify the following block diagram.

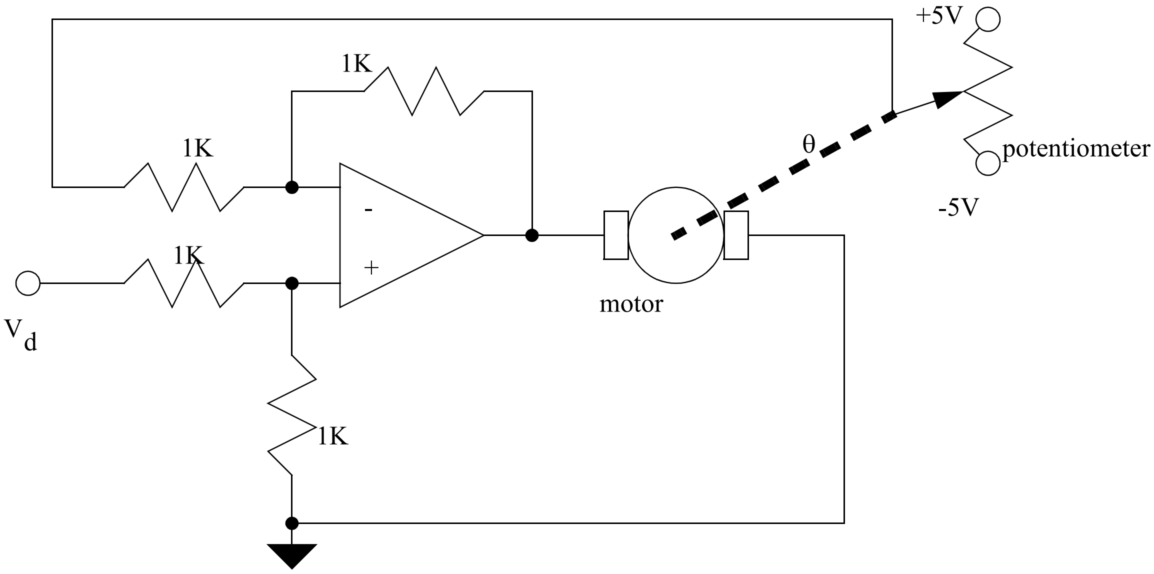


Problem 8.28 Simplify the following block diagram for a crane gantry position control system. The system also includes a negative feedback loop to control the sway of an attached load



Problem 8.29 Develop a block diagram for the system below and transfer functions for each of the blocks. The input is Vd and the output is the motor shaft position. Assume all components are ideal. The motor has a resistance of 10 ohms.

With an input voltage of 4V the motor spins at 4000RPM (steady state), and has a time constant of 0.1s. When the potentiometer is rotated +180 degrees the output is 5V, a rotation of -180 degrees results in an output of -5V.

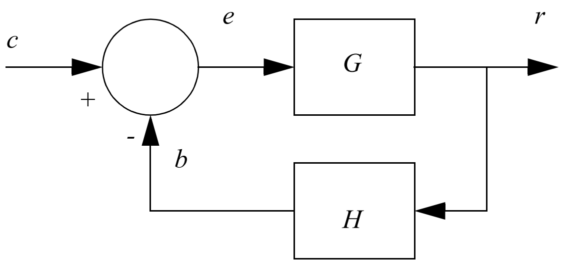


9. Feedback Control Systems

Topic 9.1 *Feedback controllers.*
Topic 9.2 *Control system design.*
Objective 9.1 *To be able to select controller parameters to meet design objectives.*

System Error

System error is often used when designing control systems. The two common types of error are system error and feedback error. The equations for calculating these errors are shown in Figure 9.1. If the feedback function 'H' has a value of '1' then these errors will be the same.



System error, $S = c - r$

Feedback error, $e = c - b$

Figure 9.1 Controller errors

An example of calculating these errors is shown in Figure 9.2. The system is a simple integrator, with a unity feedback loop. The overall transfer function for the system is calculated and then used to find the system response. The response is then compared to the input to find the system error. In this case the error will go to zero as time approaches infinity.

Given,

$$G(D) = \frac{K_p}{D} \qquad H(D) = 1$$

$$\frac{r}{c} = \frac{G}{1+GH} = \frac{K_p}{D+K_p}$$

$$\dot{r} + K_p r = K_p c$$

$$c = A \qquad \text{(a step function)}$$

The homogeneous solution is,

$$r_h = C_1 e^{-K_p t}$$

The particular solution is found with a guess,

$$r_p = C_2$$

$$\dot{r}_p = 0$$

$$0 + K_p C_2 = K_p A \qquad C_2 = A$$

The solutions can be combined and the remaining unknown found for the system at rest initially.

$$r = r_h + r_p = C_1 e^{-K_p t} + A$$

$$0 = C_1 e^0 + A$$

$$C_1 = -A$$

$$r = -Ae^{-K_p t} + A$$

The error can now be calculated.

$$S = c - r$$

$$S = A - \left(-Ae^{-K_p t} + A\right) = Ae^{-K_p t}$$

Figure 9.2 System error calculation example for a step input

Controller Transfer Functions

The PID controller, and simpler variations were discussed in earlier sections. A more complete table is given in Figure

9.3.

Type	*Transfer Function*	
Proportional (P)	$G_c = K$	
Proportional-Integral (PI)	$G_c = K\left(1 + \frac{1}{\tau D}\right)$	
Proportional-Derivative (PD)	$G_c = K(1 + \tau D)$	
Proportional-Integral-Derivative (PID)	$G_c = K\left(1 + \frac{1}{\tau D} + \tau D\right)$	
Lead	$G_c = K\left(\frac{1 + \alpha\tau D}{1 + \tau D}\right)$	$\alpha > 1$
Lag	$G_c = K\left(\frac{1 + \tau D}{1 + \alpha\tau D}\right)$	$\alpha > 1$
Lead-Lag	$G_c = K\left[\left(\frac{1 + \tau_1 D}{1 + \alpha\tau_1 D}\right)\left(\frac{1 + \alpha\tau_2 D}{1 + \tau_2 D}\right)\right]$	$\alpha > 1$ $\tau_1 > \tau_2$

Figure 9.3 Standard controller types

Feedforward Controllers

When a model of a system is well known it can be used to improve the performance of a control system by adding a feed forward function, as pictured in Figure 9.4. The feed forward function is basically an inverse model of the process. When this is used together with a more traditional feedback function the overall system can outperform more traditional controllers function, such as the PID controller.

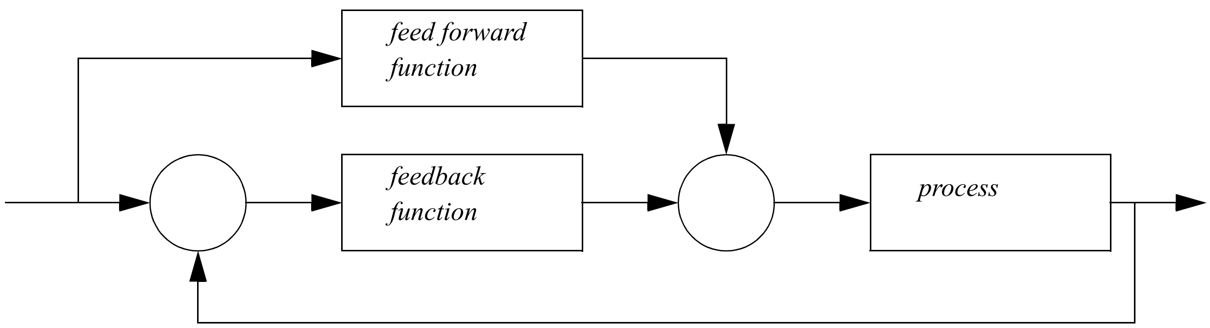


Figure 9.4 A feed forward controller

State Equation Based Systems

State variable matrices were introduced before. These can also be used to form a control system, as shown in Figure 9.5.

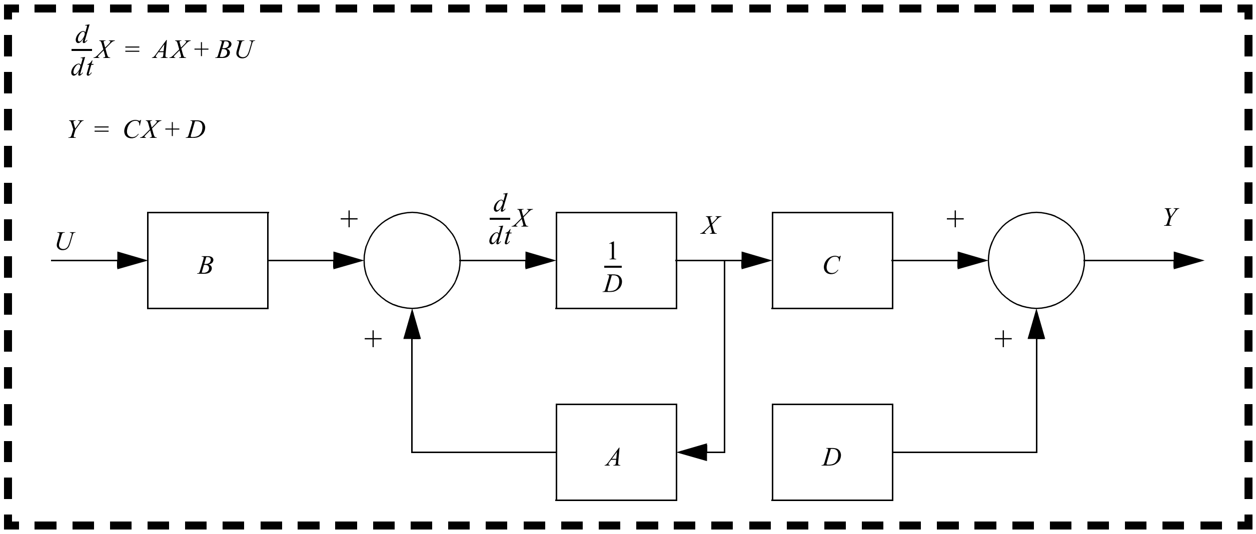


Figure 9.5 A state variable control system

An example is shown in Figure 9.6 that implements a second order state equation. The system uses two integrators to integrate the angular acceleration, then the angular velocity, to get the position.

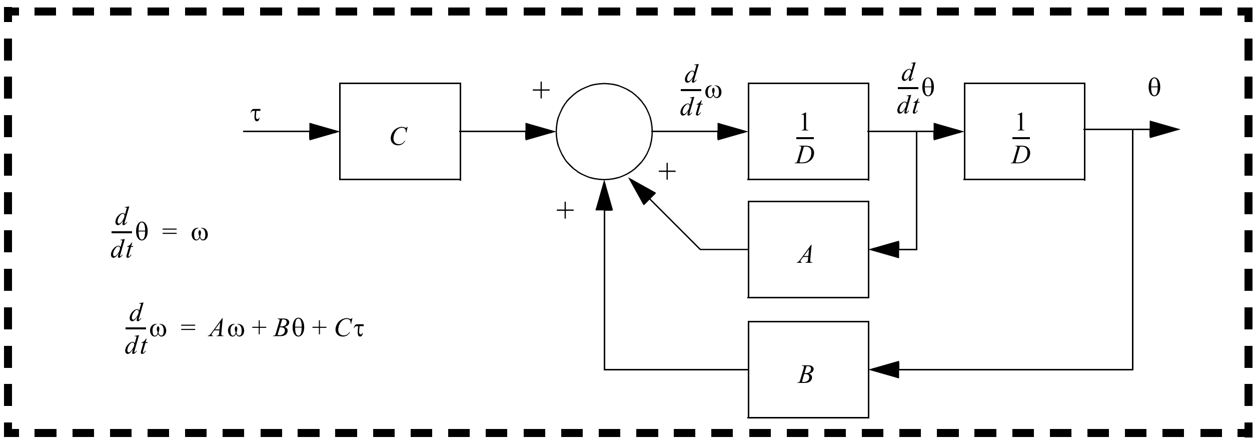


Figure 9.6 A second order state variable control system

The previous block diagrams are useful for simulating systems. These can then be used in feed forward control systems to estimate system performance and then predict a useful output value.

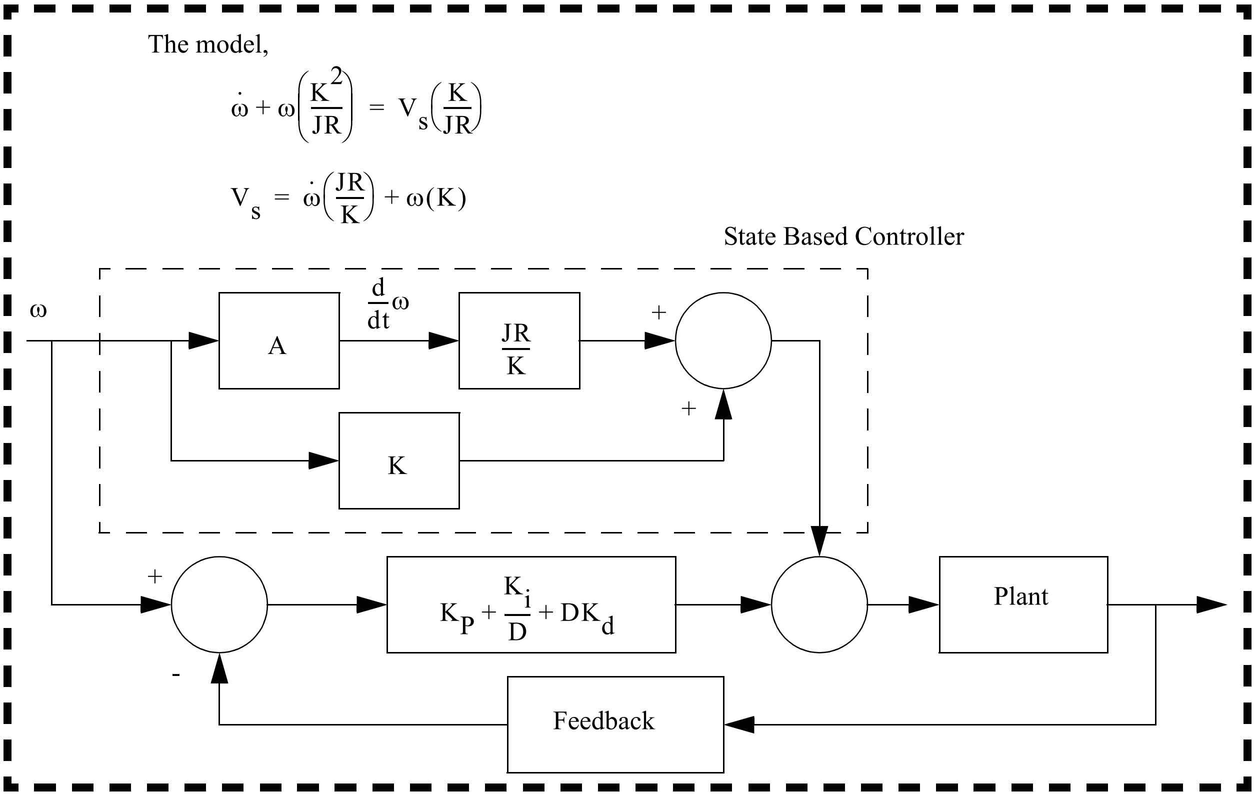


Figure 9.7 A state based feed forward controller

Cascade Controllers

When controlling a multi-step process a cascade controller can allow refined control of sub-loops within the larger control system. Most large processes will have some form of cascade control. For example, the inner loop may be for a heating oven, while the outer loop controls a conveyor feeding parts into the oven.

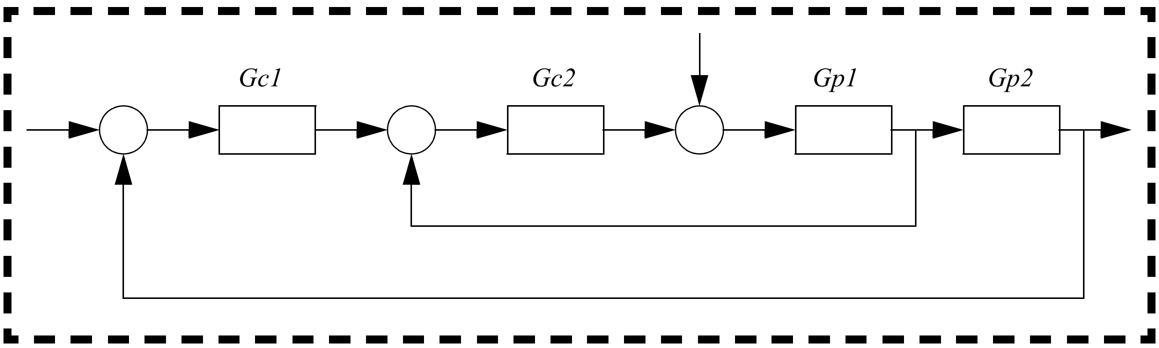


Figure 9.8 A cascade controller

9.1 Embedded Control

Previous sections have shown how to develop a controller transfer function for a desired system behavior. These transfer functions could be converted to op-amp circuits, mechanisms, or software. Figure 9.9 shows the conversion of a transfer function to a subroutine written in C. The controller transfer function has an input of ‘e’, the feedback error, and the output of ‘u’, the controller output. The transfer function is converted to a differential equation and then two state equations. Both of the state equations are rewritten using the subscript ‘i’ the represent the current and previous values of each value. These previous ‘i-1’ values become 'e_last', 'v_last', and 'u_last' in the subroutine. The derivative 'e_derivative' is calculated using the current and previous values of 'e'. A time step 'T' is provided for differentiating and integrating. The time step would be the since the last 'e' value was used to calculate the last 'u' value. The sequence of this subroutine is i) calculate the derivative of 'e', ii) find the derivatives using the state equations, iii) integrate the derivatives, and iv) store the current values for use in the next time step.

$$\frac{u}{e} = \frac{D+4}{D^2 + 10D + 100}$$

First this needs to be reduced to a differential equation.

$$\ddot{u} + 10\dot{u} + 100u = \dot{e} + 4e$$

This is then rearranged to state equations with subscripts,

$$\dot{u} = v$$

$$\dot{u}_i = v_{i-1}$$

$$\dot{v} + 10v + 100u = \dot{e} + 4e$$

$$\therefore \dot{v} = -10v - 100u + \dot{e} + 4e$$

$$\therefore \dot{v}_i = -10v_{i-1} - 100u_{i-1} + \dot{e}_i + 4e_i$$

The derivative for e can be calculated using previous values,

$$\dot{e}_i = \frac{e_i - e_{i-1}}{T}$$

```
#define  T  0.010 // time step

double controller(double e){
    double   e_derivative, u_derivative, v_derivative;
    static double e_last=0, u_last=0, v_last=0;

    e_derivative = ( e - e_last ) / T; /* the error derivative */

    /* the two state equations */
    u_derivative = v_last;
    v_derivative = -10*v_last - 100*u_last + e_derivative + 4*e;


    u = u_last + T*u_derivative;
    v = v_last + T*v_derivative;

    u_last = u;
    v_last = v;
    e_last = e;

    return u;
}
```

Figure 9.9 Converting A Transfer Functions to C Code

Using a single time step for integration and differentiation is reasonable for simpler systems with little or no noise. In practical systems more sophisticated methods are used for calculating integrals and derivatives. The accuracy of the error estimation can be increased by using a Taylor series equation, Figure 9.10. In this example the first and second order derivatives are estimated using previous error values. In this case the first and second order derivatives are used, but additional terms can be used to increase the accuracy of the error estimation. (Note that these methods are explored in depth in signals and systems courses, using 'z-transforms'.)

A Taylor series approach will provide a better estimate of the derivative.

$$e_i \approx e_{i-1} + T\dot{e}_{i-1} + \frac{T^2\ddot{e}_{i-1}}{2} + \frac{T^3\dddot{e}_{i-1}}{6} + \dots$$

$$\dot{e}_i \approx 0 + \dot{e}_{i-1} + T\ddot{e}_{i-1} + \frac{T^2\dddot{e}_{i-1}}{2} + \dots$$

$$\dot{e}_{i-1} \approx \frac{e_i - e_{i-1}}{T}$$

$$\ddot{e}_{i-1} \approx \frac{\dot{e}_{i-1} - \dot{e}_{i-2}}{T} \approx \frac{\frac{e_i - e_{i-1}}{T} - \frac{e_{i-1} - e_{i-2}}{T}}{T} = \left(\frac{1}{T^2}\right)e_i + \left(\frac{-2}{T^2}\right)e_{i-1} + \left(\frac{1}{T^2}\right)e_{i-2}$$

$$\dot{e}_i \approx \dot{e}_{i-1} + T\ddot{e}_{i-1} \approx \frac{e_i - e_{i-1}}{T} + \left(\frac{T}{T^2}\right)e_i + \left(\frac{-2T}{T^2}\right)e_{i-1} + \left(\frac{T}{T^2}\right)e_{i-2}$$

$$\dot{e}_i \approx \left(\frac{2}{T}\right)e_i + \left(-\frac{3}{T}\right)e_{i-1} + \left(\frac{1}{T}\right)e_{i-2}$$

```
double controller(double e){
    double   e_derivative, u_derivative, v_derivative;
    static double e_1, e_2; /* the previous values of e */
    static double u_last=0, v_last=0;

    e_derivative = ( 2 * e - 3 * e_1 + e_2 ) / T; /* the error derivative */
```

Figure 9.10   A Second Order Taylor Series Estimation of the Derivative Error

Other considerations include practical limits on the control systems and potential failure modes, Figure 9.11. For example the output of the control system may be from -10.0V to 10.0V. Therefore, any value of ‘u’ that falls outside of the valid range is clipped at the maximum or minimum. Another problem would be an invalid value of ‘T’ that results in a divide by zero error. Ensuring that the time step is greater than 0.0001 should prevent problems. This step is essential when dealing with system with a variable time step value.

```
#define  T  0.010 // time step
#define  u_max   10.0  // maximum u value
#define  u_min   -10.0  // minimum u value

double controller(double e){
    double   e_derivative, u_derivative, v_derivative;
    static double e_1, e_2; /* the previous values of e */
    static double u_last=0, v_last=0;

    if ( T , 0.0001 ){
        fault(); // create a fault condition if the time step is too small
    }

    e_derivative = ( 2 * e - 3 * e_1 + e_2 ) / T; /* the error derivative */

    /* the two state equations */
    u_derivative = v_last;
    v_derivative = -10*v_last - 100*u_last + e_derivative + 4*e;


    u = u_last + T*u_derivative;
    v = v_last + T*v_derivative;

    /* Clip the values at the output limits */
    if ( u > u_max ) u = u_max;
    if ( u < u_min ) u = u_min;

    u_last = u;
    v_last = v;
    e_2 = e_1;
    e_1 = e;

    return u;
}
```

Figure 9.11 A Control Subroutine With Error Detection

9.2 Summary

- Controllers can be designed to meet criteria, such as damping ratio and natural frequency.
- System errors can be used to determine the long term stability and accuracy of a controlled system.
- Other control types are possible for more advanced systems.

9.3 Problems With Solutions

Problem 9.1 The following system is a feedback controller for an elevator. It uses a desired height ‘d’ provided by a user, and the actual height of the elevator ‘h’. The difference between these two is called the error ‘e’. The PID controller will examine the value ‘e’ and then control the speed of the lift motor with a control voltage ‘c’. The elevator and controller are described with transfer functions, as shown below. all of these equations can be combined into a single system transfer equation as shown.

$error = d - h$

$$\frac{r}{e} = K_p + \frac{K_i}{D} + K_d D = \frac{2D + 1 + D^2}{D}$$ PID controller

$$\frac{h}{r} = \frac{10}{D^2 + D}$$ elevator

combine the transfer functions

$$\left(\frac{r}{e}\right)\left(\frac{h}{r}\right) = \frac{h}{e} = \frac{2D + 1 + D^2}{D}\,\frac{10}{D^2 + D} = \frac{(D + 1)^2}{D}\,\frac{10}{D(D + 1)} = \frac{10(D + 1)}{D^2}$$

$$\frac{h}{d - h} = \frac{10(D + 1)}{D^2}$$ eliminate 'e'

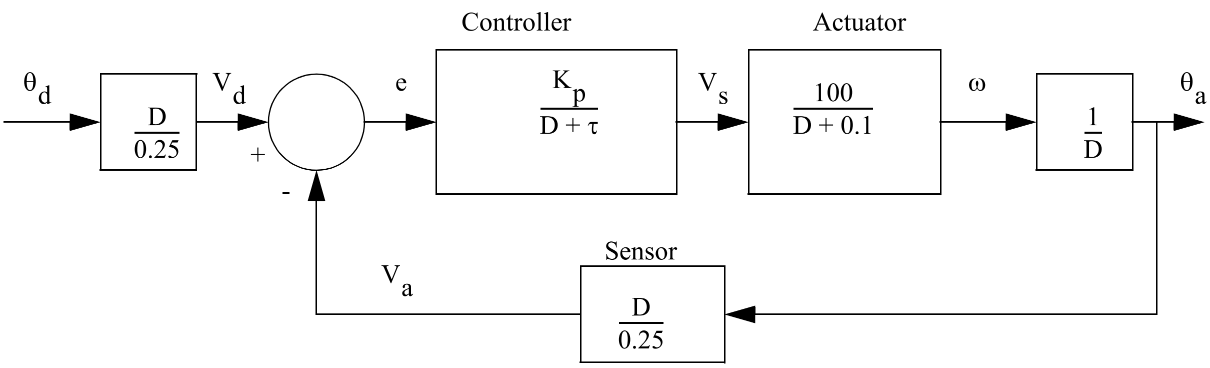
$$h = \left(\frac{10(D + 1)}{D^2}\right)(d - h)$$

$$h\left(1 + \frac{10(D + 1)}{D^2}\right) = \left(\frac{10(D + 1)}{D^2}\right)(d)$$

$$\frac{h}{d} = \left(\frac{\frac{10(D + 1)}{D^2}}{1 + \frac{10(D + 1)}{D^2}}\right) = \frac{10D + 10}{D^2 + 10D + 10}$$ system transfer function

a) Find the response of the final equation to a step input. The system starts at rest on the ground floor, and the input (desired height) changes to 20 as a step input.
b) Calculate the damping factor and natural frequency of the results in part a).

Problem 9.2 a) Simplify the block diagram as far as possible.



b) Given the transfer function below, select values for Kp and tau that will include a second order response that has a damping factor of 0.125 and a natural frequency of 10rad/s.

$$\frac{Q_a}{Q_d} = \frac{400K_p}{D^2 + D(\tau + 0.1) + (400K_p + 0.1\tau)}$$

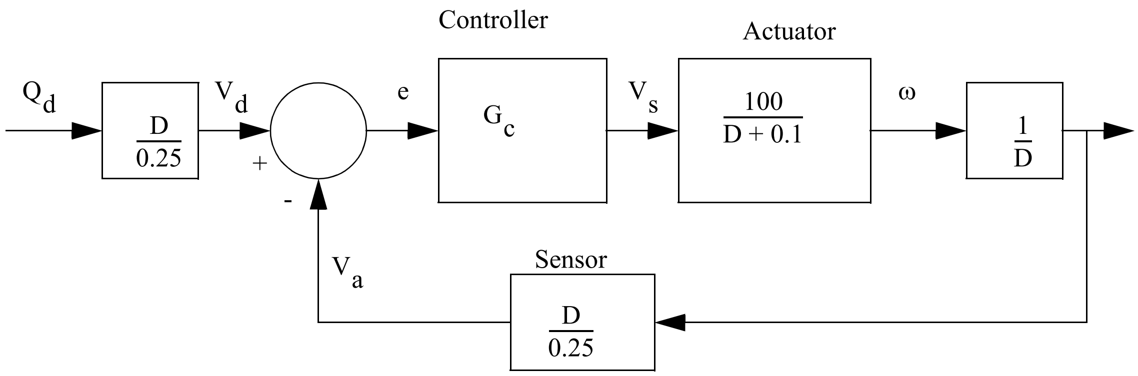
c) The function below has a step input of magnitude 1.0. Find the output as a function of time using numerical methods. Give the results in a table OR graph.

$$\frac{Q_a}{Q_d} = \frac{99.76}{D^2 + 2.5D + 99.884}$$

d) The function below has a step input of magnitude 1.0. Find the output as a function of time by integrating the differential equation (i.e., using the homogeneous and particular solutions).

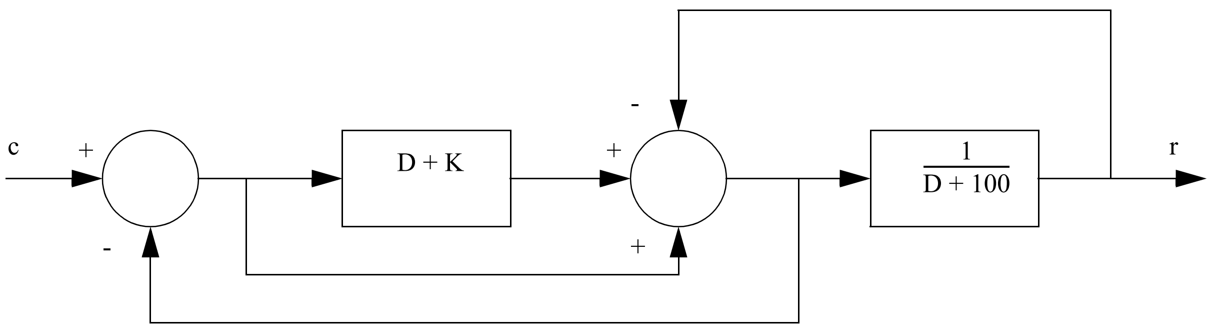
$$\frac{Q_a}{Q_d} = \frac{99.76}{D^2 + 2.5D + 99.884}$$

e) Select a controller transfer function, Gc, that will reduce the system to a first order system with a time constant of 0.5s, as shown below.



desired transfer function $\frac{Q_a}{Q_d} = \frac{1}{D+2}$

Problem 9.3 Convert the following block diagram to state equations. Pick a K value that would leave the system critically damped.



Problem 9.4 Write a function in C that implements a PID function using the following function prototype. The function accepts the desired and current values and returns the controller output.

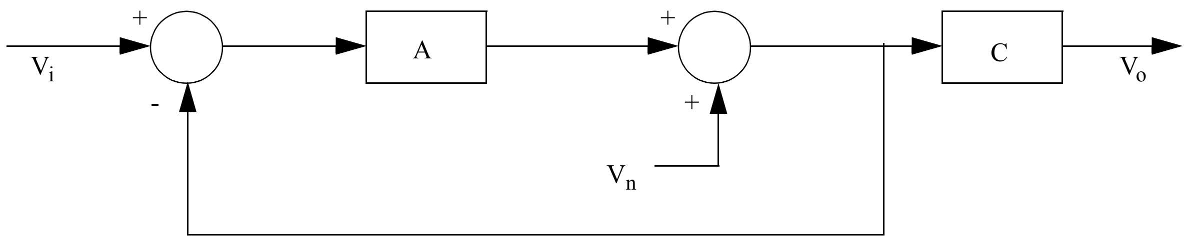
```
double pid(double v_desired, double v_feedback);
```

Problem 9.5 Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

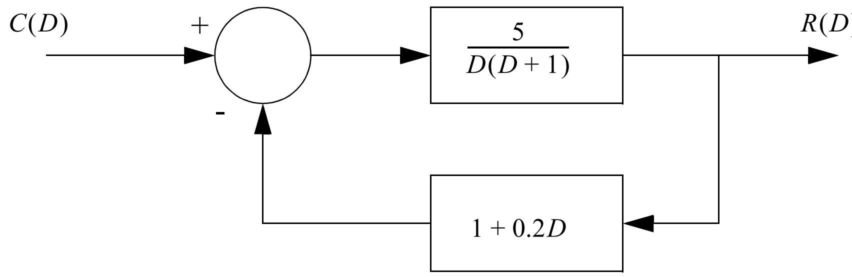
$$G = \frac{4}{D+2} = \frac{Y}{X} \qquad X(t) = 20 \quad \text{When } t >= 0$$

Problem 9.6 Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.
a) develop a transfer function for y/F.
b) find the input function F.
c) solve the input output equation to find an explicit equation of the position as a function of time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.
d) solve part c) numerically.

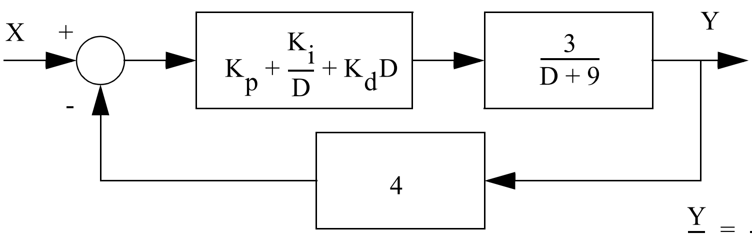
Problem 9.7 Simplify the block diagram below. (Note: Vn is an input and cannot be combined in a transfer function.)



Problem 9.8 Find the system output, feedback error and system error when the input is a ramp with the function c(t) = 0.5t. Sketch the system errors as a function of time.



Problem 9.9 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer function is given.



$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1 + 12K_d) + D(9 + 12K_p) + (12K_i)}$$

a) Verify the closed loop controller function given.
b) For the given transfer function select controller values that will result in a natural frequency of 3 rad/sec and damping factor of 0.5. (Hint: assume Kd=0.)
c) For the given transfer function, if the values are Kp=5 and Ki=1, and Kd=0, find the response equation to a unit ramp input (i.e., X=t) as a function of time by solving the differential equation to obtain an explicit solution.
d) For the given transfer function, if the values of Kp=1 and Ki=Kd=0, find the response to a unit ramp input (i.e., X=t) as a function of time using state equations and a numerical method. (email the program and graphical output to the instructor).

9.4 Problem Solutions

Answer 9.1

a) $h(t) = 2.91e^{-8.873t} - 22.91e^{-1.127t} + 20$

b) $\zeta = 1.58$ $\omega_n = 3.162\frac{rad}{s}$

Answer 9.2

a) $$\frac{Q_a}{Q_d} = \frac{400K_p}{D^2 + D(\tau + 0.1) + (400K_p + 0.1\tau)}$$

b) $$\frac{400K_p}{D^2 + D(\tau + 0.1) + (400K_p + 0.1\tau)} = \frac{(\dots)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

$$\tau + 0.1 = 2\zeta\omega_n = 2\left(\frac{1}{8}\right)10 = 2.5 \qquad \tau = 2.4$$

$$400K_p + 0.1\tau = \omega_n^2 = 100 \qquad K_p = \frac{100 - 0.1(2.4)}{400} = 0.2494$$

$$\frac{Q_a}{Q_d} = \frac{99.76}{D^2 + 2.5D + 99.884}$$

c) $$\dot{Q}_a = P_a$$

$$\dot{P}_a = P_a(-2.5) + Q_a(-99.884) + Q_d(99.76)$$

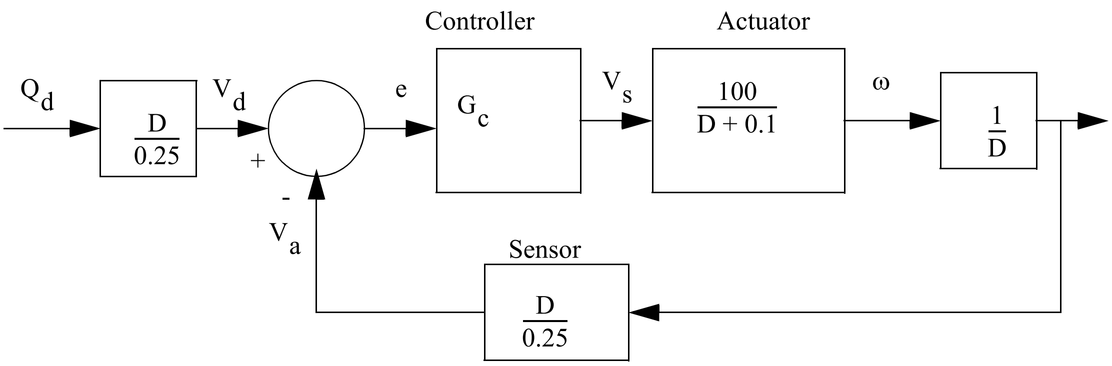
d) $$\ddot{Q}_a + \dot{Q}_a(2.5) + Q_a(99.884) = Q_d(99.76)$$

Found using Wolfram Alpha:
q'' + 2.5q' + 99.884q = 99.76, q(0)=0, q'(0)=0

$$Q_a(t) = -0.1259e^{-1.25t}\sin(9.9157t) - 0.9988e^{-1.25t}\cos(9.9157t) + 0.9988$$

$$Q_a(t) = 1.0067e^{-1.25t}\sin\left(9.9157t + \operatorname{atan}\left(\frac{0.9988}{0.1259}\right) + \pi\right) + 0.9988$$

e)



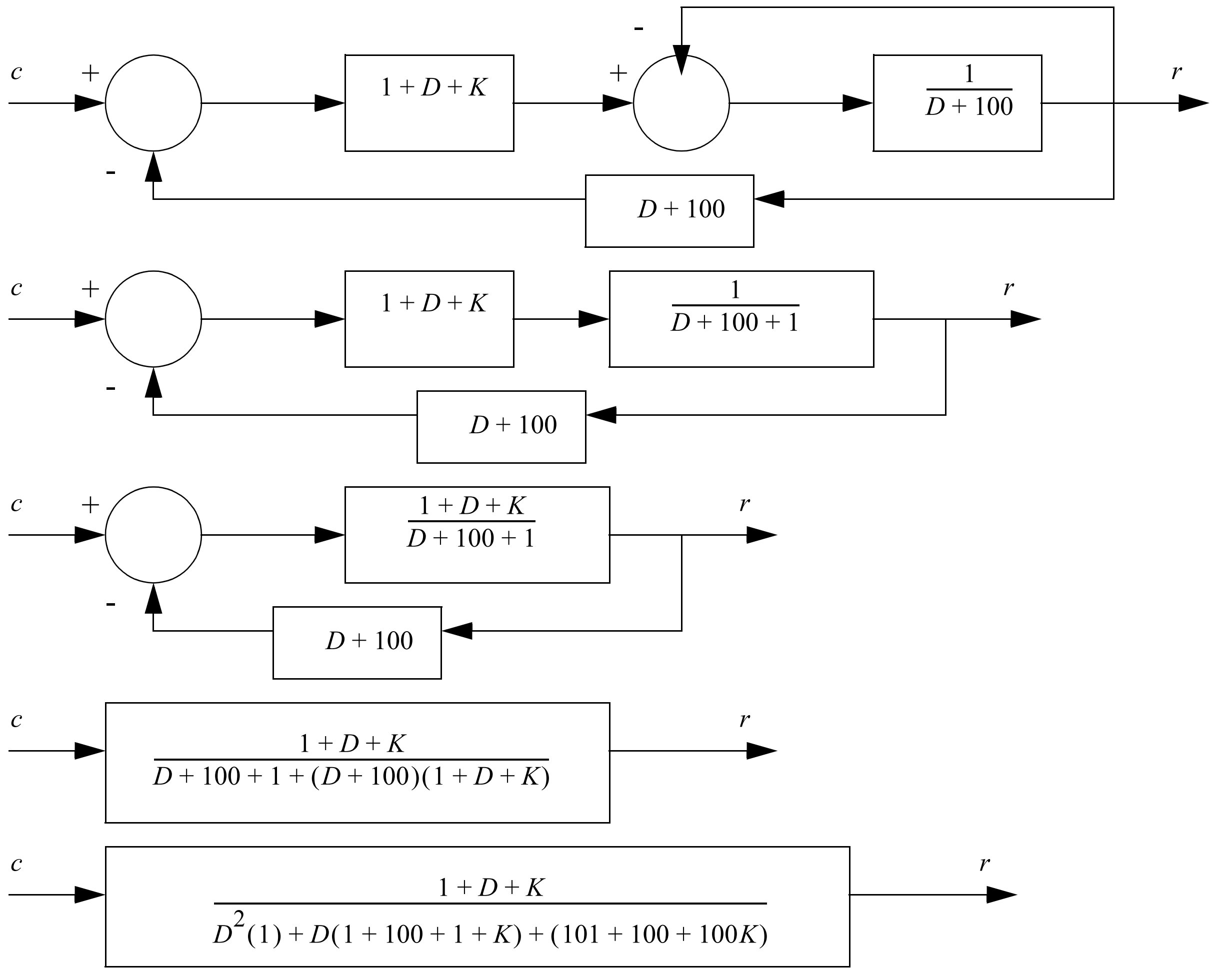
$$\frac{Q_a}{Q_d} = \frac{4G_c(100)D}{D(D+0.1) + 4DG_c(100)} = \frac{1}{D+2}$$

$$4G_c(100)(D+2) = (D+0.1) + 4G_c(100)$$

$$G_c(400D + 800 - 400) = (D + 0.1)$$

$$G_c = \left(\frac{D + 0.1}{400D + 400}\right)$$

Answer 9.3



$$\frac{r}{c} = \frac{1 + D + K}{D^2 + D(102 + K) + (201 + 100K)}$$

$$\omega_n = \sqrt{201 + 100K}$$

$$2(1)\sqrt{201 + 100K} = 102 + K$$

$$4(201) + 4(100K) = K^2 + 204K + 102^2$$

$$0 = K^2 - 196K + 9600 \qquad K = 96, 100$$

$$r(D^2 + D(102 + K) + (201 + 100K)) = c(1 + D + K)$$

$$\dot{r} = s$$

$$\dot{s} = s(-102 - K) + r(-201 - 100K) + \dot{c}(1) + c(1 + K)$$

$$\dot{s} - \dot{c} = s(-102 - K) + r(-201 - 100K) + c(1 + K)$$

$$q = s - c$$

$$\dot{q} = s(-102 - K) + r(-201 - 100K) + c(1 + K) \qquad \text{eqn 9.1}$$

$$q = s - c = \dot{r} - c$$

$$\dot{r} = c + q \qquad \text{eqn 9.2}$$

$$s = q + c \qquad \text{eqn 9.3}$$

Answer 9.4

```
#define T 0.010 // time step
#define Kp 1
#define Ki 1
#define Kd 1

double pid(double v_desired, double v_feedback){
    double   e, derivative;
    static double e_last = 0, integral_sum = 0;

    e = v_desired - v_feedback;
    derivative = (e - e_last) / T;
    integral_sum + T * e_last;

    e_last = e;

    return Kp * e + Ki * integral_sum + Kd * derivative;
}
```

Answer 9.5

$$y(t) = -40e^{-2t} + 40$$

Answer 9.6

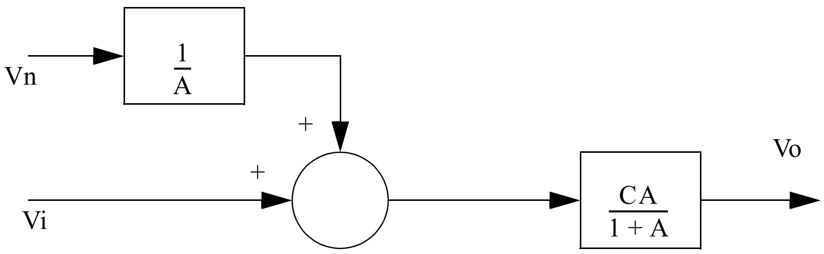
a) $$\frac{y}{F} = \frac{1}{D^2 M + DK_d + K_s}$$

b) $$F = Mg$$

c) $$y(t) = -10.13e^{-0.25t}\cos(0.968t - 0.253) + 9.81$$

d) $$\begin{bmatrix} y(t+h) \\ v(t+h) \end{bmatrix} = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + h\left[\begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 9.81 \end{bmatrix}\right]$$

Answer 9.7



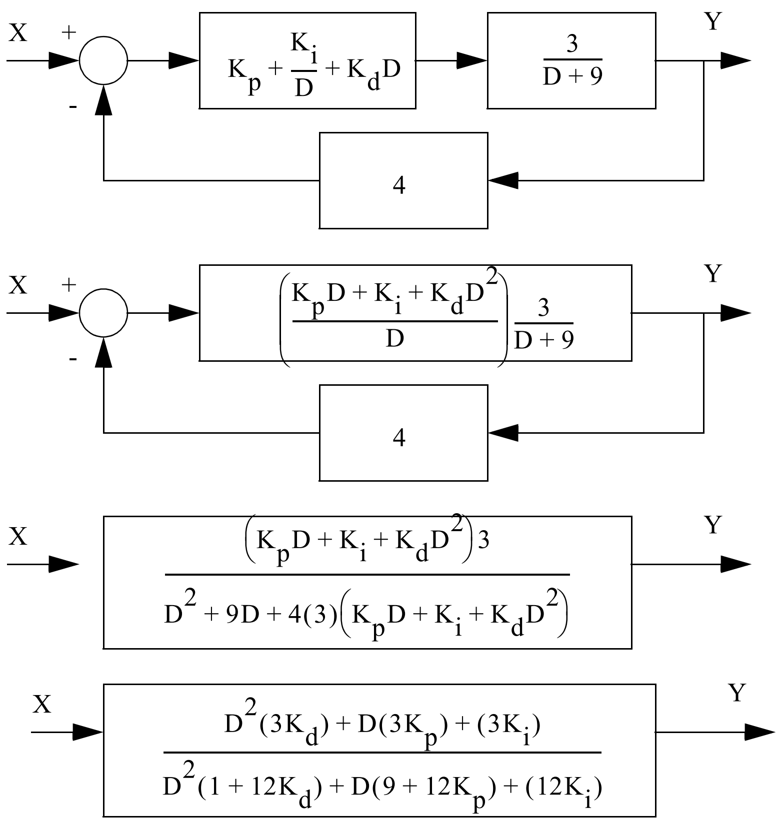
Answer 9.8

$$R(t) = 0.25e^{-t}\cos(2t + 0.644) - \frac{1}{5} + \frac{1}{2}t$$

$$S(t) = 0.25e^{-t}\cos(2t + 0.644) - \frac{1}{5}$$

$$e(t) = 0.224e^{-t}\sin(2t - 0.464) + \frac{1}{10}$$

Answer 9.9 a)



b)

$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1 + 12K_d) + D(9 + 12K_p) + (12K_i)} = \frac{\dots}{D^2 + D2\zeta\omega_n + \omega_n^2}$$

$$\omega_n^2 = \frac{12K_i}{1 + 12K_d} \qquad 3^2 = \frac{12K_i}{1 + 12K_d} \qquad 9 + 108K_d = 12K_i$$

$$2\zeta\omega_n = \frac{9 + 12K_p}{1 + 12K_d} \qquad 2(0.5)3 = \frac{9 + 12K_p}{1 + 12K_d} \qquad 3 + 36K_d = 9 + 12K_p$$

In simple terms there are two equations and three unknowns. The system can only be solved by adding one equation, or removing one unknown. Therefore I have made an arbitrary decision to set Kd = 0. However, any of the gains could be set to any value to move ahead. Note: We would prefer to avoid arbitrary solutions and under-constrained problems suggest ill-posed designs.

$$K_d = 0$$

$$9 + 108(0) = 12K_i \qquad K_i = \frac{3}{4}$$

$$3 + 36(0) = 9 + 12K_p \qquad K_p = -\frac{1}{2}$$

c)

$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(1+12K_d) + D(9+12K_p) + (12K_i)} = \frac{15D+3}{D^2+69D+12}$$

$$\ddot{Y} + 69\dot{Y} + 12Y = 15\dot{X} + 3X$$

$$X = t$$

$$\ddot{Y} + 69\dot{Y} + 12Y = 15 + 3t \qquad X = t$$

Assume initial conditions are zero. Solution simplified from WolframAlpha using y" + 69y' + 12y = 15 + 3t, y(0)=0, y'(0)=0

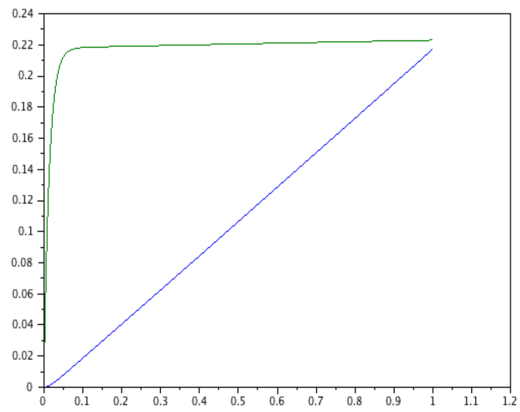
$$Y(t) = 62.500\times10^{-3}(4t-3) + 3.1654\times10^{-3}e^{-68.8256t} + 184.33\times10^{-3}e^{-0.17435t}$$

d)

$$\frac{Y}{X} = \frac{D^2(3(0)) + D(3(1)) + (3(0))}{D^2(1+12(0)) + D(9+12(1)) + (12(0))} = \frac{(3)}{D(1)+(21)}$$

$$\dot{Y} + 21Y = 3t$$

$$\dot{Y} = Y(-21) + 3t$$




```
Y0 = 0 ;
ST = [ Y0 ];
t = [0];
h = 0.001;
n_steps = 1000;
function ST_der = state_eqns(ST, t)
    Y = ST($, 1) ;
    Y_der = - 21 * Y + 3 * t;
    ST_der = [ Y_der ];
endfunction
for i = 1 : n_steps
    ST = [ ST ; ST($, :) + h * state_eqns( ST($, :), t($) ) ]
    t = [ t ; t($) + h]
end
plot(t, ST);
```

9.1 Problems Without Solutions

Problem 9.10 *Calculate the system response using a ramp input,*

Given,

$$G(D) = \frac{K_p}{D}$$

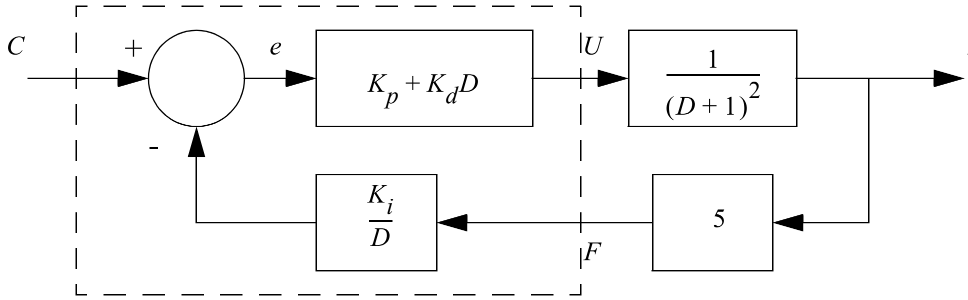
$$H(D) = 1$$

$$c = At$$

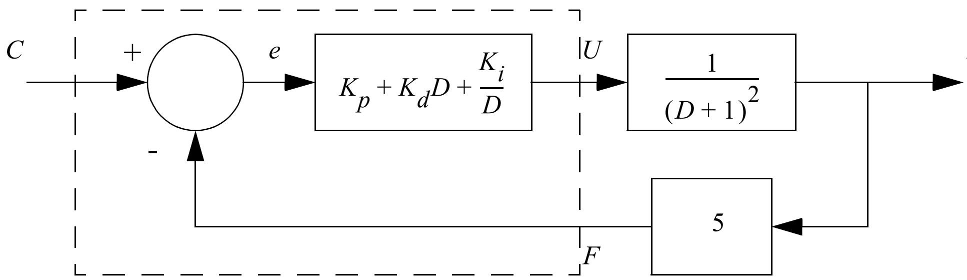
Problem 9.11 Find the system error 'e' for the given ramp input, R.

$$G(D) = \frac{1}{D^2 + 4D + 5} \qquad H(D) = 5 \qquad c = 4t$$

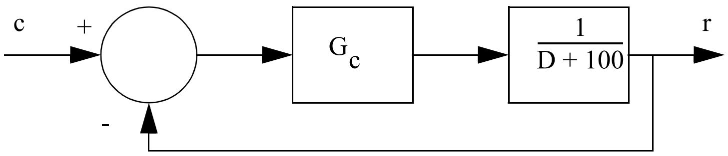
Problem 9.12 Write a C subroutine to implement the control system that is shown inside the dashed line. The subroutine arguments are the setpoint, C, and the feedback value, F. the subroutine returns the new controller output value, U.



Problem 9.13 Write C subroutines to implement the control system that is shown inside the dashed line. The subroutine arguments are the setpoint, C, and the feedback value, F. the subroutine returns the new controller output value, U.



Problem 9.14 Given the following negative feedback control system, calculate a new controller transfer function (Gc) to get the desired response for the system.



desired response

$$\frac{r}{c} = \frac{1}{D + 10}$$

Problem 9.15 Write a C subroutine that implements the control function below. (Hint: Convert it to state equations first.)

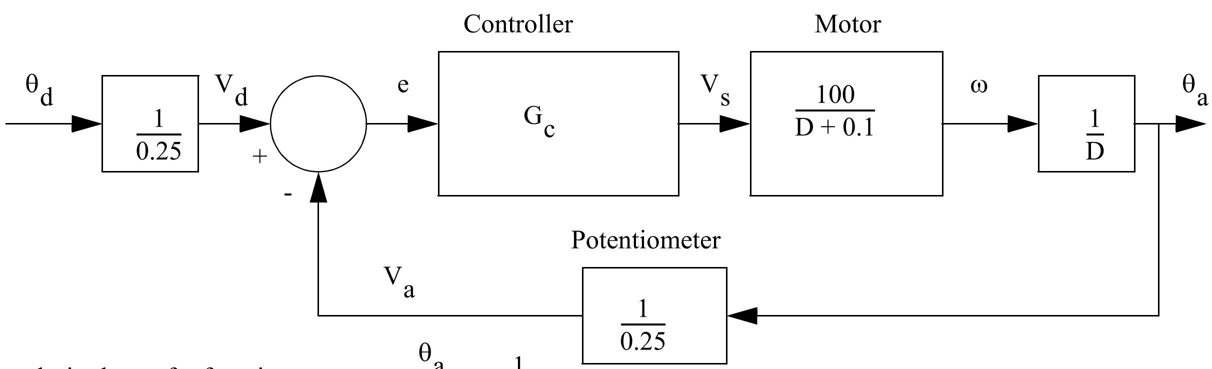
$$\frac{u}{e} = \frac{D + 10}{D^2 + 2D + 3}$$

```
#define T 0.010 // The step time is 10ms

double u(e) {
```

Problem 9.16 Draw the block diagram equivalent of the C subroutine.

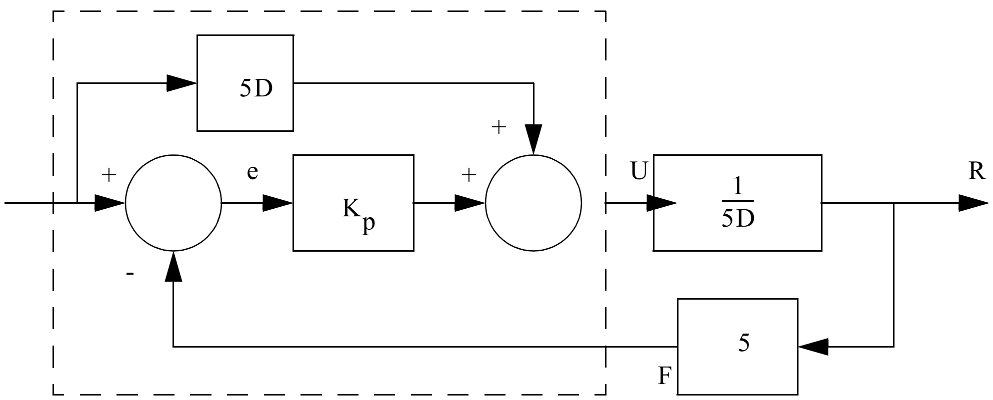
```
double U(double C, double F){
        double e;// Error value
        double F_int = 0.0;// integral of feedback value
        double e_int = 0.0;// integral of error
        double Kp = 10.0;// Proportional gain coefficient
        double Ki = 2.0;// Integral gain coefficient
        double T = 0.010;// Time step for system in seconds

        F_int = F_int + T * F;
        e = C - F_int;
        e_int = e_int + T * e;
        return Kp * e + Ki * e_int;
}
```

Problem 9.17 Select a controller transfer function, Gc, that will reduce the system to a first order system with a time constant of 0.5s, as shown below.



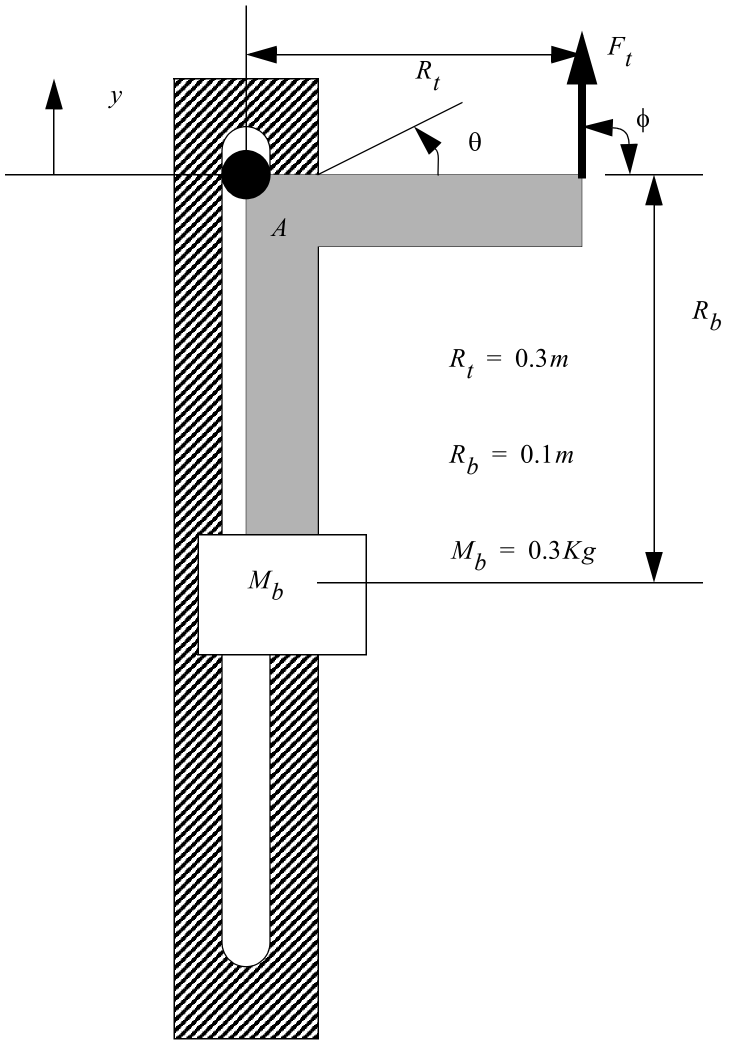
desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{D+2}$

Problem 9.18 Write a C subroutine to implement the control system that is shown inside the dashed line. The subroutine arguments are the setpoint, C, and the feedback value, F. the subroutine returns the new controller output value, U.



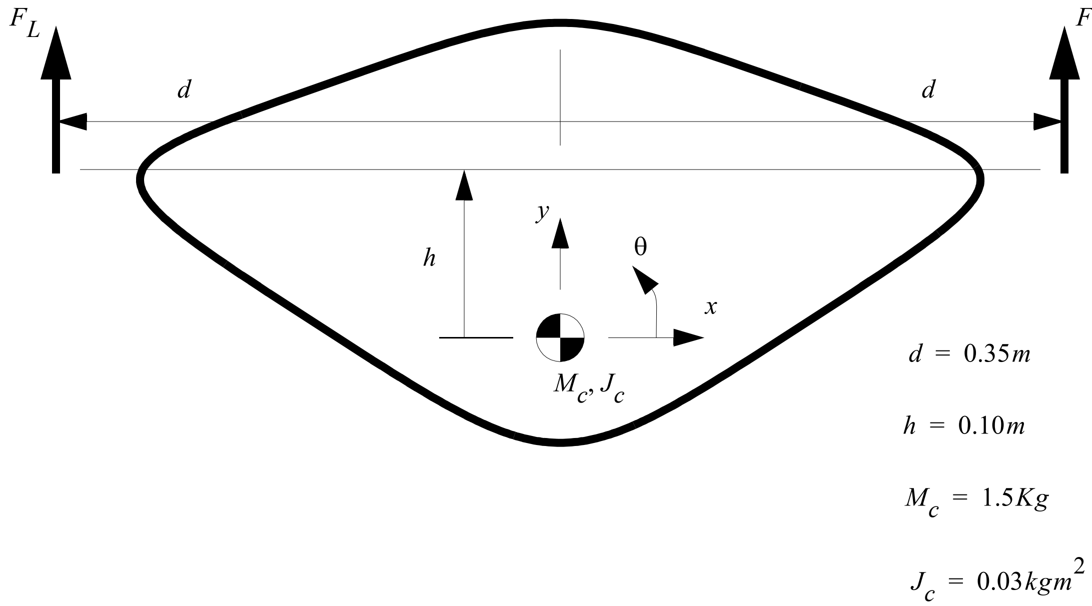
int controller(int C, int F){

Problem 9.19 A symmetrical system is approximated with a thrust force Ft, and a ballast mass Mb. The mass and forces are attached to an angled arm. The arm is free to move vertically, and rotate about point A.



a) Develop differential equations to model the system as a function of height and angle. Neglect Coriolis effects.
b) Develop a numerical integration program.
c) Assume that the thrust force, Ft, is always vertical. Numerically simulate the system if Ft = 10sin(10t)+12.
d) Develop a subroutine for modeling a brushless servo motor driven propeller. The input to the unit is a PWM value between 0% and 100%. The output was measured with as; 0%=0N, 5%=0N, 10%=3N, 20%=6N, 40%=12N, 60%=17N, 80%=21N, 100%=23N. Given a particular RPM value the subroutine should estimate the 0, 1st, and 2nd derivative of the thrust value.
e) Develop a feedback diagram for a system that uses a PID controller to control the angle ONLY.
f) Develop a feedback diagram for a system that uses a PID controller to control the height ONLY.
g) Write software in spin to implement the control algorithms.
h) Example the numerical integration program to model the height control feedback loop using a height of 0.5m.
i) Expand the system model to have two thrust motors symmetrically positioned. Each will have its own thrust force.
j) Modify the numerical analysis program to balance the two motor system.

Problem 9.20 A quadcopter system has four motors mounted on arms. For analysis and design purposes these systems can be simplified to a mass hung between two propellers. The equivalent system is pictured in the figure below. The copter body is defined as a solid mass Mc, and Jc. The position of the copter is defined using an x,y position and angle of rotation, theta. There are two motors driving propellers. The are centered at a height of 'h' above, and 'd' across from the center of mass. The motors are approximated with a thrust force, FL or FR.



a) Draw three FBDs for the x translation, y translation, and rotation.
b) Write the six state equations for the copter.
c) Write a program in Scilab to numerically integrate the system for Fr = Fl = 7.36 N.
d) Repeat the analysis in c) using Fl = 7.36 N, Fr = 7.36+1sin(3t) N.
e) Develop a Scilab subroutine for modeling a brushless servo motor driven propeller. The input to the unit is a PWM value between 0% and 100%. The output was measured with as; 0%=0N, 5%=0N, 10%=3N, 20%=6N, 40%=12N, 60%=17N, 80%=21N, 100%=23N. The subroutine should accept the PWM value and return the force value. (e.g., F = PWM_to_F(PWM))
f) Write a Scilab subroutine for the brushless DC motor accept a desired thrust force, and return a PWM value. (e.g., PWM = F_to_PWM(Thrust))

g) Use the following Pseudocode to modify the Scilab simulation. Run the program and verify.

```
t_now := 0;
y := 0;
x := 0;
theta := 0;

function state equations

function F_to_PWM

function PWM_to_F

repeat until t > 10s
        y_setpoint := 10; // eventually get_new_y(t);
        x_setpoint := 0; // eventually get_new_x(t)

        y_e := y_setpoint - y;
        x_e := x_setpoint - x;

        theta_setpoint := 1 * x_e;
        theta_e := theta_setpoint - theta;

        F_both = 20 * y_e; // eventually becomes height_PID(y_e)
        F_diff = 20 *  theta_e; // eventually angle_PID(theta_setpoint)

        Fl_wanted = F_both - F_diff
        Fr_wanted = F_both + F_diff
        PWM_l = F_to_PWM(Fl);
        PWM_r = F_to_PWM(Fr);
        Fl = PWM_to_F(PWM_1);
        Fr = PWM_to_F(PWM_r);

        // Integrate for 10ms
        loop t = t_now to t_now+10ms in 0.1ms steps
                [x, y, theta] = [x, y, theta] + h * [x', y', theta']
        t := t + 10ms

plot (t, x, y, theta)
```

h) Draw a block diagram for the program in step g).
i) Implement, add, and debug the function 'height_PID'
j) Implement, add, and debug the function 'angle_PID'
k) Write the 'get_new_x' and 'get_new_y' functions so that x=y=0 while y<10s and x=y=10 when t >= 10s.
l) Test the system and refine the PID parameters to minimize the settling time after the change at t=10s.

Problem 9.21 Develop an equation to estimate the first derivative of an error input. Use a third order Taylor's series.

10. Phasor Analysis

Topic 10.1 *Phasor forms for steady state analysis.*
Topic 10.2 *Complex and polar calculation of steady state system responses.*
Topic 10.3 *Vibration analysis.*
Objective 10.1 *To be able to analyze steady state responses using the phasor transform.*

When a system is stimulated by an input it will respond. Initially there is a substantial transient response, that is eventually replaced by a steady state response. Techniques for finding the combined steady state and transient responses were covered in earlier chapters. These include the integration of differential equations, and numerical solutions. Phasor analysis can be used to find the steady state response only. These techniques involve using the phasor transform on the system transfer function, input and output.

10.1 Phasors for Steady State Analysis

When considering the differential operator we can think of it as a complex number, as in Figure 10.1. The real component of the number corresponds to the natural decay (e-to-the-t) of the system. But, the complex part corresponds to the oscillations of the system. In other words the real part of the number will represent the transient effects of the system, while the complex part will represent the sinusoidal steady-state. Therefore to do a steady-state sinusoidal analysis we can replace the 'D' operator with $j\omega$, this is the phasor transform.

$$D = \sigma + j\omega$$

where,

D = *differential operator*
σ = *decay constant*
ω = *oscillation frequency*

Phasor transform

$$D = j\omega$$

Figure 10.1 Transient and steady-state parts of the differential operator

An example of the phasor transform is given in Figure 10.2. We start with a transfer function for a mass-spring-damper system. In this example numerical values are assumed to put the equation in a numerical form. The differential operator is replaced with $j\omega$, and the equation is simplified to a complex number in the denominator. This equation then described the overall response of the system to an input based upon the frequency of the input. A generic form of sinusoidal input for the system is defined, and also converted to phasor (complex) form. (Note: the frequency of the input does not show up in the complex form of the input, but it will be used later.) The steady state response of the system is then obtained by multiplying the transfer function by the input, to obtain the output.

A phasor transform can be applied to a transfer function for a mass-spring-damper system. Some component values are assumed.

$$M = 1000kg \qquad K_s = 2000\frac{N}{m} \qquad K_d = 3000\frac{Ns}{m}$$

$$\frac{x(D)}{F(D)} = \frac{1}{MD^2 + K_dD + K_s} = \frac{1}{1000D^2 + 3000D + 2000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{1000j^2\omega^2 + 3000j\omega + 2000} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

A given input function can also be converted to phasor form.

$$F(t) = A\sin(\omega_{input}t + \theta_{input})$$

$$F(\omega) = A(\cos\theta_{input} + j\sin\theta_{input})$$

$$F(\omega) = A\cos\theta_{input} + jA\sin\theta_{input}$$

Note: the frequency is not used when converting an oscillating signal to complex form. But it is needed for the transfer function.

The response of the steady state output 'x' can now be found for the given input.

$$x(\omega) = \frac{x(\omega)}{F(\omega)}F(\omega) = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}(A\cos\theta_{input} + jA\sin\theta_{input})$$

Figure 10.2 A phasor transform example

To continue the example in Figure 10.2 values for the sinusoidal input force are assumed. After this the method only requires the simplification of the complex expression. In particular having a complex denominator makes analysis difficult and is undesirable. To simplify this expression it is multiplied by the complex conjugate. After this, the expression is quickly reduced to a simple complex number. The complex number is then converted to polar form, and then finally back into a function of time.

Assume the input to the system is, $F(t) = 10\sin(100t + 0.5)N$

$A = 10N \qquad \omega = 100\frac{rad}{s} \qquad \theta_{input} = 0.5rad$

These can be applied to find the steady state output response,

$$x(\omega) = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))}(10\cos 0.5 + j10\sin 0.5)$$

$$x(\omega) = \frac{1}{(-9998000) + j(300000)}(8.776 + j4.794)$$

$$x(\omega) = \frac{8.776 + j4.794}{-9998000 + j300000}\frac{(-9998000 - j300000)}{(-9998000 - j300000)}$$

Note: This is known as the complex conjugate;
1. The value is equivalent to 1 so it does not change the value of the expression
2. The complex component is now negative
3. Only the denominator is used top and bottom

$$x(\omega) = \frac{(-86304248) + j(-50563212)}{1.0005\times10^{14}}$$

$$x(\omega) = (-0.863\times10^{-6}) + j(-0.505\times10^{-6})$$

$$x(\omega) = \sqrt{(-0.863\times10^{-6})^2 + (-0.505\times10^{-6})^2} - atan\left(\frac{-0.505\times10^{-6}}{-0.863\times10^{-6}}\right) + \pi$$

Note: the signs of the components indicate that the angle is in the bottom left quadrant of the complex plane, so the angle should be between 180 and 270 degrees. To correct for this pi radians are added to the result of the calculation.

$$x(\omega) = 0.9999\times10^{-6} - 3.671$$

This can then be converted to a function of time.

$$x(t) = 0.9999\times10^{-6}\sin(100t + 3.671)m$$

Figure 10.3 A phasor transform example (cont'd)

Note: when dividing and multiplying complex numbers in polar form the magnitudes can be multiplied or divided, and the angles added or subtracted.
Unfortunately when the numbers are only added or subtracted they need to be converted back to Cartesian form to perform the operations. This method eliminates the need to multiply by the complex conjugate.

$$\frac{A + jB}{C + jD} = \frac{\sqrt{A^2 + B^2}\angle atan\left(\frac{B}{A}\right)}{\sqrt{C^2 + D^2}\angle atan\left(\frac{D}{C}\right)} = \frac{\sqrt{A^2 + B^2}}{\sqrt{C^2 + D^2}}\angle\left(atan\left(\frac{B}{A}\right) - atan\left(\frac{D}{C}\right)\right)$$

$$\frac{A\angle\theta_1}{B\angle\theta_2} = \frac{A}{B}\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta_1)(B\angle\theta_2) = AB\angle(\theta_1 + \theta_2)$$

For example,

$$(1 + j)\left(\frac{2 + j}{3 + 4j}\right)$$

$$= \sqrt{2}\angle 0.7854\frac{\sqrt{5}\angle 0.4636}{\sqrt{25}\angle 0.9273}$$

$$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{25}}\angle 0.7854 + 0.4636 - 0.9273$$

$$= 0.6325\angle 0.3217$$

$$= 0.6 + j0.2$$

Figure 10.4 Calculations in polar notation

The Cartesian form of complex numbers seen in the last section are well suited to operations where complex numbers are added and subtracted. But, when complex numbers are to be multiplied and divided these become tedious and bulky. The polar form for complex numbers simplifies many calculations. The previous example started in Figure 10.2 is redone using polar notation in Figure 10.5. In this example the input is directly converted to polar form, without the need for calculation. The input frequency is substituted into the transfer function and it is then converted to polar form. After this the output is found by multiplying the transfer function by the input. The calculations for magnitudes involve simple multiplications. The angles are simply added. After this the polar form of the result is converted directly back to a function of time.

Consider the input function from the previous example in polar form it becomes,

$$F(t) = 10 \sin(100t + 0.5) N \qquad F(\omega) = 10 \angle 0.5$$

The transfer function can also be put in polar form.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))} = \frac{1}{-9998000 + j300000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1 \angle 0}{\sqrt{(-9998000)^2 + (300000)^2} \angle \left(atan\left(\frac{300000}{-9998000} \right) + \pi \right)}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1 \angle 0}{10002500 \angle 3.112} = \frac{1}{10002500} \angle (0 - 3.112) = 0.9998 \times 10^{-7} \angle -3.112$$

The output can now be calculated.

$$x(\omega) = \frac{x(\omega)}{F(\omega)} F(\omega) = (0.9998 \times 10^{-7} \angle -3.112)(10 \angle 0.5)$$

$$x(\omega) = 0.9998 \times 10^{-7} (10) \angle (-3.112 + 0.5) = 0.9998 \times 10^{-6} \angle -2.612$$

The output function can be written from this result.

$$x(\omega) = 0.9998 \times 10^{-6} \sin(100t - 2.612)$$

Note: recall that $\tan\theta = \frac{Im}{Re}$, but the $\operatorname{atan}\theta$ function in calculators and software only returns values between -90 to 90 degrees. To compensate for this the sign of the real and imaginary components must be considered to determine where the angle lies. If it lies beyond the -90 to 90 degree range the correct angle can be obtained by adding or subtracting 180 degrees.

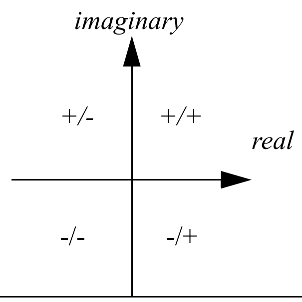
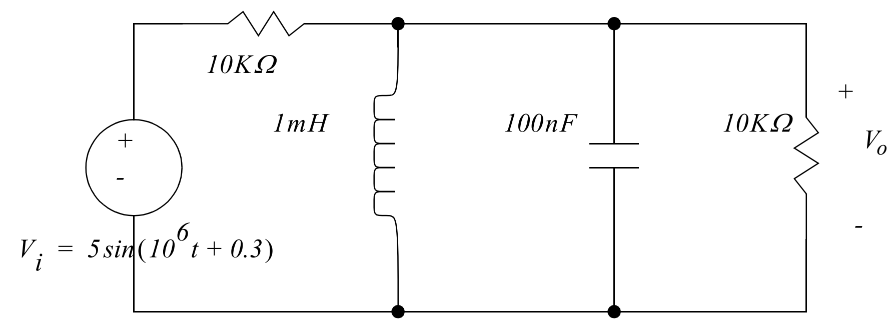


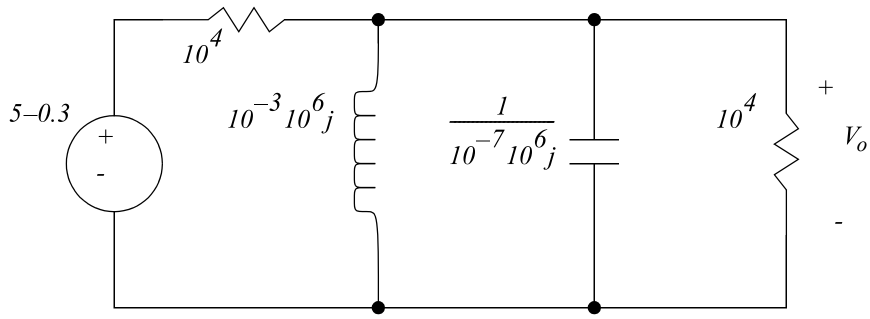
Figure 10.5 Correcting quadrants for calculated angles

Consider the circuit analysis example in Figure 10.6. In this example the component values are converted to their impedances, and the input voltage is converted to phasor form. (Note: this is a useful point to convert all magnitudes to powers of 10.) After this the three output impedances are combined to a single impedance. In this case the calculations were simpler in the cartesian form.

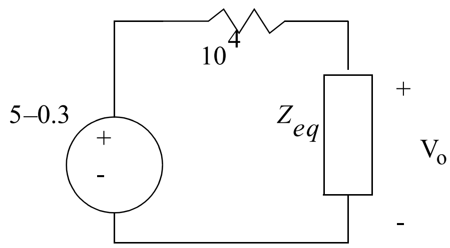
Given the circuit,



The impedances and input voltage can be written in phasor form.



The three output impedances in parallel can then be combined.



$$\frac{1}{Z_{eq}} = \frac{1}{10^{-3}10^6 j} + \frac{1}{\left(\frac{1}{10^{-7}10^6 j}\right)} + \frac{1}{10^4}$$

$$\frac{1}{Z_{eq}} = -10^{-3}j + 10^{-1}j + 10^{-4} = (10^{-4}) + j(0.099)$$

$$Z_{eq} = \frac{1}{(10^{-4}) + j(0.099)}$$

Figure 10.6 Phasor analysis of a circuit

The analysis continues in Figure 10.7 as the output is found using a voltage divider. In this case a combination of cartesian and polar forms are used to simplify the calculations. The final result is then converted back from phasor form to a function of time.

The output can be found using the voltage divider form.

$$V_o(\omega) = (5\text{–}0.3)\left(\frac{Z_{eq}}{10^4 + Z_{eq}}\right)$$

$$V_o(\omega) = (5\text{–}0.3)\left(\frac{\left(\frac{1}{(10^{-4}) + j(0.099)}\right)}{10^4 + \left(\frac{1}{(10^{-4}) + j(0.099)}\right)}\right)$$

$$V_o(\omega) = (5\text{–}0.3)\left(\frac{1}{(10^4)((10^{-4}) + j(0.099)) + 1}\right)$$

$$V_o(\omega) = (5\text{–}0.3)\left(\frac{1}{2 + j990}\right) = \frac{5\text{–}0.3}{\sqrt{2^2 + 990^2}\text{–}atan\frac{990}{2}} = \frac{5\text{–}0.3}{990\text{–}1.5687761}$$

$$V_o(\omega) = \frac{5}{990}\text{–}(0.3 - 1.5687761) = 5.05\text{¥}10^{-3}\text{—}1.269$$

Finally, the output voltage can be written.

$$V_o(t) = 5.05\sin(10^6 t - 1.269)mV$$

Figure 10.7 Phasor analysis of a circuit (cont'd)

Phasor analysis is applicable to systems that are linear. This means that the principle of superposition applies. Therefore, if an input signal has more than one frequency component then the system can be analyzed for each component, and then the results simply added. The example considered in Figure 10.2 is extended in Figure 10.8. In this example the input has a static component, as well as frequencies at 0.5 and 20 rad/s. The transfer function is analyzed for each of these frequencies components. The output components are found by multiplying the inputs by the response at the corresponding frequency. The results are then converted back to functions of time, and added together.

Given the transfer function for the system,

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

Note: these are gains and phase shifts that will be used heavily in Bode plots later.

and an input with multiple frequency components,

$$F(t) = 1000 + 20sin(20t) + 10sin((0.5t)(N))$$

the transfer function for each frequency can be calculated,

$$\frac{x(0)}{F(0)} = \frac{1}{(2000 - 1000(0)^2) + j(3000(0))} = \frac{1}{2000} = 0.0005–0$$

$$\frac{x(20)}{F(20)} = \frac{1}{(2000 - 1000(20)^2) + j(3000(20))} = \frac{1}{-398000 + j60000} = \frac{1–0}{402497–2.992}$$

$$\frac{x(0.5)}{F(0.5)} = \frac{1}{(2000 - 1000(0.5)^2) + j(3000(0.5))} = \frac{1}{1750 + j1500} = \frac{1–0}{2305–0.709}$$

These can then be multiplied by the input components to find output components.

$$x(0) = (0.0005–0)1000–0 = 0.5–0$$

$$x(20) = \left(\frac{1–0}{402497–2.992}\right)20–0 = 0.497\times10^{-4}—2.992$$

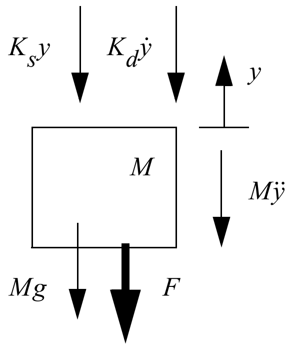
$$x(0.5) = \left(\frac{1–0}{2305–0.709}\right)10–0 = 4.34\text{¥}10^{-3}—0.709$$

Therefore the output is,

$$x(t) = 0.5 + 49.7\text{¥}10^{-6}sin(20t - 2.992) + 4.34\text{¥}10^{-3}sin(0.5t - 0.709)$$

Figure 10.8 A example for a signal with multiple frequency components (based on the example in Figure 10.2)

Phasor analysis is a useful application of Fourier Transforms. The substitution of jw for D is valid as long as the system is described with a transfer function. If the system cannot be rearranged into a transfer function the Fourier Transform properties must be considered. [Figure number] shows an example of a a mass-spring-damper system exposed to gravity. Without the gravity term this equation would be easily converted to a transfer function. However, the constant 'Mg' term prevents this conversion. Theoretically this is multiplied by the unit impulse function during the transform. The unit impulse function is circled in the equation. This theoretical function has a unit area of 1, is infinitely high, and infinitely short. The only time is will exist is when the frequency is 0 rad/s. So, when we deal with any other frequencies, this term will be zero. For our purposes, the constant term can be canceled, simplifying the equation, allowing the creation of a transfer function. If you have a chance to study the Fourier transform in the future you will explore this property. However, for now, you may simply cancel the term and recognize that the result for w=0 will be incorrect.



$$M\ddot{y} + K_d\dot{y} + K_s y = -F - Mg$$

$$y(MD^2 + K_d D + K_s) = F(-1) + (-Mg)$$

Note: The 'Mg' term prevents the equation from being rearranged into a transfer function. However, for the phasor (Fourier) transform the term is replaced with a unit impulse function. For our phasor application this means that we can set it to zero. The proof of this property goes beyond our treatment of the topic, but the rule will be used to eliminate constant forcing functions.

$$y(M(j\omega)^2 + K_d j\omega + K_s) = F(-1) + \delta(\omega)(-Mg)$$

$$y(M(j\omega)^2 + K_d j\omega + K_s) = F(-1)$$

$$\frac{y}{F} = \frac{-1}{M(j\omega)^2 + K_d j\omega + K_s} \equiv \frac{-1}{MD^2 + K_d D + K_s}$$

Figure 10.9 Creating a Phasor Transfer Function from a Complicated Differential Equation

10.2 Phasors in Software

Phasor based analysis requires complex numbers. Mathematics software, such as Matlab and Scilab, include complex numbers as standard definitions. Figure 10.10 is an example of a Scilab program used to find the gain and phase shift for a transfer function 'G' at a frequency of 100Hz. In the example 'j' is defined as the complex number to avoid confusion with 'i' normally used for electrical current. (Note: In Scilab the complex number is '%i'.) For simplicity the 'D' operator is replaced with the complex number, and frequency in radians per second. The transfer function 'G' is calculated and the result is a complex number. The magnitude of 'G' is the gain, which is converted to dB. The phase shift is the complex plane angle of 'G'.

$$G = \frac{D+5}{D^2 + 5D + 100}$$

```
f = 100; // give it a frequency
j = sqrt(-1); // instead of i
D = j * 2 * %pi * f; // the phasor substitution
G = (D + 5) / (D ^ 2 + 5 * D + 100); // the transfer function
20*log10(abs(G)) // gain in dB
180 * atan2(imag(G), real(G)) / %pi // phase angle in degrees
```

Figure 10.10 Scilab Analysis for Phasors

Traditional programming languages may require the use of a mathematical library, or additional programming. Figure 10.11 is a simple set of C functions for complex numbers. A data structure for complex numbers is defined with an angle and magnitude. (Note: This data structure could have also been defined as a real and imaginary component.) Functions are then written to perform basic mathematical operations. The complex structure and functions can be reused for other calculations and programs.

```
#include <math.h>
#define PI 3.14159

struct complex {
        double mag;
        double angle;
};

void add(struct *complex R, struct *complex A, struct *complex
   B){
        double x, y;
        x = A->mag * cos(A->angle) + B->mag * cos(B->angle);
        y = A->mag * sin(A->angle) + B->mag * sin(B->angle);
        R->mag = sqrt(x*x + y*y);
        R->mag = atan2(x, y);
}

void multiply(struct *complex R, struct *complex A, struct
   *complex B){
        R->mag = A->mag * B->mag;
        R->angle = A->angle + B->angle;
}

void divide(struct *complex R, struct *complex A, struct *com-
   plex B){
        R->mag = A->mag / B->mag; // no divide by zero check
        R->angle = A->angle - B->angle;
}

void define(struct *complex R, double mag, double angle){
        R->mag = mag;
        R->angle = angle;
}
```

Figure 10.11 C Functions for Complex Numbers

The transfer function ‘G’ from the previous example is implemented in Figure 10.12 with the C functions. The routine is called and then each of the mathematical operations is performed with a separate function call. This function could be simplified with overloaded operators in C++ or Java. The main function calls the transfer function calculation using a frequency of 75Hz. The calculation results are returned in the ‘result’ structure. The resulting magnitude and angle are printed without being converted to dB for gain and degrees for angle.

```
void G(struct *complex R, double f){
        double w;
        struct complexD, temp1, temp2, temp3;

        w = 2 * PI * f;
        define(D, w, PI/2.0); // D = jw

        define(&temp1, 5.0, 0.0); // The numerator
        add(&temp1, &D, &temp1);

        multiply(&temp2, &D, &D); // The denominator
        define(&temp3, 5.0, 0.0);
        multiply(&temp3, &D, &temp3);
        add(&temp2, &temp2, &temp3);
        define(&temp3, 100.0, 0.0);
        add(&temp2, &temp2, &temp3);

        divide(R, &temp1, &temp2);
}

void main(){
        struct complex result;

        G(&result, 75.0);
        printf("mag=%f, angle =%f \n", result.mag,
   result.angle);
}
```

Figure 10.12 A C Program Example of Phasor Analysis

10.3 Vibrations

Oscillating displacements and forces in mechanical systems will cause vibrations. In some cases these become a nuisance, or possibly lead to premature wear and failure in mechanisms. A common approach to dealing with these problems is to design vibration isolators. The equations for transmissibility and isolation are shown in Figure 10.13. These equations can compare the ratio of forces or displacements through an isolator. The calculation is easy to perform with a transfer function or Bode plot.

Transmissibility for a simple component can be a ration of input to output force, or displacement, at different frequencies. At a frequency of 0 rad/s this value is normally 1, or 100%.

$$T = \left|\frac{F_{out}(\omega)}{F_{in}(\omega)}\right| = \left|\frac{x_{out}(\omega)}{x_{in}(\omega)}\right|$$

For a damped system the transmissibility will range between 0% and 100%. However, in an underdamped system the value can be greater than 100% near the frequencies of the resonant peaks.

For realistic systems it is often impractical to put the inputs and outputs in the same units. In these cases we can compare the output magnitude at 0 rad/s and the frequency in question.

$$T = \left|\frac{F_{out}(\omega)}{F_{out}(0)}\right| = \left|\frac{x_{out}(\omega)}{x_{out}(0)}\right|$$

Isolation is the compliment of transmission. So when the transmissibility is 90%, the isolation is 10%.

$$I = (1 - T)$$

Figure 10.13 Transmissibility

An example of a transmissibility calculation is shown in [fig number]. The example begins with a transfer function that has been derived for some system where the input is ‘F’ and the output is ‘x’. In the previous sections of this chapter, phasors (D=jw) were used to find the transfer function ratio given a specific frequency. The magnitude of this ratio is the magnifier between input and output sine waves. The angle of the ratio is the offset between the input and output sine waves. For the given transfer function the input and output values are not the same, meaning they do not have the same units. As a result the analysis must compare the gain at a frequency of zero, and the target frequency. Therefore two values must be calculated to find the transmissibility at a frequency. In this example the input ‘F’ is eliminated by giving it a value of 1.0, but it is not necessary to do this, the value of the transfer function alone is sufficient.

In the second part of the example the transmissibility is calculated for multiple frequencies. These are then plotted to show how the system behaves over a range of frequencies. The frequency range is set to cover three decades, where each decade is a multiple of 10 starting at 0.1 rad/s. The transfer function is calculated in the function to simplify the remainder of the program. Before the loop starts the initial gain is calculated. All of the frequencies and magnitudes are stored in arrays for plotting later. In the loop the current output magnitude is divided by the initial magnitude to normalize it to a scale of 1. The resulting plot is done using a logarithmic scale to make it readable. If a logarithmic scale is not used the graph is often a line pushed to one side of the plot that is unusable for analysis and engineering design.

The given transfer function has an input force F, resulting in a displacement x. The Phasor transform is used for the transfer function, and the input F is set to 1 to simplify calculations.

$$\frac{x}{F} = \frac{5D+6}{D^2+7D+8}$$

A manual calculation is shown for the transmissibility for an input at 100rad/s. This is done by calculating the output magnitude at w=0 and w=100. The input value F is arbitrary because it will be canceled out, so it is given a value of 1 for simplicity.

$$|x(\omega)| = \left|1\left(\frac{5D+6}{D^2+7D+8}\right)\right| = \left|\frac{5j\omega+6}{-\omega^2+7j\omega+8}\right| = \frac{\sqrt{6^2+5^2}}{\sqrt{(8-\omega^2)^2+7^2}}$$

$$T(100) = \frac{|x(100)|}{|x(0)|} = \frac{0.0499}{0.75} = 0.0665 = 6.65\%$$

A program can also be used to do this over a range of frequencies. The results are plotted using log scales because the differences are multiplied, whereas an additive problem would use linear axes. Note that the denominator of the transfer function is slightly underdamped, so the transmissibility rises above 1.0 near the damped frequency.

```
w_start = 0.1; w_end = 100; w_multiplier = 1.5
F = 1; // Set it to anything, but 1 works
j = sqrt(-1);
function [x] = calc(w)
    D = j * w;
    x = F * (5 * D + 6) / (D^2 + 7 * D + 8)
endfunction
w_now = w_start;
mag_init = calc(w_now);
frequency_rad = [w_now];
transmissibility = [1]; // everything is normalized
w_now = w_now * w_multiplier;
while w_now < w_end
    frequency_rad = [frequency_rad; w_now];
    transmissibility = [transmissibility; calc(w_now) / mag_init]
    w_now = w_now * w_multiplier;
end
plot2d(frequency_rad, transmissibility, logflag="ll");
```

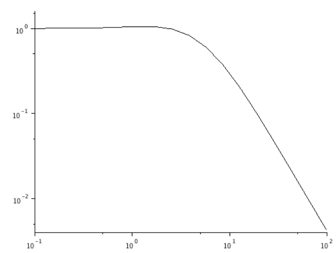


Figure 10.14 Transmissibility Example

10.4 Summary

- Phasor transforms and phasor representations can be used to find the steady state response of a system to a given input.
- Vibration analysis determines frequency components in mechanical systems.

10.5 Problems With Solutions

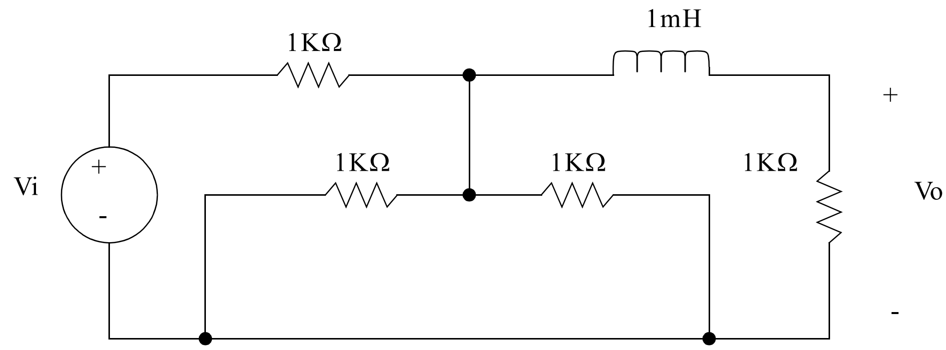
Problem 10.1 Simplify the following expressions.

a) $\frac{16}{(4j+4)^2}$

b) $\frac{3j+5}{(4j+3)^2}$

c) $(3+5j)4j$ where, $j = \sqrt{-1}$

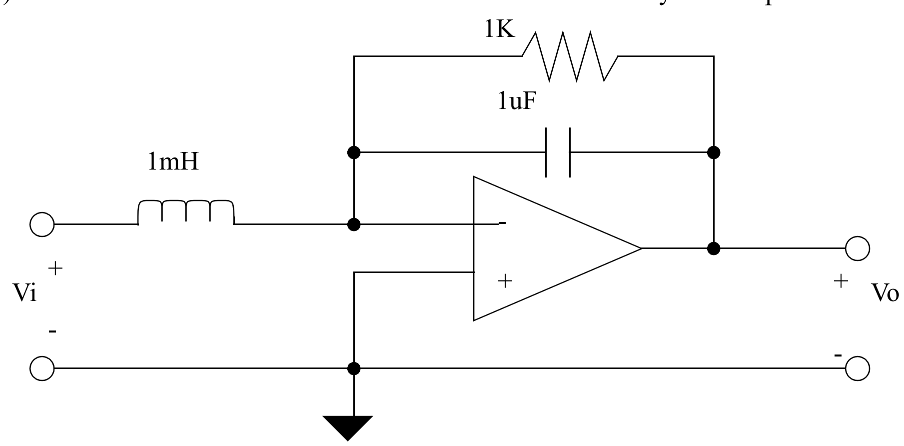
Problem 10.2 Develop a transfer function for the system pictured below and then find the response to an input voltage of Vi = 10sin(1,000,000 t) using phasor transforms.



Problem 10.3 A single d.o.f. model with a weight of 1.2 KN and a stiffness of 340 N/m has a steady-state harmonic excitation force applied at 95 rpm (revolutions per minute). What damper value will give a vibration isolation of 92%? Use phasors to do the calculations.

Problem 10.4 Four helical compression springs are used, one at each corner of a piece of equipment. The spring rate is 240 N/m for each spring and the vertical static deflection of the equipment is 10mm. Calculate the mass of the equipment and determine the amount of isolation the springs would afford if the equipment operating frequency is twice the natural frequency of the system.

Problem 10.5 a) For the circuit below find the transfer function and the steady state response for an input of Vi=5sin(1000t)V.



b) Verify the results in part a) by explicitly solving the differential equation.

Problem 10.6 Given an input of F=5sin(62.82t), find the output, x, using phasors for the following transfer functions.

a) $$\frac{x}{F} = \frac{D^2}{(D + 200\pi)^2}$$

b) $$\frac{x}{F} = \frac{D(D + 2\pi)}{(D + 200\pi)^2}$$

c) $$\frac{x}{F} = \frac{D^2(D + 2\pi)}{(D + 200\pi)^2}$$

10.6 Problem Solutions

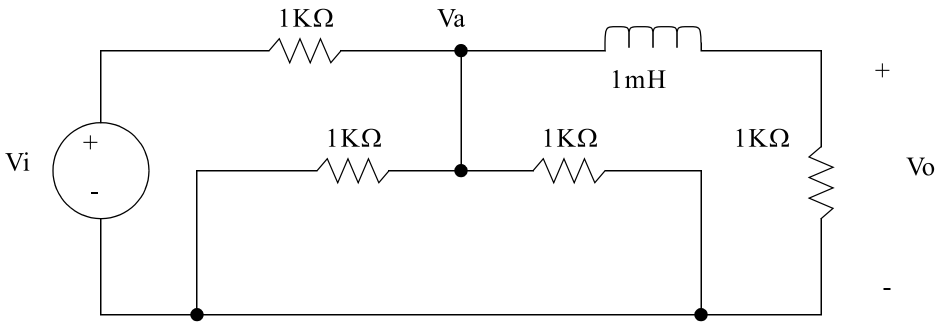
Answer 10.1

a) $$\frac{16}{(4j+4)^2} = \frac{16}{-16+32j+16} = \frac{16}{32j} = -0.5j$$

b) $$\frac{3j+5}{(4j+3)^2} = \frac{3j+5}{-16+24j+9} = \frac{3j+5}{-7+24j}\left(\frac{-7-24j}{-7-24j}\right) = \frac{-35-141j+72}{49+576} = \frac{37-141j}{625}$$

c) $$(3+5j)4j = 12j + 20j^2 = 12j - 20$$

Answer 10.2



$$\sum I_{Va} = \frac{V_a - V_i}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a - V_o}{0.001D} = 0$$

$$V_a\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right) \qquad \text{eqn 10.1}$$

$$\sum I_{Vo} = \frac{V_o - V_a}{0.001D} + \frac{V_o}{1K\Omega} = 0$$

$$V_o\left(\frac{1}{0.001D} + \frac{1}{1K\Omega}\right) = V_a\left(\frac{1}{0.001D}\right)$$

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right) = V_a \qquad \text{eqn 10.2}$$

substitute (2) into (1)

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right)$$

$$\frac{V_o}{V_i} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) - \left(\frac{1}{0.001D}\right)}$$

for the given input of Vi(t) = 10sin(1,000,000 t).

$$\frac{V_o}{10 + 0j} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001j10^6 + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001(j10^6)}\right) - \left(\frac{1}{0.001(j10^6)}\right)}$$

$$\frac{V_o}{10 + 0j} = \frac{10^{-3}}{(j + 1)(3 \text{¥} 10^{-3} - j10^{-3}) + j(10^{-3})}$$

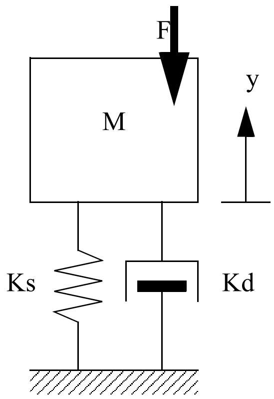
$$\frac{V_o}{10 + 0j} = \frac{1}{(j + 1)(3 - j) + j}$$

$$\frac{V_o}{10 + 0j} = \left(\frac{1}{4 + 3j}\right)\left(\frac{4 - 3j}{4 - 3j}\right)$$

$$V_o = 10\left(\frac{4 - 3j}{25}\right) = 2\text{—}0.644$$

$$V_o(t) = 2.00\sin(10^6 t - 0.644)V$$

Answer 10.3



$$\sum F_y = -K_s y - K_d y D - F - Mg = M y D^2$$

$$\frac{y}{F} = \frac{-1}{D^2 M + DK_d + K_s}$$

$$F_{floor} = K_s y + K_d y D$$

$$\frac{F_{floor}}{y} = K_s + K_d D$$

$$\frac{F_{floor}}{F} = \left(\frac{F_{floor}}{y}\right)\left(\frac{y}{F}\right) = \frac{-(K_s + K_d D)}{D^2 M + DK_d + K_s}$$

$$\frac{F_{floor}}{F} = \frac{-(K_s + K_d \omega j)}{-\omega^2 M + \omega j K_d + K_s}$$

$$\left|\frac{F_{floor}}{F}\right| = \frac{\sqrt{K_s^2 + (K_d \omega)^2}}{\sqrt{\left(K_s - \omega^2 M\right)^2 + (\omega K_d)^2}}$$

For 92% isolation, there is 100-92 = 8% transmission, at 95rpm.

$$K_s = 340\frac{N}{m} \qquad M = \frac{1200N}{9.81\frac{N}{kg}} = 122kg$$

$$\omega = \left(95\frac{rev}{min}\right)\left(\frac{1min}{60sec}\right)\left(\frac{2\pi rad}{rev}\right) = 9.95\frac{rad}{s}$$

$$0.08 = \frac{\sqrt{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}}{\sqrt{\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2}}$$

$$\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2 = \frac{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}{0.08^2}$$

$$(1.377878\times10^8)\frac{N^2}{m^2} + 99.0025\frac{rad^2}{s^2}K_d^2 = \frac{115600}{0.0064}\frac{N^2}{m^2} + K_d^2\frac{99.0025}{0.0064}\frac{rad^2}{s^2}$$

$$\left(\frac{1.197253\times10^8}{15370.138}\right)\frac{N^2 s^2}{m^2 rad^2} = K_d^2 \qquad K_d = 88.3\frac{Ns}{m}$$

Answer 10.4

a)

$$F = K_s y$$

$$Mg = 4(240)0.010 \qquad M = 0.979kg$$

b)

$$\ddot{y}(M) + \dot{y}(K_d) + y(K_s) = Mg + F$$

$$\ddot{y} + \dot{y}\left(\frac{K_d}{M}\right) + y\left(\frac{K_s}{M}\right) = g + \frac{F}{M}$$

$$\omega_n = \sqrt{\frac{K_s}{M}} = \sqrt{\frac{4(240)}{0.979}} = 31.31$$

$$\omega_o = 2\omega_n = 62.62$$

$$y = \frac{Mg + F}{D^2 M + DK_d + K_s}$$

$$\omega_{off} = 0 \qquad |y_{off}| = 0.011046$$

$$\omega_{on} = 62.62 \qquad |y_{on}| = 0.0036833$$

$$I = \frac{|y_{off}| - |y_{on}|}{|y_{off}|} = \frac{0.011046 - 0.0036833}{0.011046} = 0.667 = 66.7\%$$

Answer 10.5

a)

$$V_o(t) = 3536\sin(1000t + 0.785)$$

b)

$$V_o(t) = -5000 + 2500e^{-1000t} + 3536\sin\left(1000t + \frac{\pi}{4}\right)$$

Answer 10.6

a)

$$x(t) = 49.5\times10^{-3}\sin(62.82t + 2.942)$$

b)

$$x(t) = 49.7\times10^{-3}\sin(62.82t + 2.843)$$

c)

$$x(t) = 3.124\sin(62.82t - 1.870)$$

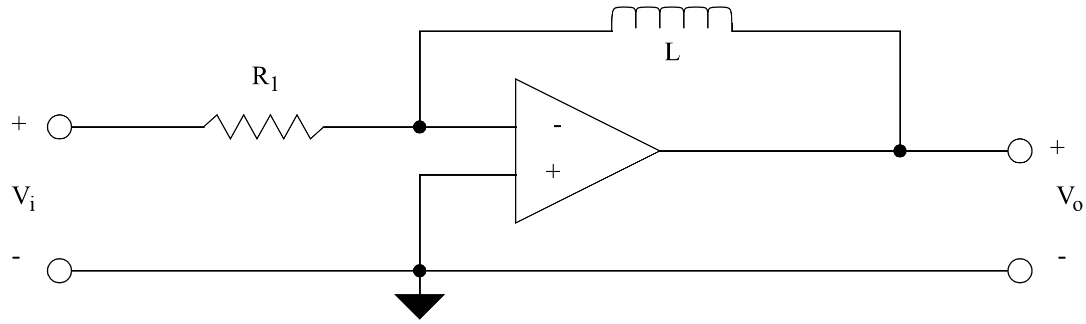
10.1 Problems Without Solutions

Problem 10.7 Given the transfer function for a vibration isolator below, find a value of K that will give 50% isolation for a 10Hz vibration.

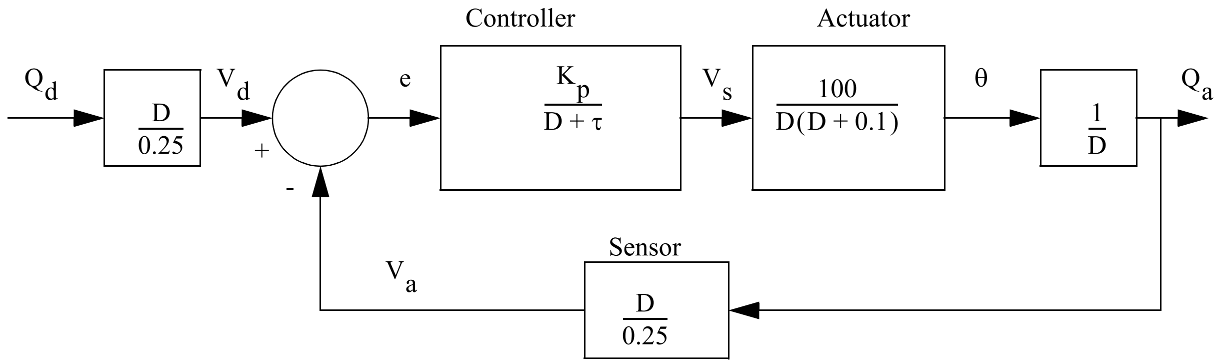
$$\frac{x_{out}}{x_{in}} = \frac{5D + 10}{104D^2 + (20K + 200)D + 4}$$

Problem 10.8 Write the differential equation for the following schematic. If the constants for the system are L=0.001H, R1 =

100000, what will the steady system response be for an input of 10sin(20000t)?



Problem 10.9 a) Simplify the block diagram as far as possible.



b) Select values for Kp and tau that will include a second order response that has a damping factor of 0.125 and a natural frequency of 10rad/s.

Hint: $$\frac{(\ )}{\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right)(D + A)}$$

c) For a step input of magnitude 1.0 find the output as a function of time using numerical methods. Give the results in a table OR graph.

d) For a step input of magnitude 1 find the output as a function of time by integrating the differential equation (i.e., using the homogeneous and particular solutions).

e) Compare the values found in steps c) and d).

f) For the transfer function; 1) apply a phasor transform and express the gain and phase angle as a function of frequency, 2) calculate a set of values and present them in a table, 3) use the values calculated in step 4) to develop a frequency response plot on semi-log paper, 5) draw a straight line approximation of the Bode plot on semi-log paper.

10.2 Complex Number Review

When dealing with oscillating systems, sine and cosine functions can be used to model system responses. Trigonometric identities are used with algebra and calculus to combine input functions and differential equations to derive output functions. These calculations are routine, but the number of terms can be make a solution very long and complex. On the other hand, complex numbers add complexity to the algebra, but eliminates trigonometry from the core calculations when only looking for oscillation responses. In other words. Some pain with complex numbers gives a great gain in simplifying calculations. This section outlines the important properties of complex numbers as applied to oscillating systems.

In traditional mathematics the imaginary, or complex, number is indicated with an ‘i’. In engineering calculations ‘j’ is used instead, so that it is not used for the variable for electrical current. The basic properties of complex numbers are shown in Figure 10.15. In simple terms a complex number may have both a real and imaginary part. In a 2D Cartesian system these are best visualized as perpendicular vector components. When adding and subtracting complex numbers the real and imaginary components don’t interact, making it trivial. When multiplying complex numbers the result is up to four terms, that can be simplified to another complex number with two terms. Division is the difficult operation with Cartesian complex numbers. The time proven method is to use complex conjugate of the denominator. A complex conjugate has the same real and imaginary magnitude, but the sign is flipped on the complex term. When any complex number is multiplied by itself, the result is always real. In this case the complex conjugate of the denominator ‘M’ is ‘M*’. Multiplying the top and bottom is the same as multiplying by ‘1’. After the multiplication the denominator only contains real terms, and these can be used to divide the real and imaginary parts. The result is a single complex number. Although this seems a little involved, it is a simple method once used.

The Complex Number: $j = \sqrt{-1}$ $\qquad$ $j^2 = -1$

Complex Numbers: $a + bj$ $\qquad$ where, a and b are both real numbers

Complex Conjugates (denoted by adding an asterisk ‘*’ the variable):

$$N = a + bj \qquad N^* = a - bj$$

Basic Properties:

$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

$$(a + bj)\ \text{Þ}\ (c + dj) = (ac - bd) + (ad + bc)j$$

$$\frac{N}{M} = \frac{a + bj}{c + dj} = \frac{N}{M}\left(\frac{M^*}{M^*}\right) = \left(\frac{a + bj}{c + dj}\right)\left(\frac{c - dj}{c - dj}\right) = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)j$$

Figure 10.15 Basic complex number properties

The complex numbers shown in Figure 10.16in both Cartesian and polar forms. While the Cartesian form shows the number with two components, the polar form shows the number as a magnitude and direction. While the representations change, the numbers are equivalent. The numbers can be converted between forms using the properties of right angle triangles. In polar form there is no ‘j’, but there is a real number, and an angle. The angle follows the angle sign that is like a less than sign with a horizontal bottom leg. So, the obvious question at this point is, “does the Polar form have any benefits?” The answer is yes, it simplifies division, and multiplication.

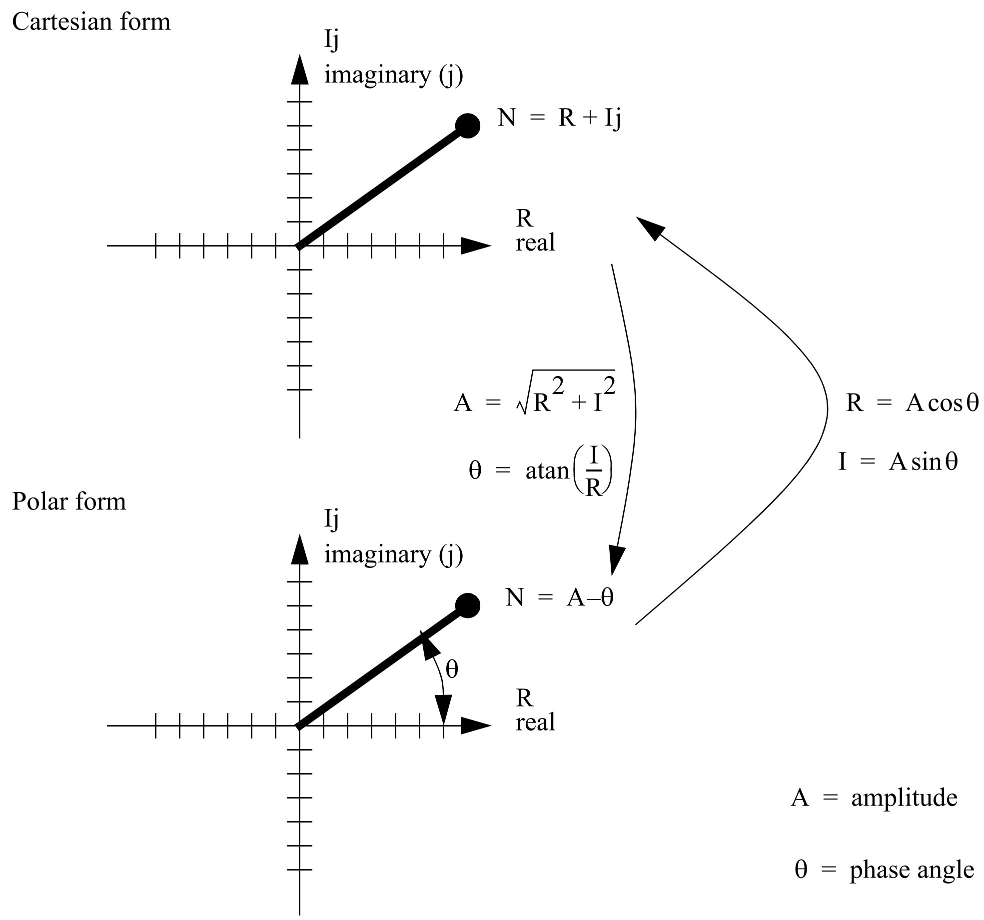


Figure 10.16 Cartesian and polar complex number forms

Euler's formula is shown in Figure 10.17. The polar notation can be directly converted to the complex exponent form. Once in this form exponents are added for multiplication, and subtracted for division. To multiply two polar complex numbers, the magnitudes 'A' are multiplied, and the angles added. Division is done by dividing the 'A' values and subtracting the angle in the denominator. The final equation, DeMoivre's theorem, will occasionally be used to find exponents and roots of complex numbers. To add and complex numbers in polar form, they are normally converted to Cartesian form first.

$$e^{j\theta} = \cos\theta + j\sin\theta$$ Euler's formula

$$A\angle\theta = A(\cos\theta + j\sin\theta) = Ae^{j\theta}$$

$$(A_1\angle\theta_1)(A_2\angle\theta_2) = (A_1A_2)\angle(\theta_1 + \theta_2)$$

$$\frac{(A_1\angle\theta_1)}{(A_2\angle\theta_2)} = \left(\frac{A_1}{A_2}\right)\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta)^n = (A^n)\angle(n\theta)$$ DeMoivre's theorem

Figure 10.17 Euler's formula for multiplication and division of polar complex numbers

11. Analog Inputs and Outputs

Topic 11.1 *Analog inputs and outputs.*
Topic 11.2 *Sampling issues; aliasing, quantization error, resolution.*
Objective 11.1 *To understand the basics of conversion to and from analog values.*

An analog value is continuous, not discrete, as shown in Figure 11.1. In the previous chapters, techniques were discussed for designing continuous control systems. In this chapter we will examine analog inputs and outputs so that we may design continuous control systems that use computers.

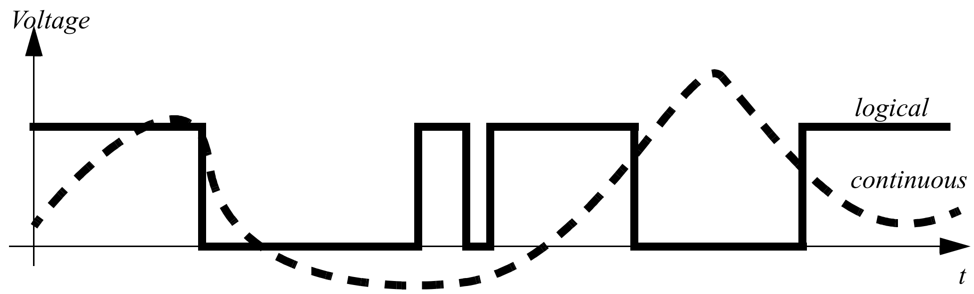


Figure 11.1 Logical and Continuous Values

Typical analog inputs and outputs for computers are listed below. Actuators and sensors that can be used with analog inputs and outputs will be discussed in later chapters.

- Inputs:
    - Oven temperature
    - Fluid pressure
    - Fluid flow rate
- Outputs:
    - Fluid valve position
    - Motor position
    - Motor velocity

A basic analog input is shown in Figure 11.2. In this type of system a physical value is converted to a voltage, current or other value by a transducer. A signal conditioner converts the signal from the transducer to a voltage or current that is read by the analog input.

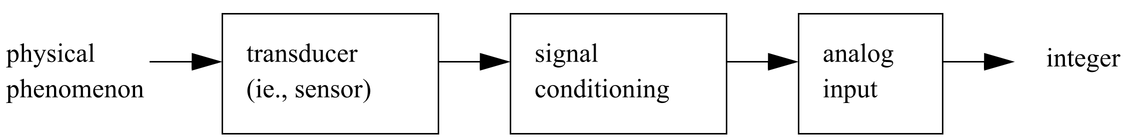


Figure 11.2 Analog inputs

Analog to digital and digital to analog conversion uses integers within the computer. Integers limit the resolution of the numbers to a discrete, or quantized range. The effect of using integers is shown in Figure 11.3 where the desired or actual analog value is continuous, but the possible integer values are quantified with a 'staircase' set of values. Consider when a continuous analog voltage is being read, it must be quantized into an available integer value. Likewise, a desired analog output value is limited to available quantized values. In general the difference between the analog and quantized integer value is an error based upon the resolution of the analog I/O.

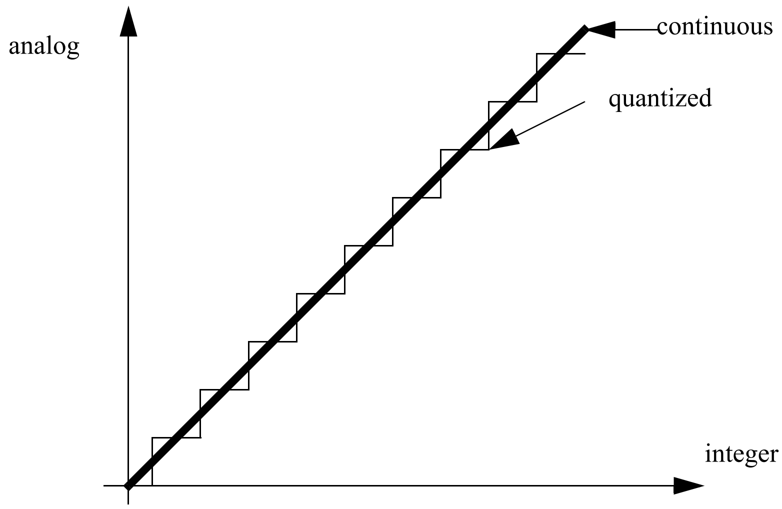


Figure 11.3 Quantization error

11.1 Analog Inputs

To input an analog voltage (into a computer) the continuous voltage value must be *sampled* and then converted to a numerical value by an A/D (Analog to Digital) converter (also known as ADC). Figure 11.4 shows a continuous voltage changing over time. There are three samples shown on the figure. The process of sampling the data is not instantaneous, so each sample has a start and stop time. The time required to acquire the sample is called the *sampling time*. A/D converters can only acquire a limited number of samples per second. The time between samples is called the sampling period *T*, and the inverse of the sampling period is the sampling frequency (also called sampling rate). The sampling time is often much smaller than the sampling period. The sampling frequency is specified when buying hardware, but a common sampling rate is 100KHz.

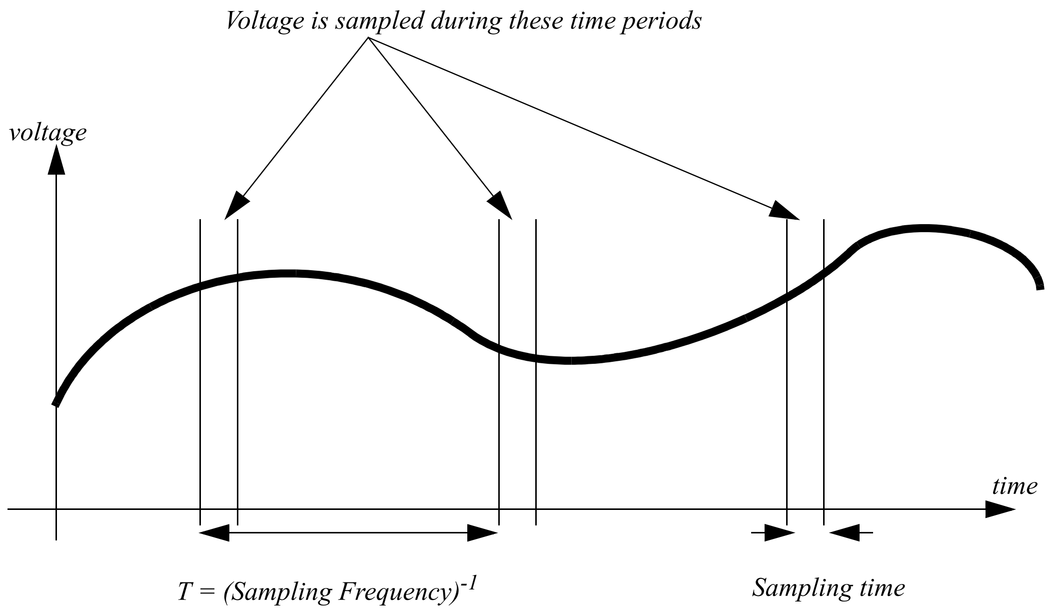


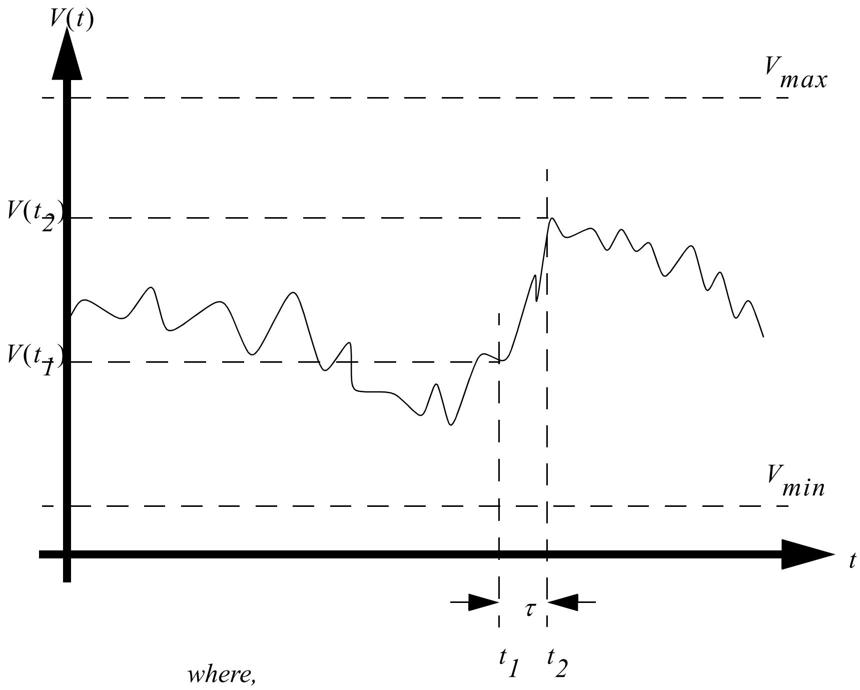
Figure 11.4 Sampling an Analog Voltage

A more realistic drawing of sampled data is shown in Figure 11.5. This data is noisier, and even between the start and end of the data sample there is a significant change in the voltage value. The data value sampled will be somewhere between the voltage at the start and end of the sample. The maximum (Vmax) and minimum (Vmin) voltages are a function of the control hardware. These are often specified when purchasing hardware, but reasonable ranges are;

- 0V to 5V
- 0V to 10V

- -5V to 5V
- -10V to 10V

The number of bits of the A/D converter is the number of bits in the result word. If the A/D converter is *8 bit* then the result can read up to 256 different voltage levels. Most A/D converters have 12 bits, 16 bit converters are used for precision measurements.



where,

$V(t)$ = *the actual voltage over time*

τ = *sample interval for A/D converter*

t = *time*

t_1, t_2 = *time at start,end of sample*

$V(t_1), V(t_2)$ = *voltage at start, end of sample*

V_{min}, V_{max} = *input voltage range of A/D converter*

N = *number of bits in the A/D converter*

Figure 11.5 Parameters for an A/D Conversion

The parameters defined in Figure 11.5 can be used to calculate values for A/D converters. These equations are summarized in Figure 11.6. Equation 1 relates the number of bits of an A/D converter to the resolution. Equation 2 gives the error that can be expected with an A/D converter given the range between the minimum and maximum voltages, and the resolution (this is commonly called the quantization error). Equation 3 relates the voltage range and resolution to the voltage input to estimate the integer that the A/D converter will record. Finally, equation 4 allows a conversion between the integer value from the A/D converter, and a voltage in the computer.

$$R = 2^N \qquad \text{eqn 11.1}$$

$$V_{ERROR} = \pm\left(\frac{V_{max} - V_{min}}{2R}\right) \qquad \text{eqn 11.2}$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)(R-1)\right] \qquad \text{eqn 11.3}$$

$$V_C = \left(\frac{V_I}{R-1}\right)(V_{max} - V_{min}) + V_{min} \qquad \text{eqn 11.4}$$

where,

R = *resolution of A/D converter*

V_I = *the integer value representing the input voltage*

V_C = *the voltage calculated from the integer value*

V_{ERROR} = *the maximum quantization error*

Figure 11.6 A/D Converter Equations

Consider a simple example, a 10 bit A/D converter is set up to read voltages between -10V and 10V. This gives a resolution of 1024, where 0 is -10V and 1023 is +10V. Because there are only 1024 steps there is a maximum error of ±9.8mV. If a voltage of 4.564V is input, the A/D converter converts the voltage to an integer value of 746. When we convert this back to a voltage the result is 4.570V. The resulting quantization error is 4.570V-4.564V=+0.006V. This error can be reduced by selecting an A/D converter with more bits. Each bit halves the quantization error.

Given,

$$N = 10$$

$$V_{max} = 10V$$

$$V_{min} = -10V$$

$$V_{in} = 4.564V$$

Calculate,

$$R = 2^N = 1024$$

$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) = 0.0098V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)R\right] = 746$$

$$V_C = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} = 4.570V$$

Figure 11.7 Sample Calculation of A/D Values

If the voltage being sampled is changing too fast we may get false readings, as shown in Figure 11.8. In the upper graph the waveform completes seven cycles, and 9 samples are taken. The bottom graph plots out the values read. The sampling frequency was too low, so the signal read appears to be different than it actually is, this is called aliasing.

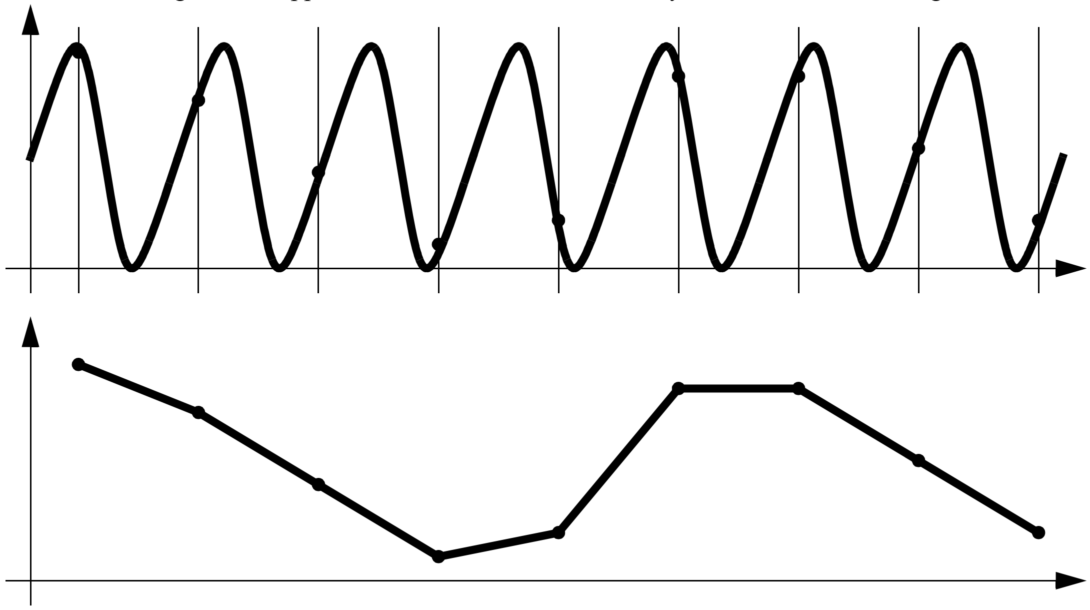


Figure 11.8 Low Sampling Frequencies Cause Aliasing

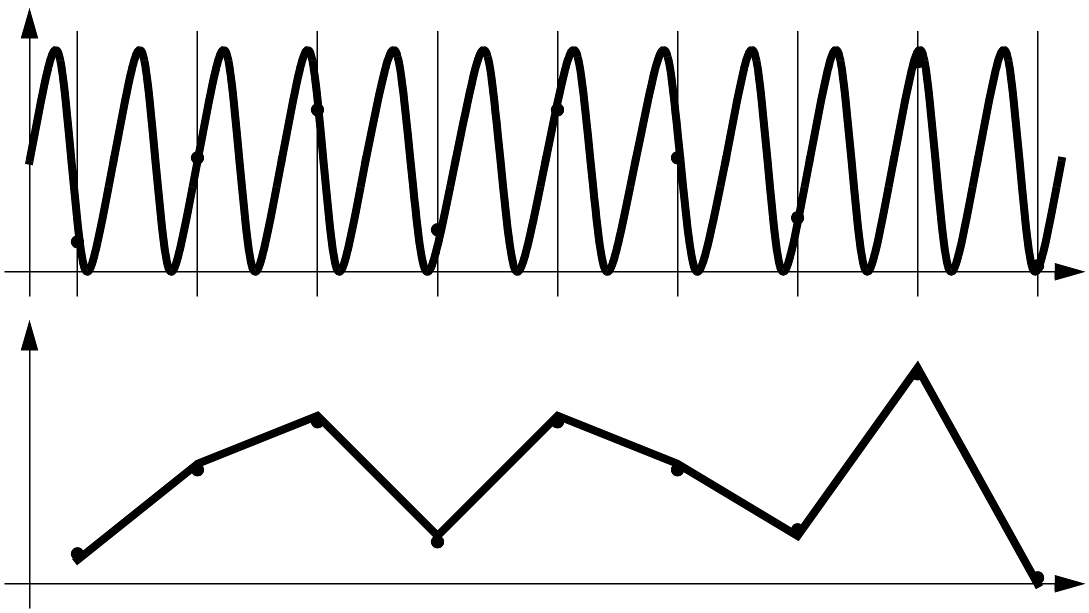


Figure 11.9 Very Low Sampling Frequencies Produce Apparently Random

The Nyquist criterion specifies that sampling frequencies should be at least twice the frequency of the signal being measured, otherwise aliasing will occur. The example in Figure 11.8 violated this principle, so the signal was aliased. If this happens in real applications the process will appear to operate erratically. In practice the sample frequency should be 4 or more times faster than the system frequency.

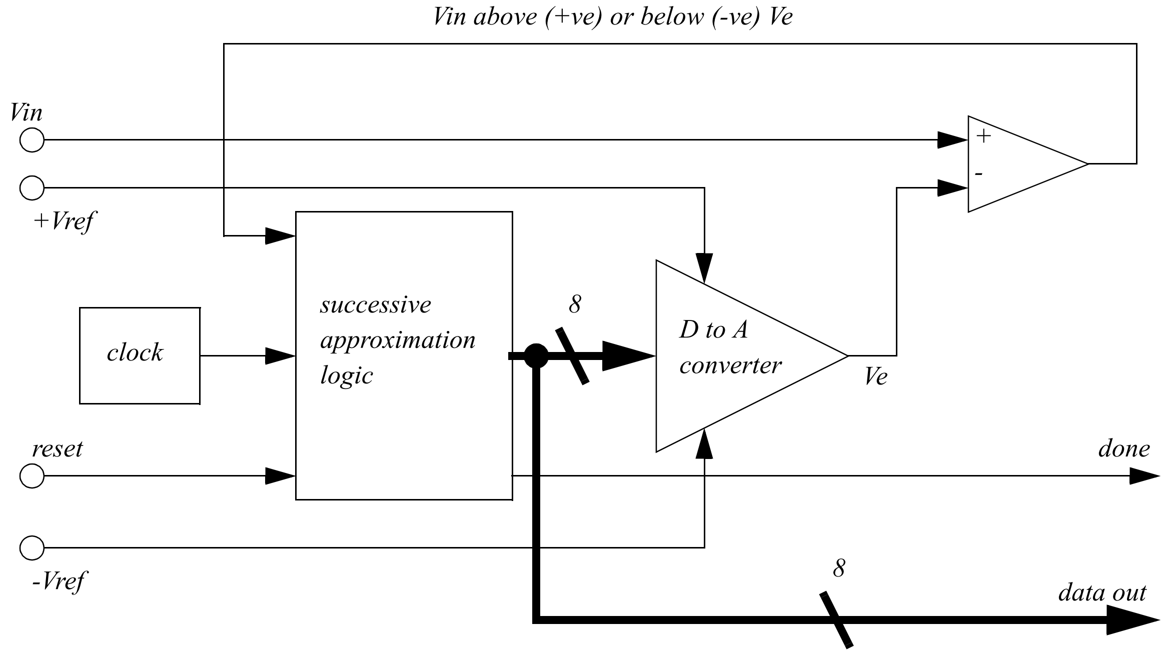
$$f_{AD} > 2f_{signal}$$

where,

f_{AD} = sampling frequency

f_{signal} = maximum frequency of the input

ASIDE: This device is an 8 bit A/D converter. The main concept behind this is the successive approximation logic. Once the reset is toggled the converter will start by setting the most significant bit of the 8 bit number. This will be converted to a voltage Ve that is a function of the +/-Vref values. The value of Ve is compared to Vin and a simple logic check determines which is larger. If the value of Ve is larger the bit is turned off. The logic then repeats similar steps from the most to least significant bits. Once the last bit has been set on/off and checked the conversion will be complete, and a done bit can be set to indicate a valid conversion value.



Quite often an A/D converter will multiplex between various inputs. As it switches the voltage will be sampled by a sample and hold circuit. This will then be converted to a digital value. The sample and hold circuits can be used before the multiplexer to collect data values at the same instant in time.

Figure 11.10 A Successive Approximation A/D Converter

There are other practical details that should be considered when designing applications with analog inputs;

- Noise - Since the sampling window for a signal is short, noise will have added effect on the signal read. For example, a momentary voltage spike might result in a higher than normal reading. Shielded data cables are commonly used to reduce the noise levels.
- Delay - When the sample is requested, a short period of time passes before the final sample value is obtained.
- Multiplexing - Most analog input cards allow multiple inputs. These may share the A/D converter using a technique called multiplexing. If there are 4 channels using an A/D converter with a maximum sampling rate of 100Hz, the maximum sampling rate per channel is 25Hz.
- Signal Conditioners - Signal conditioners are used to amplify, or filter signals coming from transducers, before they are read by the A/D converter.
- Resistance - A/D converters normally have high input impedance (resistance), so they affect circuits they are measuring.
- Single Ended Inputs - Voltage inputs to the A/D converter can use a single common for multiple inputs, these types of inputs are called single ended inputs. These tend to be more prone to noise.
- Double Ended Inputs - Each double ended input has its own common. This reduces problems with electrical noise, but also tends to reduce the number of inputs by half.
- Sampling Rates - The maximum number of samples that can be read each second. If reading multiple channels with a multiplexer, this may be reduced.
- Quantization Error - Analog IO is limited by the binary resolution of the converter. This means that the output is at discrete levels, instead of continuous values.
- Triggers - often external digital signals are used to signal the start of data collection.
- Range - the typical voltages that the card can read. Typical voltage ranges are -10V to 10V, 0V to 10V, 0V to 5V,

1V to 5V, -5V to 5V, 4mA to 20mA.
- DMA - a method to write large blocks of memory directly to computer memory. This is normally used for high speed data captures.
- Filters - some A/D input cards will provide built in functions to filter the incoming data to remove high frequency noise components.
- Input impedance - most analog inputs have very high input resistances, in the range of Mohms.

11.1 Analog Outputs

Analog outputs are much simpler than analog inputs. To set an analog output an integer is converted to a voltage. This process is very fast, and does not experience the timing problems with analog inputs. But, analog outputs are subject to quantization errors. Figure 11.11 gives a summary of the important relationships. These relationships are almost identical to those of the A/D converter.

$$R = 2^N \qquad \text{eqn 11.5}$$

$$V_{ERROR} = \pm\left(\frac{V_{max} - V_{min}}{2R}\right) \qquad \text{eqn 11.6}$$

$$V_I = INT\left[\left(\frac{V_{desired} - V_{min}}{V_{max} - V_{min}}\right)R\right] \qquad \text{eqn 11.7}$$

$$V_{output} = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} \qquad \text{eqn 11.8}$$

where,

R = *resolution of A/D converter*

V_{ERROR} = *the maximum quantization error*

V_I = *the integer value representing the desired voltage*

V_{output} = *the voltage output using the integer value*

$V_{desired}$ = *the desired output voltage*

Figure 11.11 Analog Output Relationships

Assume we are using an 8 bit D/A converter that outputs values between 0V and 10V. We have a resolution of 256, where 0 results in an output of 0V and 255 results in 10V. The quantization error will be 20mV. If we want to output a voltage of 6.234V, we would specify an output integer of 160, this would result in an output voltage of 6.250V. The quantization error would be 6.250V-6.234V=0.016V.

Given,

$$N = 8$$

$$V_{max} = 10V$$

$$V_{min} = 0V$$

$$V_{desired} = 6.234V$$

Calculate,

$$R = 2^N = 256$$

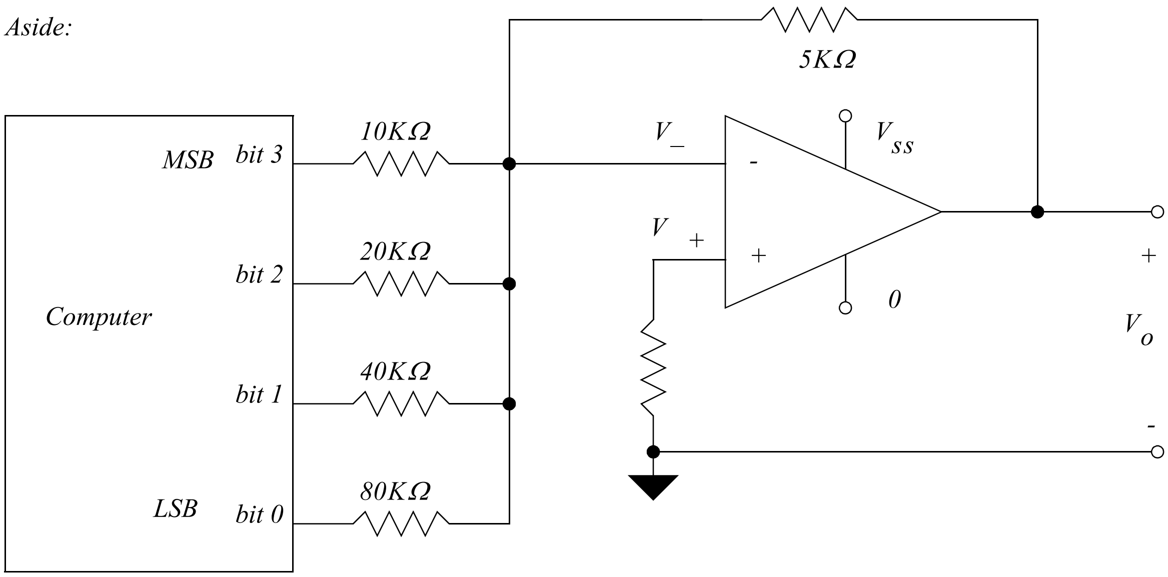
$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) = 0.020V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)R\right] = 160$$

$$V_C = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} = 6.250V$$

The current output from a D/A converter is normally limited to a small value, typically less than 20mA. This is enough for instrumentation, but for high current loads, such as motors, a current amplifier is needed. This type of interface will be discussed later. If the current limit is exceeded for 5V output, the voltage will decrease (so don't exceed the rated voltage). If the current limit is exceeded for long periods of time the D/A output may be damaged.

Aside:



First we write the obvious,

$$V_+ = 0 = V_-$$

Next, sum the currents into the inverting input as a function of the output voltage and the input voltages from the computer,

$$\frac{V_{b_3}}{10K\Omega} + \frac{V_{b_2}}{20K\Omega} + \frac{V_{b_1}}{40K\Omega} + \frac{V_{b_0}}{80K\Omega} = \frac{V_o}{5K\Omega}$$

$$\backslash V_o = 0.5V_{b_3} + 0.25V_{b_2} + 0.125V_{b_1} + 0.0625V_{b_0}$$

Consider an example where the binary output is 1110, with 5V for on,

$$\backslash V_o = 0.5(5V) + 0.25(5V) + 0.125(5V) + 0.0625(0V) = 4.375V$$

Figure 11.12 A Digital-To-Analog Converter

11.1 Noise Reduction

Signals from sensors are often not in a form that can be directly input to a controller. In these cases it may be necessary to buy or build signal conditioners. Normally, a signal conditioner is an amplifier, but it may also include noise filters, and circuitry to convert from current to voltage. This section will discuss the electrical and electronic interfaces between sensors and controllers.

Analog signal are prone to electrical noise problems. This is often caused by electromagnetic fields on the factory floor inducing currents in exposed conductors. Some of the techniques for dealing with electrical noise include;

- Twisted pairs - the wires are twisted to reduce the noise induced by magnetic fields.
- Shielding - shielding is used to reduce the effects of electromagnetic interference.
- Single/double ended inputs - shared or isolated reference voltages (commons).

When a signal is transmitted through a wire, it must return along another path. If the wires have an area between them the magnetic flux enclosed in the loop can induce current flow and voltages. If the wires are twisted, a few times per inch, then the amount of noise induced is reduced. This technique is common in signal wires and network cables.

A shielded cable has a metal sheath, as shown in Figure 11.13. This sheath needs to be connected to the measuring device to allow induced currents to be passed to ground. This prevents electromagnetic waves to induce currents (and voltages) in the signal wires.

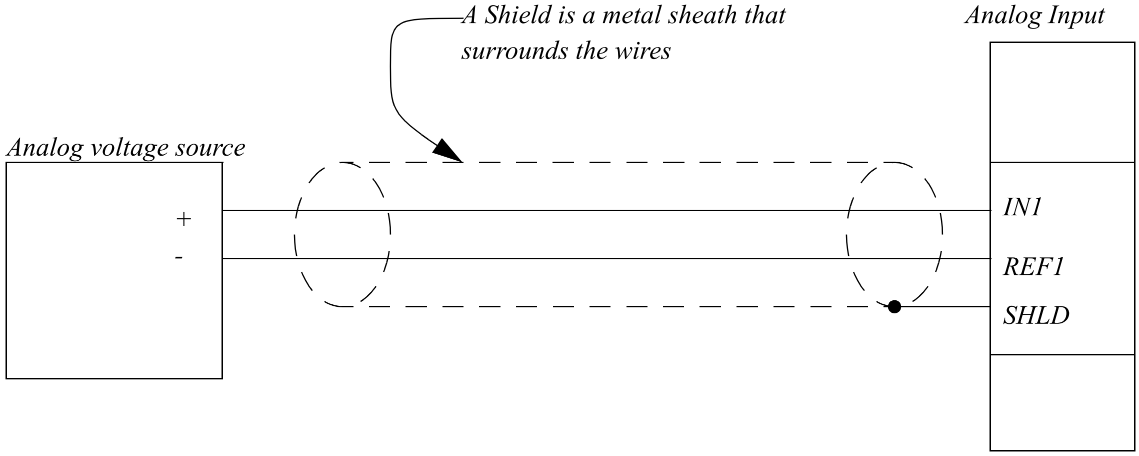
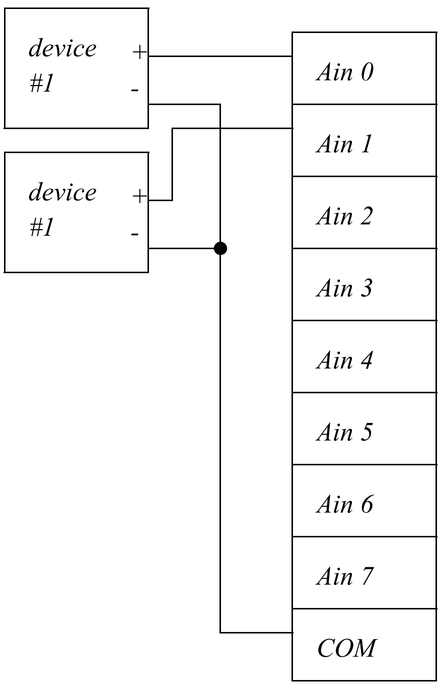
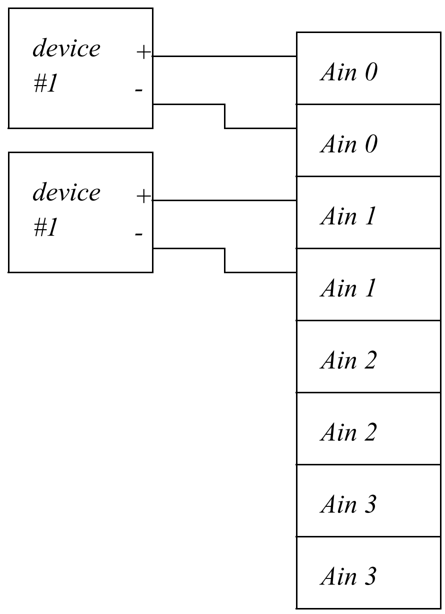


Figure 11.13 Cable Shielding

When connecting analog voltage sources to a controller the common, or 0V reference voltage, can be connected different ways, as shown in Figure 11.14. The least expensive method uses one shared common for all analog signals, this is called single ended. The more accurate method is to use separate commons for each signal, this is called double ended. Most analog input cards allow a choice between one or the other. But, when double ended inputs are used the number of available inputs is halved. Most analog output cards are double ended.



Single ended - with this arrangement the signal quality can be poorer, but more inputs are available.



Double ended - with this arrangement the signal quality can be better, but fewer inputs are available.

Figure 11.14 Single and Double Ended Inputs

Shielding

When a changing magnetic field cuts across a conductor, it will induce a current flow. The resistance in the circuits will convert this to a voltage. These unwanted voltages result in erroneous readings from sensors, and signal to outputs. Shielding will reduce the effects of the interference. When shielding and grounding are done properly, the effects of electrical noise will be negligible. Shielding is normally used for; all logical signals in noisy environments, high speed counters or high speed circuitry, and all analog signals.

There are two major approaches to reducing noise; shielding and twisted pairs. Shielding involves encasing conductors and electrical equipment with metal. As a result electrical equipment is normally housed in metal cases. Wires are normally put in cables with a metal sheath surrounding both wires. The metal sheath may be a thin film, or a woven metal mesh. Shielded wires are connected at one end to "drain" the unwanted signals into the cases of the instruments. "Shielding for a Thermocouple" on page 411 shows a thermocouple connected with a thermocouple. The cross section of the wire contains two insulated conductors. Both of the wires are covered with a metal foil, and final covering of insulation finishes the cable. The wires are connected to the thermocouple as expected, but the shield is only connected on the amplifier end to the case. The case is then connected to the shielding ground, shown here as three diagonal lines.

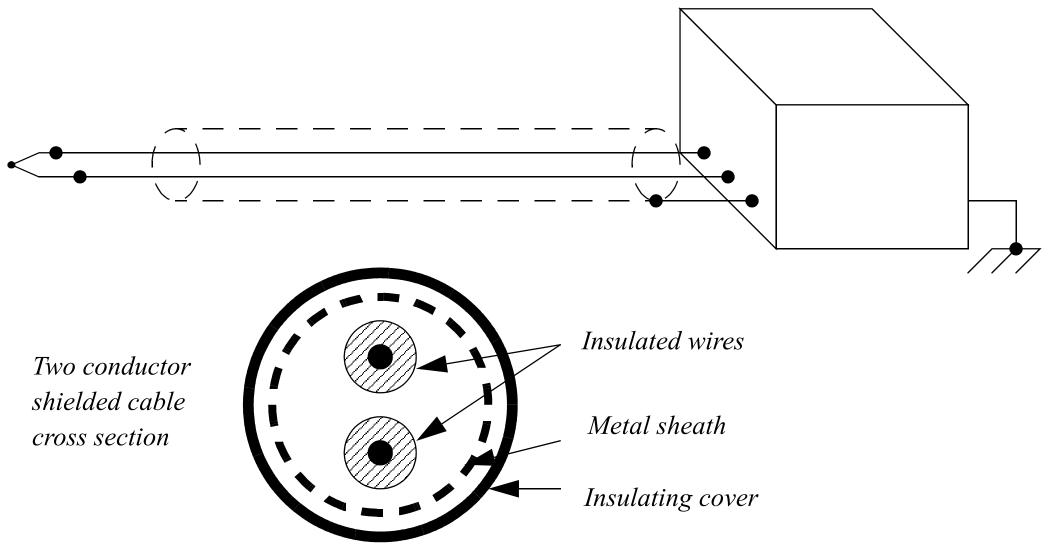


Figure 11.15 Shielding for a Thermocouple

A twisted pair is shown in "A Twisted Pair" on page 411. The two wires are twisted at regular intervals, effectively forming small loops. In this case the small loops reverse every twist, so any induced currents are cancel out for every two twists.

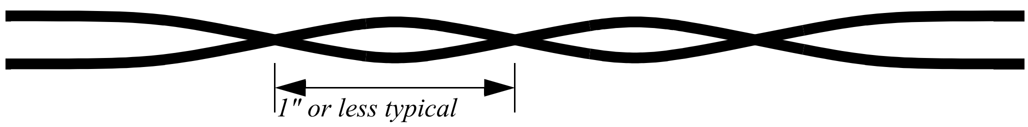


Figure 11.16 A Twisted Pair

When designing shielding, the following design points will reduce the effects of electromagnetic interference.

- Avoid "noisy" equipment when possible.
- Choose a metal cabinet that will shield the control electronics.
- Use shielded cables and twisted pair wires.
- Separate high current, and AC/DC wires from each other when possible.
- Use current oriented methods such as sourcing and sinking for logical I/O.
- Use high frequency filters to eliminate high frequency noise.
- Use power line filters to eliminate noise from the power supply.

Grounding

- Ground voltages are based upon the natural voltage level in the physical ground (the earth under your feet). This will vary over a distance. Most buildings and electrical systems use a ground reference for the building. Between different points on the same building ground voltage levels may vary as much as a few hundred millivolts. This can lead to significant problems with voltage readings and system safety.
- A signal can be floating, or connected to a ground
- If floating a system normally has a self contained power source, or self reference such as a battery, strain gauge or thermocouple. These are usually read with double ended outputs. The potential for floating voltage levels can be minimized by connecting larger resistors (up to 100K) from the input to ground.
- A grounded system uses a single common (ground) for all signals. These are normally connected to a single ended inputs.
- The analog common can also be connected to the ground with a large resistor to drain off induced voltages.
- Cable shields or grounds are normally only connected at one side to prevent ground loops.

11.2 Case Study

- Data smoothing by averaging inputs
- Conversion back to an input voltage
- Use it to calculate an output voltage for control.

11.3 Summary

- A/D conversion will convert a continuous value to an integer value.
- D/A conversion is easier and faster and will convert a digital value to an analog value.
- Resolution limits the accuracy of A/D and D/A converters.
- Sampling too slowly will alias the real signal.
- Analog inputs are sensitive to noise.
- Analog shielding should be used to improve the quality of electrical signals.

11.4 Problems With Solutions

Problem 11.1 If given a 12 bit analog input with a range of -10V to 10V. If we put in 2.735V, what will the integer value be after the A/D conversion? What is the error? What voltage can we calculate?

Problem 11.2 What is the difference between an A/D input and D/A output?

Problem 11.3 An analog voltage that has a range of -10V to 10V and is to be read to a precision of +/-0.05V. How is the minimum number of bits required for the A/D converter?

Problem 11.4 Write a (complete) program for the ATmega32 that will read a potentiometer connected to PA0. The potentiometer is connected across a 2V voltage source. When the potentiometer is above the middle of the range (>1V) pin PB0 will be turned on.

11.5 Problem Solutions

Answer 11.1

$$N = 12 \qquad R = 4096 \qquad V_{min} = -10V \qquad V_{max} = 10V \qquad V_{in} = 2.735V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)(R-1)\right] = 2607$$

$$V_C = \left(\frac{V_I}{R-1}\right)(V_{max} - V_{min}) + V_{min} = 2.733V$$

$$error = V_c - V_{in} = (-0.002)V$$

Answer 11.2 An A/D converter will convert an analog input voltage (or current) to an integer value. A D/A output will convert an integer value to an output voltage.

Answer 11.3

$$R = \frac{10V - (-10V)}{0.1V} = 200 \qquad \begin{array}{l} 7 \text{ bits}, R = 128 \\ 8 \text{ bits}, R = 256 \end{array}$$

The minimum number of bits is 8.

Answer 11.4

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)(R-1)\right] = INT\left[\left(\frac{1.0 - 0}{2.54 - 0}\right)(2^{10} - 1)\right] = 403$$

```
#include <avr/io.h>

int main(void)
{
	ADMUX = 0xC0; // set the input to channel 0
	ADCSRA = 0xE0;// turn on ADC and set to free running
	DDRB = (unsigned char)255; // set port B to outputs

	while (1) {
		if(ADCW > 403){
			PORTB = 0x01; // set output
		} else {
			PORTB = 0x00; // clear output
		}
	}

	return 0;
}
```

11.6 Problems Without Solutions

Problem 11.5 An analog output needs to be between -4V and 8V, in 0.005V intervals. How many bits are required for the D/A converter?

Problem 11.6 Discuss methods for reducing electrical noise in analog inputs.

Problem 11.7 Write a C program for an ATMega 32 microcontroller to collect an analog data value every half a second and print it to the screen. The program should run for 10 seconds and then stop.

Problem 11.8 Write a C program for an ATMega 32 microcontroller to read a 10 bit analog input every 20ms and update an 8 bit PWM output. The relationship between input and output should be such that when the analog input is 0 the PWM output should be 0, and when the analog input is 1023 the PWM output should be 255.

12. Continuous Sensors

Topic 12.1	*Continuous sensor issues; accuracy, resolution, etc.*
Topic 12.2	*Angular measurement; potentiometers, encoders and tachometers.*
Topic 12.3	*Linear measurement; potentiometers, LVDTs, Moire fringes and accelerometers.*
Topic 12.4	*Force measurement; strain gages and piezoelectric.*
Topic 12.5	*Liquid and fluid measurement; pressure and flow.*
Topic 12.6	*Temperature measurement; RTDs, thermocouples and thermistors.*
Topic 12.7	*Other sensors.*
Topic 12.8	*Continuous signal inputs and wiring.*
Objective 12.1	*To understand the common continuous sensor types.*
Objective 12.2	*To understand interfacing issues.*

Continuous sensors convert physical phenomena to measurable signals, typically voltages or currents. Consider a simple temperature measuring device, there will be an increase in output voltage proportional to a temperature rise. A computer could measure the voltage, and convert it to a temperature. The basic physical phenomena typically measured with sensors include;

- Angular or linear position
- Acceleration
- Temperature
- Pressure or flow rates
- Stress, strain or force
- Light intensity
- Sound

Most of these sensors are based on subtle electrical properties of materials and devices. As a result the signals often require *signal conditioners*. These are often amplifiers that boost currents and voltages to larger voltages.

Sensors are also called transducers. This is because they convert an input phenomena to an output in a different form. This transformation relies upon a manufactured device with limitations and imperfection. As a result sensor limitations are often characterized with;

- Accuracy - This is the maximum difference between the indicated and actual reading. For example, if a sensor reads a force of 100N with a ±1% accuracy, then the force could be anywhere from 99N to 101N.
- Resolution - Used for systems that step through readings. This is the smallest increment that the sensor can detect, this may also be incorporated into the accuracy value. For example if a sensor measures up to 10 inches of linear displacements, and it outputs a number between 0 and 100, then the resolution of the device is 0.1 inches.
- Repeatability - When a single sensor condition is made and repeated, there will be a small variation for that particular reading. If we take a statistical range for repeated readings (e.g., ±3 standard deviations) this will be the repeatability. For example, if a flow rate sensor has a repeatability of 0.5cfm, readings for an actual flow of 100cfm should rarely be outside 99.5cfm to 100.5cfm.
- Linearity - In a linear sensor the input phenomenon has a linear relationship with the output signal. In most sensors this is a desirable feature. When the relationship is not linear, the conversion from the sensor output (e.g., voltage) to a calculated quantity (e.g., force) becomes more complex.
- Precision - This considers accuracy, resolution and repeatability or one device relative to another.
- Range - Natural limits for the sensor. For example, a sensor for reading angular rotation may only rotate 200 degrees.
- Dynamic Response - The frequency range for regular operation of the sensor. Typically sensors will have an upper operation frequency, occasionally there will be lower frequency limits. For example, our ears hear best between 10Hz and 16KHz.
- Environmental - Sensors all have some limitations over factors such as temperature, humidity, dirt/oil, corrosives and pressures. For example many sensors will work in relative humidities (RH) from 10% to 80%.
- Calibration - When manufactured or installed, many sensors will need some calibration to determine or set the relationship between the input phenomena, and output. For example, a temperature reading sensor may need to be zeroed or adjusted so that the measured temperature matches the actual temperature. This may require special equipment, and need to be performed frequently.
- Cost - Generally more precision costs more. Some sensors are very inexpensive, but the signal conditioning equipment costs are significant.

This section describes sensors that will be of use for industrial measurements. The sections have been divided by the phenomena to be measured. Where possible details are provided.

12.1 Spatial Sensors

Angular Position - Potentiometers

Potentiometers measure the angular position of a shaft using a variable resistor. A potentiometer is shown in Figure 12.1. The potentiometer is resistor, normally made with a thin film of resistive material. A wiper can be moved along the surface of the resistive film. As the wiper moves toward one end there will be a change in resistance proportional to the distance moved. If a voltage is applied across the resistor, the voltage at the wiper interpolate the voltages at the ends of the resistor.

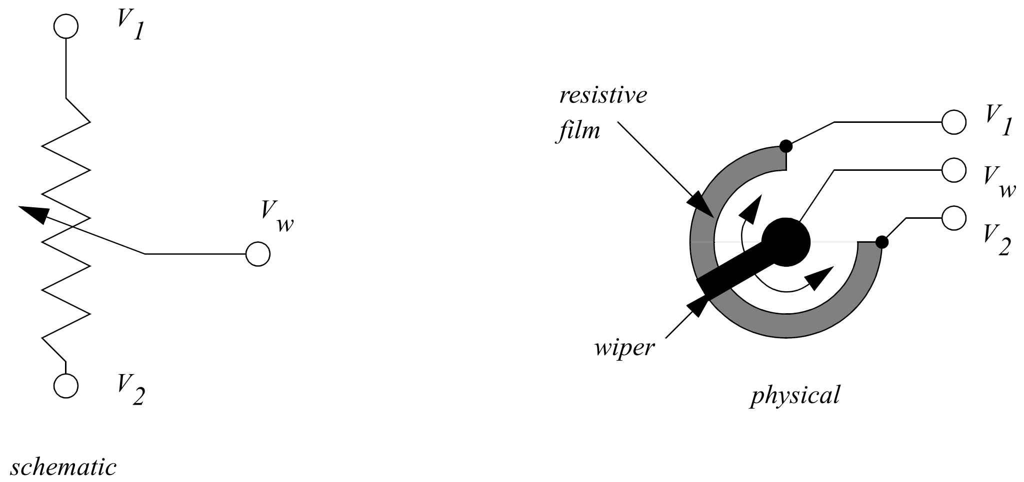
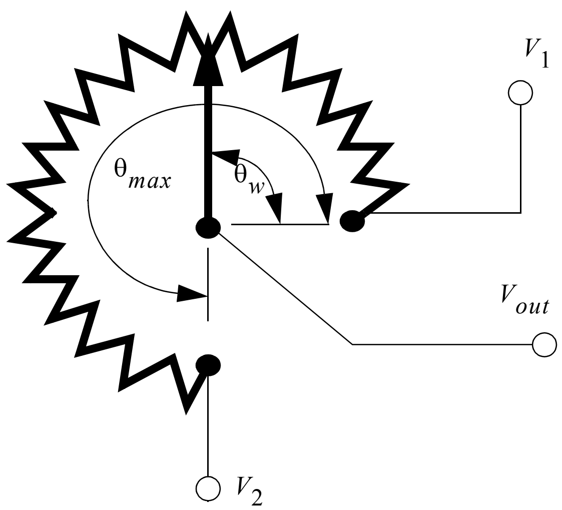


Figure 12.1 A Potentiometer

The potentiometer in Figure 12.2 is being used as a voltage divider. As the wiper rotates the output voltage will be proportional to the angle of rotation.



$$V_{out} = (V_2 - V_1)\left(\frac{\theta_w}{\theta_{max}}\right) + V_1$$

Figure 12.2 A Potentiometer as a Voltage Divider

Potentiometers are popular because they are inexpensive, and don't require special signal conditioners. But, they have limited accuracy, normally in the range of 1% and they are subject to mechanical wear.

Potentiometers measure absolute position, and they are calibrated by rotating them in their mounting brackets, and then tightening them in place. The range of rotation is normally limited to less than 360 degrees or multiples of 360 degrees. Some potentiometers can rotate without limits, and the wiper will jump from one end of the resistor to the other.

Faults in potentiometers can be detected by designing the potentiometer to never reach the ends of the range of motion. If an output voltage from the potentiometer ever reaches either end of the range, then a problem has occurred, and the machine can be shut down. Two examples of problems that might cause this are wires that fall off, or the potentiometer rotates in its mounting.

Angular Displacement - Encoders

Encoders use rotating disks with optical windows, as shown in Figure 12.3. The encoder contains an optical disk with fine windows etched into it. Light from emitters passes through the openings in the disk to detectors. As the encoder shaft is rotated, the light beams are broken. The encoder shown here is a quadrature encode, and it will be discussed later.

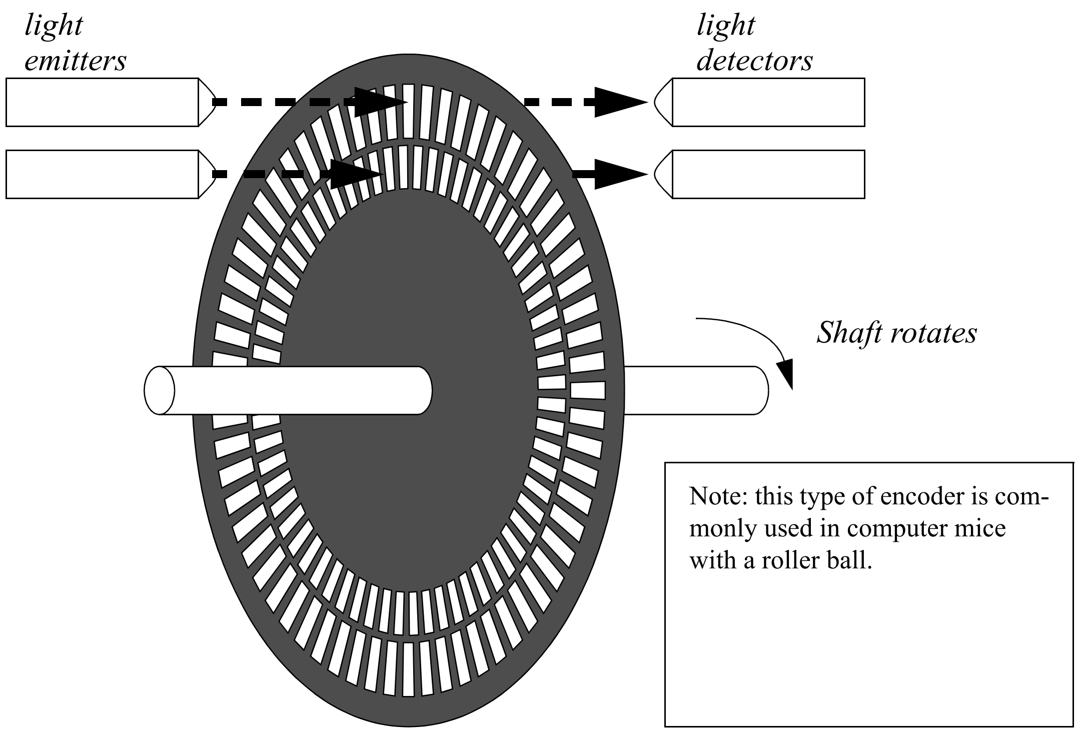


Figure 12.3 An Encoder Disk

There are two fundamental types of encoders; absolute and incremental. An absolute encoder will measure the position of the shaft for a single rotation. The same shaft angle will always produce the same reading. The output is normally a binary or Gray code number. An incremental (or relative) encoder will output two pulses that can be used to determine displacement. Logic circuits or software is used to determine the direction of rotation, and count pulses to determine the displacement. The velocity can be determined by measuring the time between pulses.

Encoder disks are shown in Figure 12.4. The absolute encoder has two rings, the outer ring is the most significant digit of the encoder, the inner ring is the least significant digit. The relative encoder has two rings, with one ring rotated a few degrees ahead of the other, but otherwise the same. Both rings detect position to a quarter of the disk. To add accuracy to the absolute encoder more rings must be added to the disk, and more emitters and detectors. To add accuracy to the relative encoder we only need to add more windows to the existing two rings. Typical encoders will have from 2 to thousands of windows per ring.

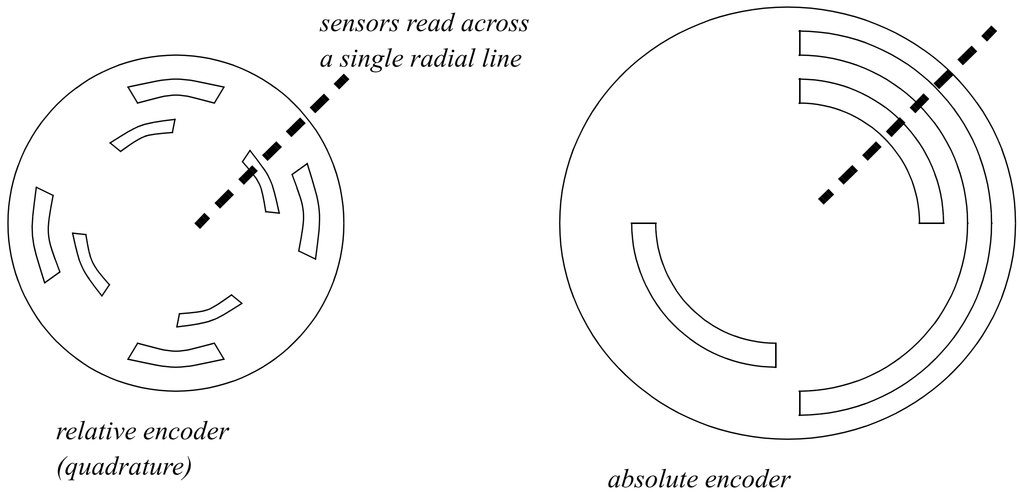
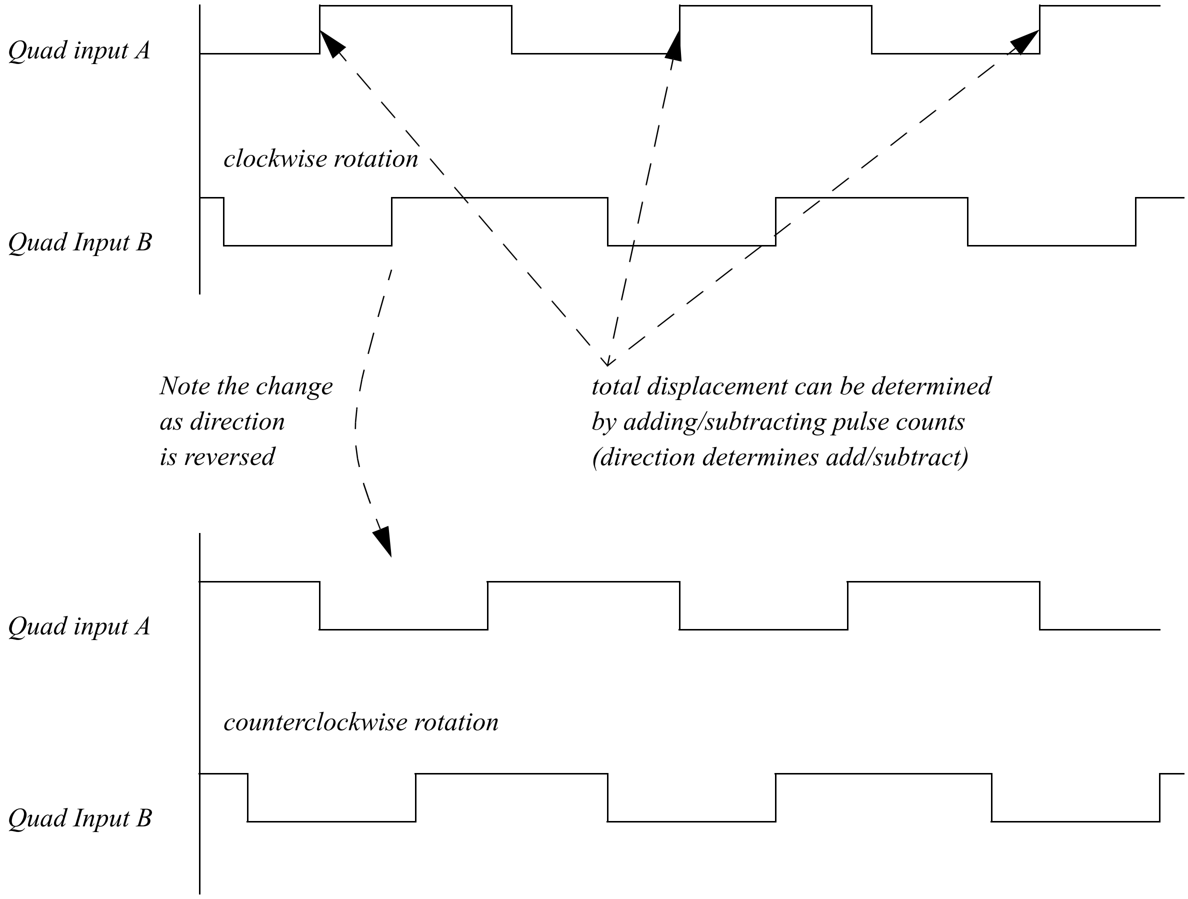


Figure 12.4 Encoder Disks

When using absolute encoders, the position during a single rotation is measured directly. If the encoder rotates multiple times then the total number of rotations must be counted separately.

When using a relative encoder, the distance of rotation is determined by counting the pulses from one of the rings. If the encoder only rotates in one direction then a simple count of pulses from one ring will determine the total distance. If the encoder can rotate both directions a second ring must be used to determine when to subtract pulses. The quadrature scheme, using two rings, is shown in Figure 12.5. The signals are set up so that one is out of phase with the other. Notice that for different directions of rotation, input *B* either leads or lags *A*.



Note: To determine direction we can do a simple check. If both are off or on, the first to change state determines direction. Consider a point in the graphs above where both A and B are off. If A is the first input to turn on the encoder is rotating clockwise. If B is the first to turn on the rotation is counterclockwise.

Aside: A circuit (or program) can be built for this circuit using an up/down counter. If the positive edge of input A is used to trigger the clock, and input B is used to drive the up/down count, the counter will keep track of the encoder position.

Figure 12.5 Quadrature Encoders

Interfaces for encoders are commonly available for PLCs and as purchased units. Newer PLCs will also allow two normal inputs to be used to decode encoder inputs.

Normally absolute and relative encoders require a calibration phase when a controller is turned on. This normally involves moving an axis until it reaches a logical sensor that marks the end of the range. The end of range is then used as the zero position. Machines using encoders, and other relative sensors, are noticeable in that they normally move to some extreme position before use.

Angular Velocity - Tachometers

Tachometers measure the velocity of a rotating shaft. A common technique is to mount a magnet to a rotating shaft. When the magnetic moves past a stationary pick-up coil, current is induced. For each rotation of the shaft there is a pulse in the coil, as shown in Figure 12.6. When the time between the pulses is measured the period for one rotation can be found, and the frequency calculated. This technique often requires some signal conditioning circuitry.

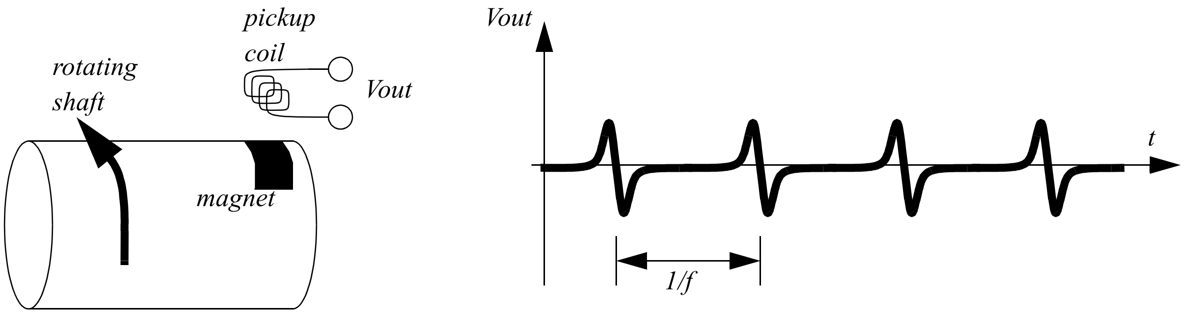


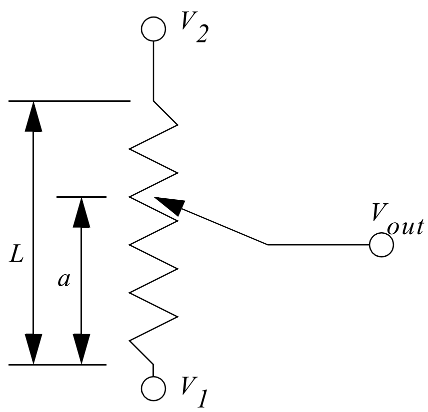
Figure 12.6 A Magnetic Tachometer

Another common technique uses a simple permanent magnet DC generator (note: you can also use a small DC motor). The generator is hooked to the rotating shaft. The rotation of a shaft will induce a voltage proportional to the angular velocity. This technique will introduce some drag into the system, and is used where efficiency is not an issue.

Both magnetic and optical tachometers are well understood, commonly used, and inexpensive.

Linear Position - Potentiometers

Rotational potentiometers were discussed before, but potentiometers are also available in linear/sliding form. These are capable of measuring linear displacement over long distances. Figure 12.7 shows the output voltage when using the potentiometer as a voltage divider.



$$V_{out} = V_1 + (V_2 - V_1)\left(\frac{a}{L}\right)$$

Figure 12.7 Linear Potentiometer

Linear/sliding potentiometers have the same general advantages and disadvantages of rotating potentiometers.

Linear Position - Linear Variable Differential Transformers (LVDT)

Linear Variable Differential Transformers (LVDTs) measure linear displacements over a limited range. The basic device is shown in Figure 12.8. It consists of outer coils with an inner moving magnetic core. High frequency alternating current (AC) is applied to the center coil. This generates a magnetic field that induces a current in the two outside coils. The core will pull the magnetic field towards it, so in the figure more current will be induced in the left hand coil. The outside coils are wound in opposite directions so that when the core is in the center the induced currents cancel, and the signal out is zero (0Vac). The magnitude of the *signal out* voltage on either line indicates the position of the core. Near the center of motion the change in voltage is proportional to the displacement. But, further from the center the relationship becomes nonlinear.

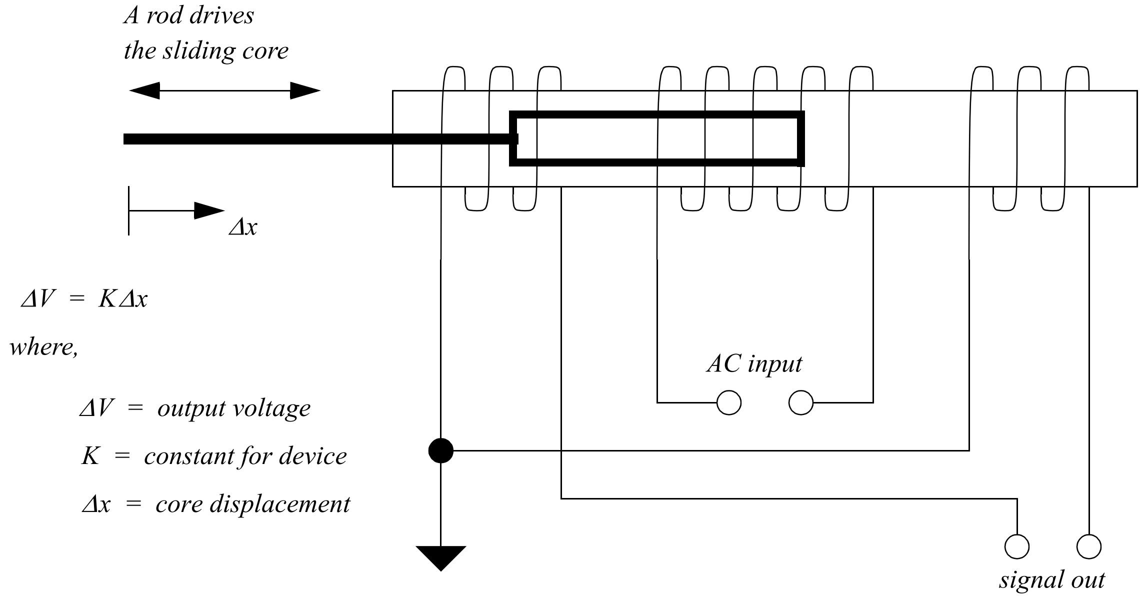


Figure 12.8 An LVDT

Aside: The circuit below can be used to produce a voltage that is proportional to position. The two diodes convert the AC wave to a half wave DC wave. The capacitor and resistor values can be selected to act as a low pass filter. The final capacitor should be large enough to smooth out the voltage ripple on the output.

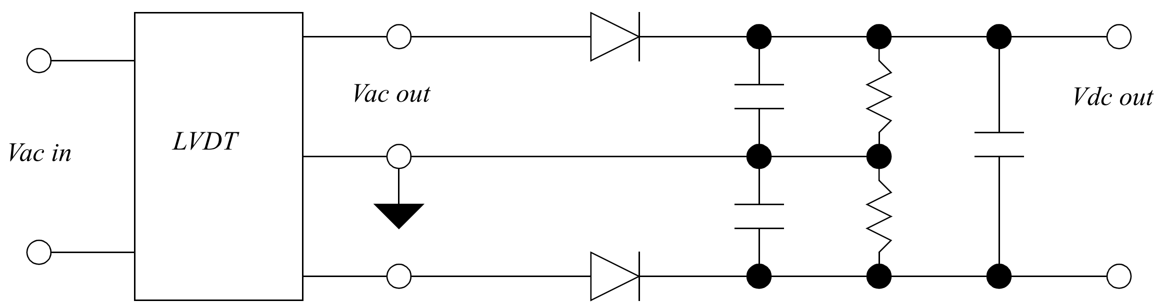
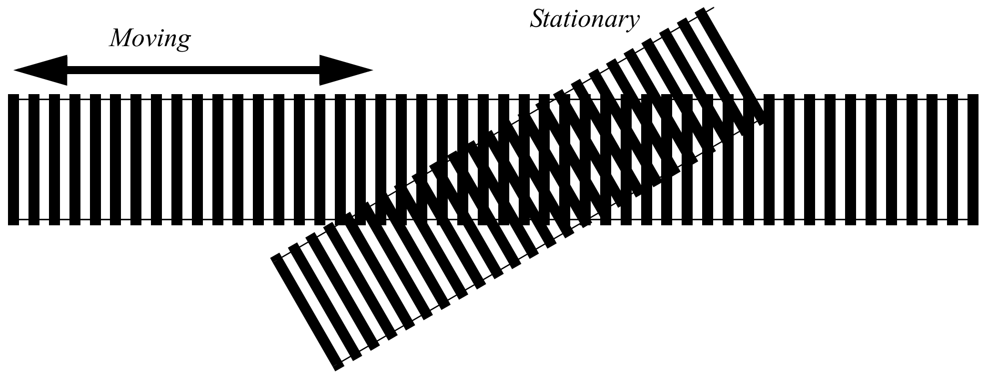


Figure 12.9 A Simple Signal Conditioner for an LVDT

These devices are more accurate than linear potentiometers, and have less friction. Typical applications for these devices include measuring dimensions on parts for quality control. They are often used for pressure measurements with Bourdon tubes and bellows/diaphragms. A major disadvantage of these sensors is the high cost, often in the thousands.

Linear Displacement - Moire Fringes

High precision linear displacement measurements can be made with Moire Fringes, as shown in Figure 12.10. Both of the strips are transparent (or reflective), with black lines at measured intervals. The spacing of the lines determines the accuracy of the position measurements. The stationary strip is offset at an angle so that the strips interfere to give irregular patterns. As the moving strip travels by a stationary strip the patterns will move up, or down, depending upon the speed and direction of motion.



Note: you can recreate this effect with the strips below. Photocopy the pattern twice, overlay the sheets and hold them up to the light. You will notice that shifting one sheet will cause the stripes to move up or down.

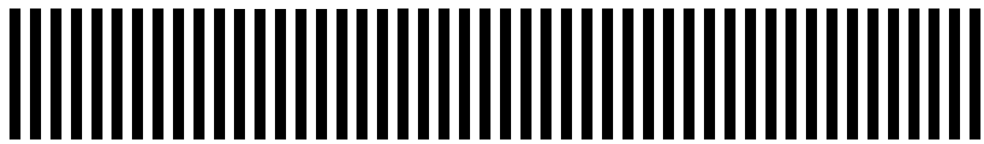


Figure 12.10 The Moire Fringe Effect

A device to measure the motion of the moire fringes is shown in Figure 12.11. A light source is collimated by passing it through a narrow slit to make it one slit width. This is then passed through the fringes to be detected by light sensors. At least two light sensors are needed to detect the bright and dark locations. Two sensors, close enough, can act as a quadrature pair, and the same method used for quadrature encoders can be used to determine direction and distance of motion.

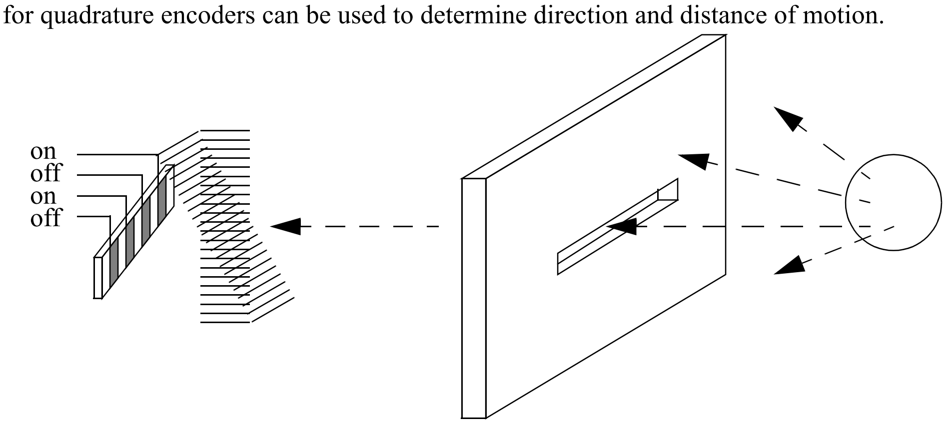


Figure 12.11 Measuring Motion with Moire Fringes

These are used in high precision applications over long distances, often meters. They can be purchased from a number of suppliers, but the cost will be high. Typical applications include Coordinate Measuring Machines (CMMs).

Acceleration and Gravity - Accelerometers

Accelerometers measure acceleration using a mass suspended on a force sensor, as shown in Figure 12.12. When the sensor accelerates, the inertial resistance of the mass will cause the force sensor to deflect. By measuring the deflection the acceleration can be determined. In this case the mass is cantilevered on the force sensor. A base and housing enclose the sensor. A small mounting stud (a threaded shaft) is used to mount the accelerometer.

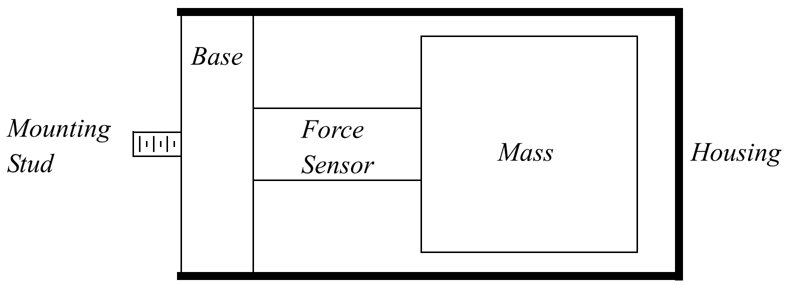


Figure 12.12 A Cross Section of an Accelerometer

Accelerometers are dynamic sensors, typically used for measuring vibrations between 10Hz to 10KHz. Temperature variations will affect the accuracy of the sensors. Standard accelerometers can be linear up to 100,000 m/s**2: high shock designs can be used up to 1,000,000 m/s**2. There is often a trade-off between a wide frequency range and device sensitivity (note: higher sensitivity requires a larger mass). "Piezoelectric Accelerometer Sensitivities" on page 423 shows the sensitivity of two accelerometers with different resonant frequencies. A smaller resonant frequency limits the maximum frequency for the reading. The smaller frequency results in a smaller sensitivity. The units for sensitivity is charge per m/s**2.

resonant freq. (Hz)	*sensitivity*
22 KHz	*4.5 pC/(m/s**2)*
180KHz	*.004*

Figure 12.13 Piezoelectric Accelerometer Sensitivities

The force sensor is often a small piece of piezoelectric material (discussed later in this chapter). The piezoelectric material can be used to measure the force in shear or compression. Piezoelectric based accelerometers typically have parameters such as,

- -100 to 250°C operating range
- 1mV/g to 30V/g sensitivity
- operate well below one forth of the natural frequency

The accelerometer is mounted on the vibration source as shown in Figure 12.14. The accelerometer is electrically isolated from the vibration source so that the sensor may be grounded at the amplifier (to reduce electrical noise). Cables are fixed to the surface of the vibration source, close to the accelerometer, and are fixed to the surface as often as possible to prevent noise from the cable striking the surface. Background vibrations can be detected by attaching control electrodes to *non-vibrating* surfaces. Each accelerometer is different, but some general application guidelines are;

- The control vibrations should be less than 1/3 of the signal for the error to be less than 12%).
- Mass of the accelerometers should be less than a tenth of the measurement mass.
- These devices can be calibrated with shakers, for example a 1g shaker will hit a peak acceleration of 9.81 m/s**2.

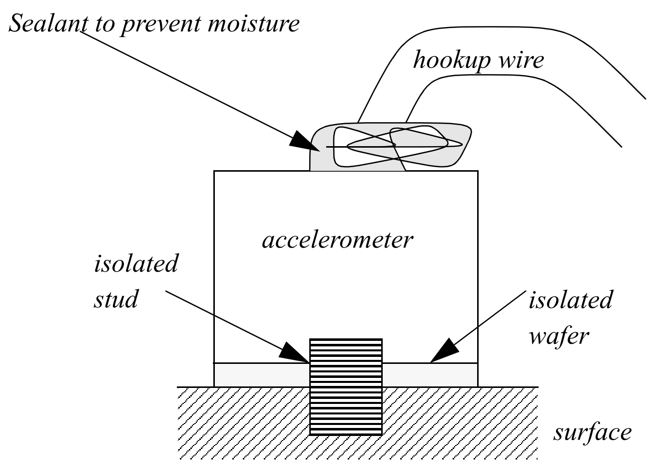


Figure 12.14 Mounting an Accelerometer

Equipment normally used when doing vibration testing is shown in Figure 12.15. The sensor needs to be mounted on the equipment to be tested. A preamplifier normally converts the charge generated by the accelerometer to a voltage. The voltage can then be analyzed to determine the vibration frequencies.

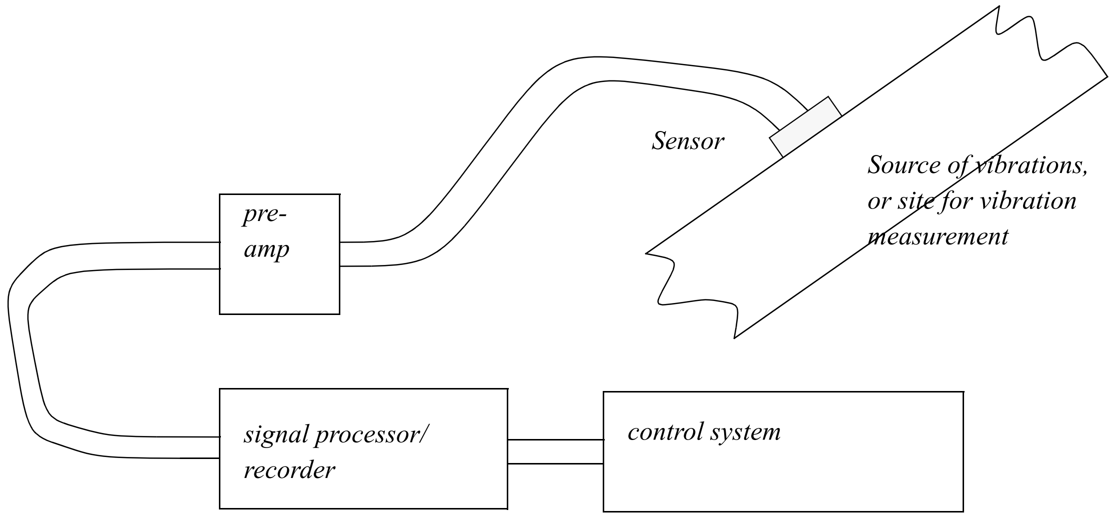


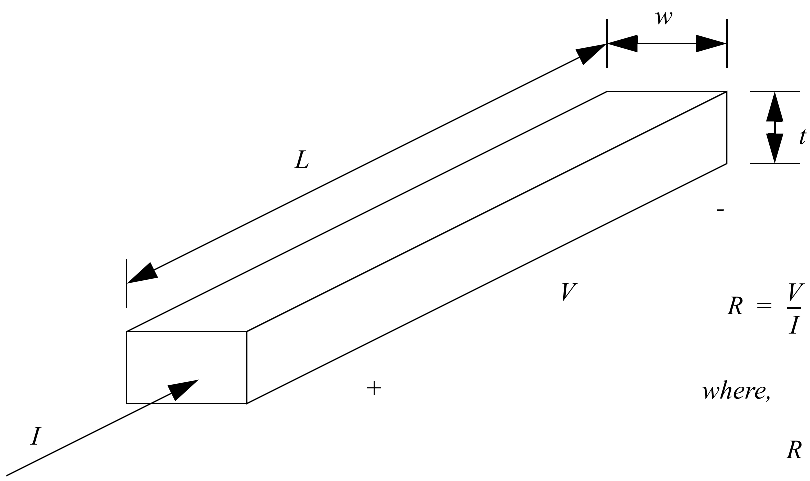
Figure 12.15 Typical Connection for Accelerometers

Accelerometers are commonly used for control systems that adjust speeds to reduce vibration and noise. Computer Controlled Milling machines now use these sensors to actively eliminate chatter, and detect tool failure. The signal from accelerometers can be integrated to find velocity and acceleration.

Currently accelerometers cost hundreds or thousands per channel. But, advances in micromachining are already beginning to provide integrated circuit accelerometers at a low cost. Their current use is for airbag deployment systems in automobiles.

Forces and Moments - Strain Gages

Strain gages measure strain in materials using the change in resistance of a wire. The wire is glued to the surface of a part, so that it undergoes the same strain as the part (at the mount point). Figure 12.16 shows the basic properties of the undeformed wire. Basically, the resistance of the wire is a function of the resistivity, length, and cross sectional area.



$$R = \frac{V}{I} = \rho\frac{L}{A} = \rho\frac{L}{wt}$$

where,

R = *resistance of wire*

V, I = *voltage and current*

L = *length of wire*

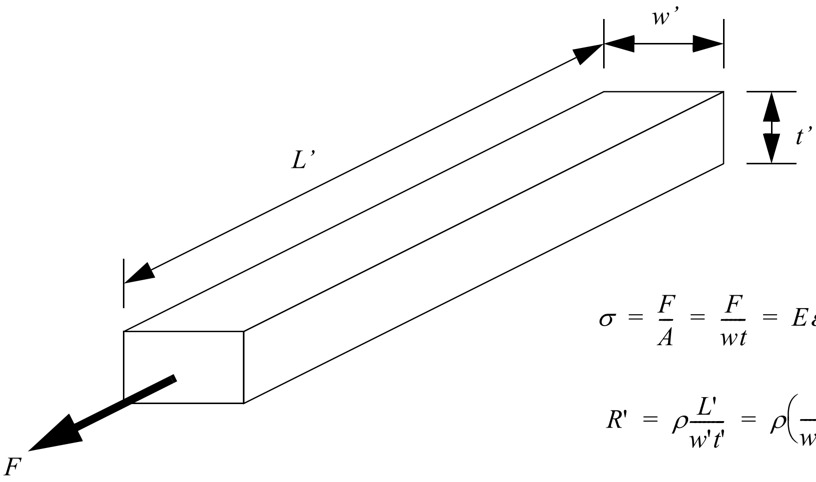
w, t = *width and thickness*

A = *cross sectional area of conductor*

ρ = *resistivity of material*

Figure 12.16 The Electrical Properties of a Wire

After the wire in Figure 12.16 has been deformed it will take on the new dimensions and resistance shown in Figure 12.17. If a force is applied as shown, the wire will become longer, as predicted by Young's modulus. But, the cross sectional area will decrease, as predicted by Poison's ratio. The new length and cross sectional area can then be used to find a new resistance.



$$\sigma = \frac{F}{A} = \frac{F}{wt} = E\varepsilon \qquad \varepsilon = \frac{F}{Ewt}$$

$$R' = \rho\frac{L'}{w't'} = \rho\left(\frac{L(1+\varepsilon)}{w(1-\nu\varepsilon)t(1-\nu\varepsilon)}\right)$$

$$\Delta R = R' - R = R\left[\frac{(1+\varepsilon)}{(1-\nu\varepsilon)(1-\nu\varepsilon)} - 1\right]$$

where,

ν = *poissons ratio for the material*

F = *applied force*

E = *Youngs modulus for the material*

σ, ε = *stress and strain of material*

Aside: Gauge factor, as defined below, is a commonly used measure of stain gauge sensitivity.

$$GF = \frac{\left(\frac{\Delta R}{R}\right)}{\varepsilon}$$

Figure 12.17 The Electrical and Mechanical Properties of the Deformed Wire

Aside: Changes in strain gauge resistance are typically small (large values would require strains that would cause the gages to plastically deform). As a result, Wheatstone bridges are used to amplify the small change. In this circuit the variable resistor R2 would be tuned until Vo = 0V. Then the resistance of the strain gage can be calculated using the given equation.

$$R_{strain} = \frac{R_2 R_1}{R_3} \quad \text{when } Vo = 0V$$

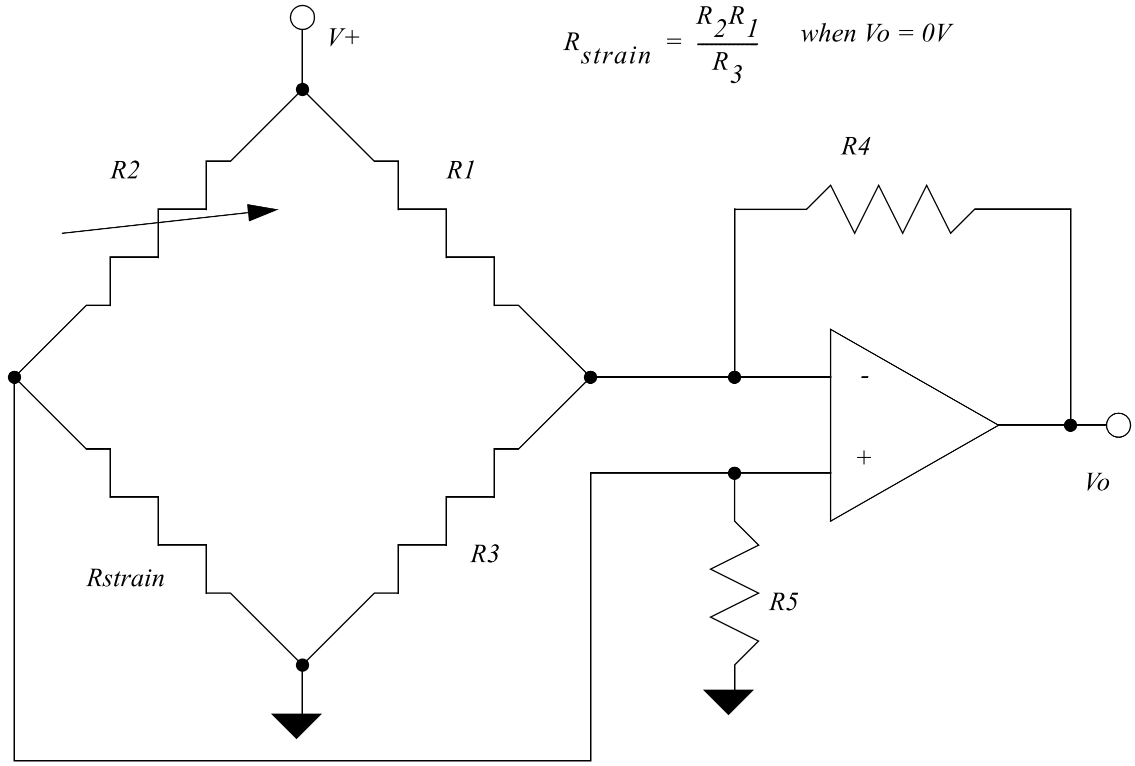


Figure 12.18 Measuring Strain with a Wheatstone Bridge

A strain gage must be small for accurate readings, so the wire is actually wound in a uniaxial or rosette pattern, as shown in Figure 12.19. When using uniaxial gages the direction is important, it must be placed in the direction of the normal stress. (Note: the gages cannot read shear stress.) Rosette gages are less sensitive to direction, and if a shear force is present the gage will measure the resulting normal force at 45 degrees. These gauges are sold on thin films that are glued to the surface of a part. The process of mounting strain gages involves surface cleaning. application of adhesives, and soldering leads to the strain gages.

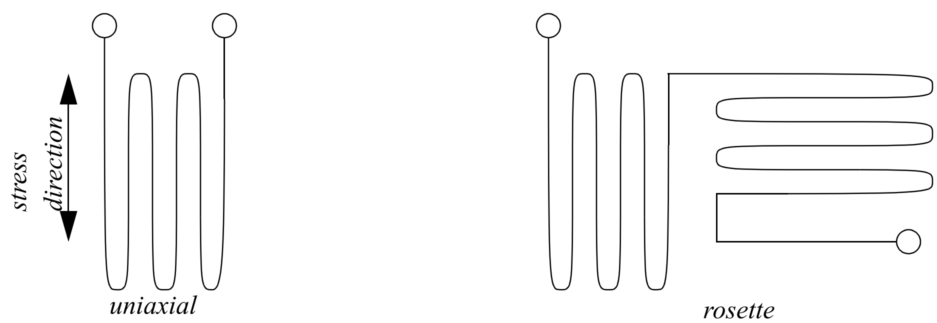


Figure 12.19 Wire Arrangements in Strain Gages

A design techniques using strain gages is to design a part with a narrowed neck to mount the strain gage on, as shown in Figure 12.20. In the narrow neck the strain is proportional to the load on the member, so it may be used to measure force. These parts are often called *load cells*.

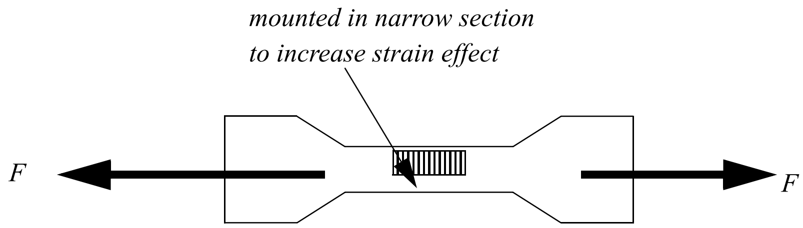
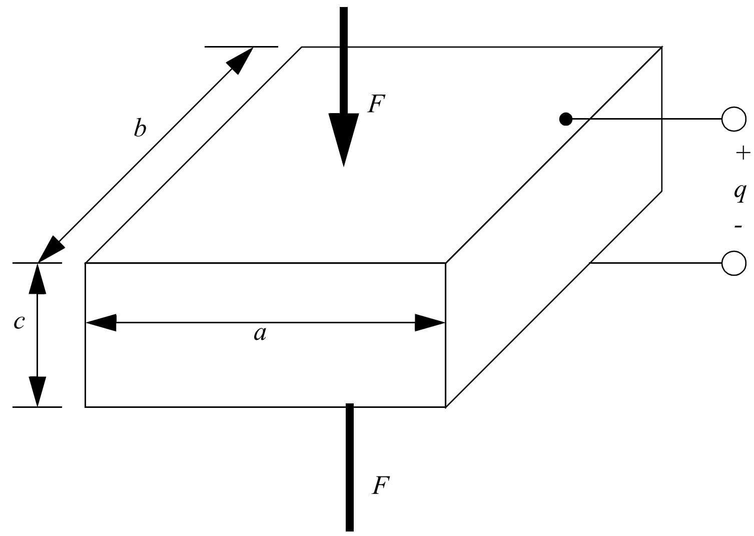


Figure 12.20 Using a Narrow to Increase Strain

Strain gages are inexpensive, and can be used to measure a wide range of stresses with accuracies under 1%. Gages require calibration before each use. This often involves making a reading with no load, or a known load applied. An example application includes using strain gages to measure die forces during stamping to estimate when maintenance is needed.

Forces and Moments - Piezoelectrics

When a crystal undergoes strain it displaces a small amount of charge. In other words, when the distance between atoms in the crystal lattice changes some electrons are forced out or drawn in. This also changes the capacitance of the crystal. This is known as the Piezoelectric effect. Figure 12.21 shows the relationships for a crystal undergoing a linear deformation. The charge generated is a function of the force applied, the strain in the material, and a constant specific to the material. The change in capacitance is proportional to the change in the thickness.



$$C = \frac{\varepsilon a b}{c} \qquad i = \varepsilon g \frac{d}{dt} F$$

where,

C = *capacitance change*

a, b, c = *geometry of material*

ε = *dielectric constant (quartz typ. 4.06*10**-11 F/m)*

i = *current generated*

F = *force applied*

g = *constant for material (quartz typ. 50*10**-3 Vm/N)*

E = *Youngs modulus (quartz typ. 8.6*10**10 N/m**2)*

Figure 12.21 The Piezoelectric Effect

These crystals are used for force sensors, but they are also used for applications such as microphones and pressure sensors. Applying an electrical charge can induce strain, allowing them to be used as actuators, such as audio speakers.

When using piezoelectric sensors charge amplifiers are needed to convert the small amount of charge to a larger voltage. These sensors are best suited to dynamic measurements, when used for static measurements they tend to *drift* or slowly lose charge, and the signal value will change.

12.2 Liquids and Gases

There are a number of factors to be considered when examining liquids and gases.

- Flow velocity
- Density
- Viscosity
- Pressure

There are a number of differences factors to be considered when dealing with fluids and gases. Normally a fluid is considered incompressible, while a gas normally follows the ideal gas law. Also, given sufficiently high enough temperatures, or low enough pressures a fluid can be come a gas.

$$PV = nRT$$

where,

$$P = \text{the gas pressure}$$

$$V = \text{the volume of the gas}$$

$$n = \text{the number of moles of the gas}$$

$$R = \text{the ideal gas constant} =$$

$$T = \text{the gas temperature}$$

When flowing, the flow may be smooth, or laminar. In case of high flow rates or unrestricted flow, turbulence may result. The Reynold's number is used to determine the transition to turbulence. The equation below is for calculation the Reynold's number for fluid flow in a pipe. A value below 2000 will result in laminar flow. At a value of about 3000 the fluid flow will become uneven. At a value between 7000 and 8000 the flow will become turbulent.

$$R = \frac{VD\rho}{u}$$

where,

$$R = \text{Reynolds number}$$

$$V = \text{velocity}$$

$$D = \text{pipe diameter}$$

$$\rho = \text{fluid density}$$

$$u = \text{viscosity}$$

Pressure Gages - Mechanical

Figure 12.22 shows different two mechanisms for pressure measurement. The Bourdon tube uses a circular pressure tube. When the pressure inside is higher than the surrounding air pressure (14.7psi approx.) the tube will straighten. A position sensor, connected to the end of the tube, will be elongated when the pressure increases.

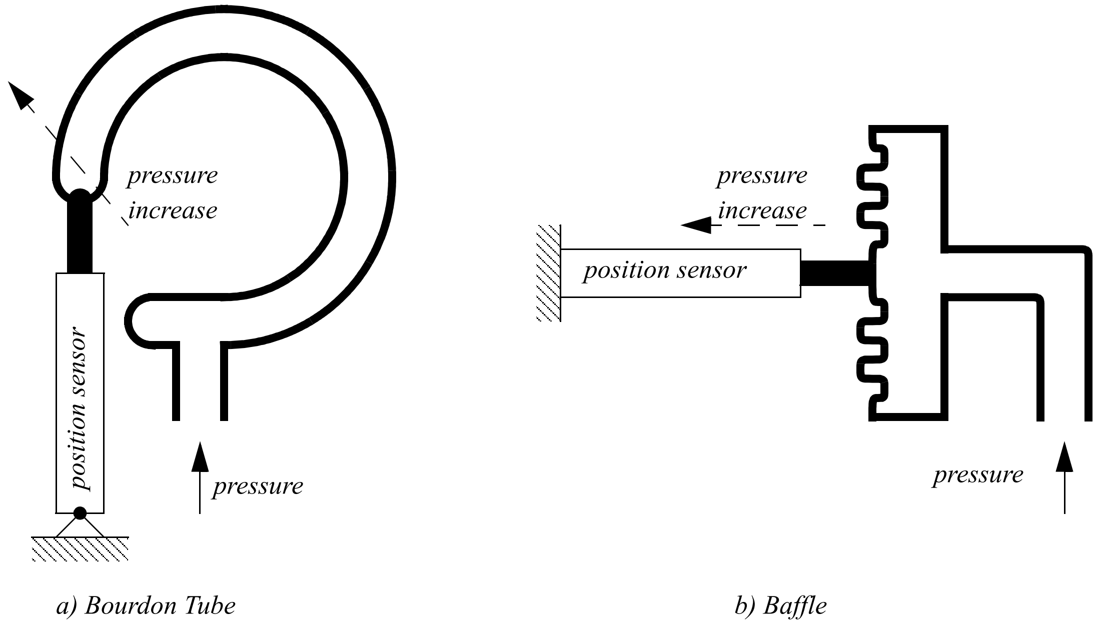


Figure 12.22 Pressure Transducers

These sensors are very common and have typical accuracies of 0.5%.

Flow Rates - Venturi Valves

When a flowing fluid or gas passes through a narrow pipe section (neck) the pressure drops. If there is no flow the pressure before and after the neck will be the same. The faster the fluid flow, the greater the pressure difference before and after the neck. This is known as a Venturi valve. Figure 12.23 shows a Venturi valve being used to measure a fluid flow rate. The fluid flow rate will be proportional to the pressure difference before and at the neck (or after the neck) of the valve.

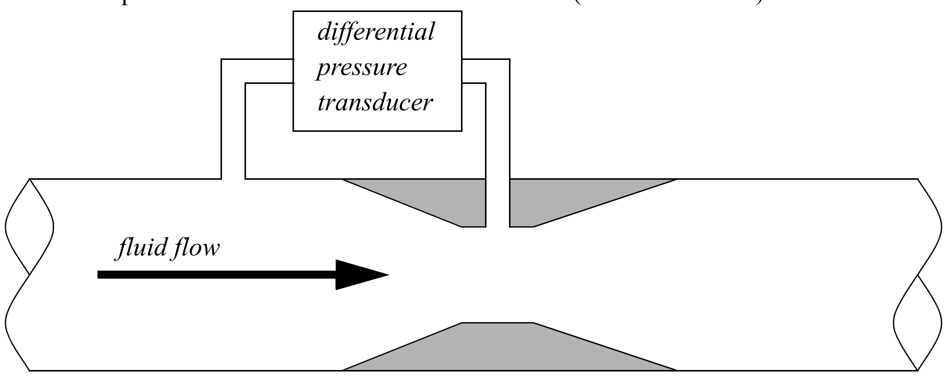


Figure 12.23 A Venturi Valve

Aside: Bernoulli's equation can be used to relate the pressure drop in a venturi valve.

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = C$$

where,

p = pressure

ρ = density

v = velocity

g = gravitational constant

z = height above a reference

C = constant

Consider the center line of the fluid flow through the valve. Assume the fluid is incompressible, so the density does not change. And, assume that the center line of the valve does not change. This gives us a simpler equation, as shown below, that relates the velocity and pressure before and after it is compressed.

$$\frac{p_{before}}{\rho} + \frac{v_{before}^2}{2} + gz = C = \frac{p_{after}}{\rho} + \frac{v_{after}^2}{2} + gz$$

$$\frac{p_{before}}{\rho} + \frac{v_{before}^2}{2} = \frac{p_{after}}{\rho} + \frac{v_{after}^2}{2}$$

$$p_{before} - p_{after} = \rho\left(\frac{v_{after}^2}{2} - \frac{v_{before}^2}{2}\right)$$

The flow velocity v in the valve will be larger than the velocity in the larger pipe section before. So, the right hand side of the expression will be positive. This will mean that the pressure before will always be higher than the pressure after, and the difference will be proportional to the velocity squared.

Figure 12.24 The Pressure Relationship for a Venturi Valve

Venturi valves allow pressures to be read without moving parts, which makes them very reliable and durable. They work well for both fluids and gases. It is also common to use Venturi valves to generate vacuums for actuators, such as suction cups.

Flow Rate - Coriolis Flow Meter

Fluid passes through thin tubes, causing them to vibrate. As the fluid approaches the point of maximum vibration it accelerates. When leaving the point it decelerates. The result is a distributed force that causes a bending moment, and hence twisting of the pipe. The amount of bending is proportional to the velocity of the fluid flow. These devices typically have a large constriction on the flow, and result is significant loses. Some of the devices also use bent tubes to increase the sensitivity, but this also increases the flow resistance. The typical accuracy for a Coriolis flowmeter is 0.1%.

Flow Rate - Magnetic Flow Meter

A magnetic sensor applies a magnetic field perpendicular to the flow of a conductive fluid. As the fluid moves, the electrons in the fluid experience an electromotive force. The result is that a potential (voltage) can be measured perpendicular to the direction of the flow and the magnetic field. The higher the flow rate, the greater the voltage. The typical accuracy for these sensors is 0.5%.

These flow meters don't oppose fluid flow, and so they don't result in pressure drops.

Flow Rate - Ultrasonic Flow Meter

A transmitter emits a high frequency sound at point on a tube. The signal must then pass through the fluid to a detector where it is picked up. If the fluid is flowing in the same direction as the sound it will arrive sooner. If the sound is against the flow it will take longer to arrive. In a transit time flow meter two sounds are used, one traveling forward, and the other in the opposite direction. The difference in travel time for the sounds is used to determine the flow velocity.

A Doppler flowmeter bounces a sound wave off particle in a flow. If the particle is moving away from the emitter and detector pair, then the detected frequency will be lowered, if it is moving towards them the frequency will be higher.

The transmitter and receiver have a minimal impact on the fluid flow, and therefore don't result in pressure drops.

Flow Rate - Vortex Flow Meter

Fluid flowing past a large (typically flat) obstacle will shed vortices. The frequency of the vortices will be proportional to the flow rate. Measuring the frequency allows an estimate of the flow rate. These sensors tend be low cost and are popular for low accuracy applications.

Quantity Measurement - Positive Displacement Meters

In some cases more precise readings of flow rates and volumes may be required. These can be obtained by using a positive displacement meter. In effect these meters are like pumps run in reverse. As the fluid is pushed through the meter it produces a measurable output, normally on a rotating shaft.

Flow Rate - Pitot Tubes

Gas flow rates can be measured using Pitot tubes, as shown in Figure 12.25. These are small tubes that project into a flow. The diameter of the tube is small (typically less than 1/8") so that it doesn't affect the flow.

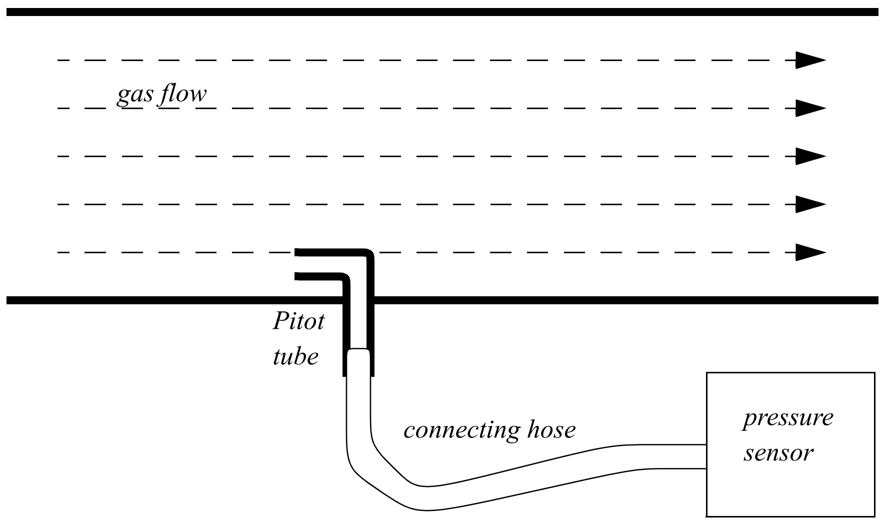


Figure 12.25 Pitot Tubes for Measuring Gas Flow Rates

12.3 Temperature

Temperature measurements are very common with control systems. The temperature ranges are normally described with the following classifications.

- Very low temperatures <-60 deg C - e.g. superconductors in MRI units
- Low temperature measurement -60 to 0 deg C - e.g. freezer controls
- Fine temperature measurements 0 to 100 deg C - e.g. environmental controls
- High temperature measurements <3000 deg F - e.g. metal refining/processing
- Very high temperatures > 2000 deg C - e.g. plasma systems

Resistive Temperature Detectors (RTDs)

When a metal wire is heated the resistance increases. So, a temperature can be measured using the resistance of a wire. Resistive Temperature Detectors (RTDs) normally use a wire or film of platinum, nickel, copper or nickel-iron alloys. The metals are wound or wrapped over an insulator, and covered for protection. The resistances of these alloys are shown in “RTD Properties” on page 433.

Material	*Temperature Range C (F)*	*Typical Resistance (ohms)*
Platinum	*-200 - 850 (-328 - 1562)*	*100*
Nickel	*-80 - 300 (-112 - 572)*	*120*
Copper	*-200 - 260 (-328 - 500)*	*10*

Figure 12.26 RTD Properties

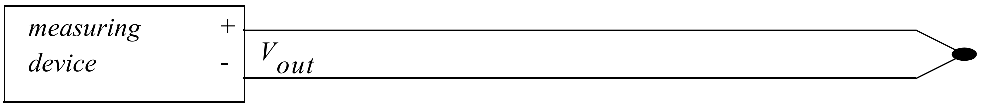
These devices have positive temperature coefficients that cause resistance to increase linearly with temperature. A platinum RTD might have a resistance of 100 ohms at 0C, that will increase by 0.4 ohms/°C. The total resistance of an RTD might double over the temperature range.

A current must be passed through the RTD to measure the resistance. (Note: a voltage divider can be used to convert the resistance to a voltage.) The current through the RTD should be kept to a minimum to prevent self heating. These devices are more linear than thermocouples, and can have accuracies of 0.05%. But, they can be expensive

Thermocouples

Each metal has a natural potential level, and when two different metals touch there is a small potential difference, a voltage. (Note: when designing assemblies, dissimilar metals should not touch, this will lead to corrosion.) Thermocouples use a junction of dissimilar metals to generate a voltage proportional to temperature. This principle was discovered by T.J. Seebeck.

The basic calculations for thermocouples are shown in Figure 12.27. This calculation provides the measured voltage using a reference temperature and a constant specific to the device. The equation can also be rearranged to provide a temperature given a voltage.



$$V_{out} = \alpha(T - T_{ref})$$

$$\backslash T = \frac{V_{out}}{\alpha} + T_{ref}$$

where,

α = *constant (V/C)* $50\frac{\mu V}{°C}$ *(typical)*

T, T_{ref} = *current and reference temperatures*

Figure 12.27 Thermocouple Calculations

The list in Table 1 shows different junction types, and the normal temperature ranges. Both thermocouples, and signal conditioners are commonly available, and relatively inexpensive. For example, most PLC vendors sell thermocouple input cards that will allow multiple inputs into the PLC.

Table 13: Thermocouple Types

ANSI Type	Materials	Temperature Range (°F)	Voltage Range (mV)
T	copper/constantan	-200 to 400	-5.60 to 17.82
J	iron/constantan	0 to 870	0 to 42.28
E	chromel/constantan	-200 to 900	-8.82 to 68.78
K	chromel/aluminum	-200 to 1250	-5.97 to 50.63
R	platinum-13%rhodium/platinum	0 to 1450	0 to 16.74
S	platinum-10%rhodium/platinum	0 to 1450	0 to 14.97
C	tungsten-5%rhenium/tungsten-26%rhenium	0 to 2760	0 to 37.07

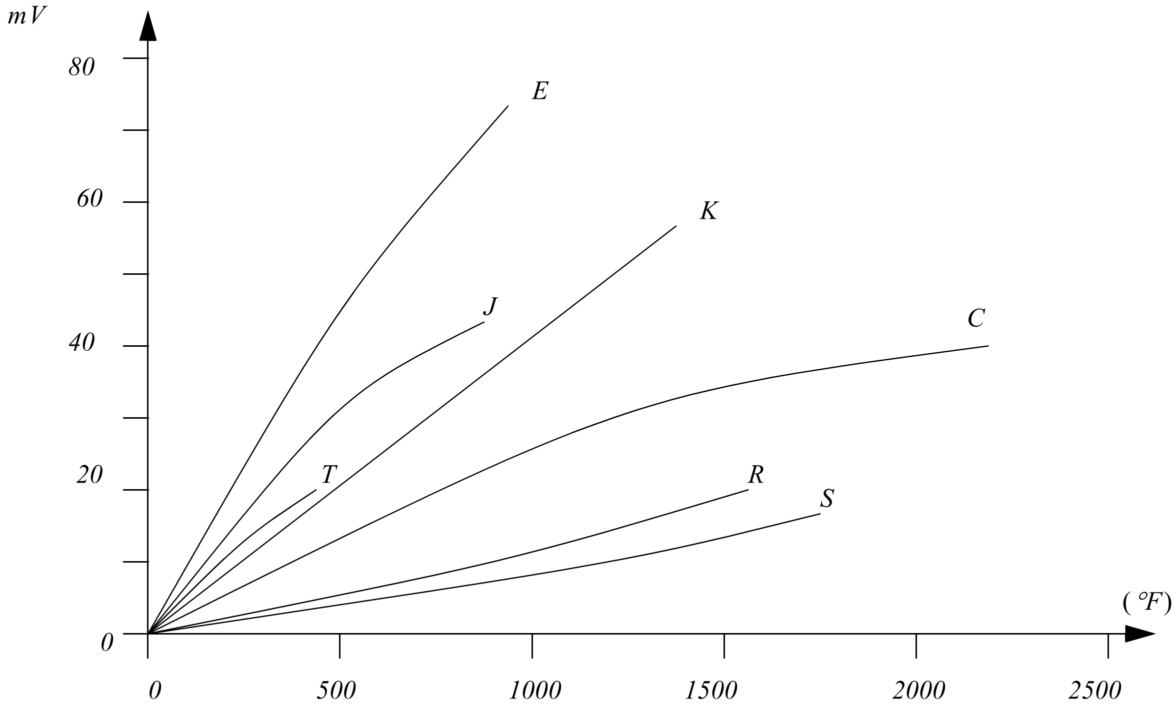


Figure 12.28 Thermocouple Temperature Voltage Relationships (Approximate)

The junction where the thermocouple is connected to the measurement instrument is normally cooled to reduce the thermocouple effects at those junctions. When using a thermocouple for precision measurement, a second thermocouple can be kept at a known temperature for reference. A series of thermocouples connected together in series produces a higher voltage and is called a thermopile. Readings can approach an accuracy of 0.5%.

Thermistors

Thermistors are non-linear devices, their resistance will decrease with an increase in temperature. (Note: this is because the extra heat reduces electron mobility in the semiconductor.) The resistance can change by more than 1000 times. The basic calculation is shown in Figure 12.29.

often metal oxide semiconductors The calculation uses a reference temperature and resistance, with a constant for the device, to predict the resistance at another temperature. The expression can be rearranged to calculate the temperature given the resistance.

$$R_t = R_o e^{\beta\left(\frac{1}{T} - \frac{1}{T_o}\right)}$$

$$\backslash T = \frac{\beta T_o}{T_o \ln\left(\frac{R_t}{R_o}\right) + \beta}$$

where,

R_o, R_t = *resistances at reference and measured temps.*

T_o, T = *reference and actual temperatures*

β = *constant for device*

Figure 12.29 Thermistor Calculations

Aside: The circuit below can be used to convert the resistance of the thermistor to a voltage using a Wheatstone bridge and an inverting amplifier.

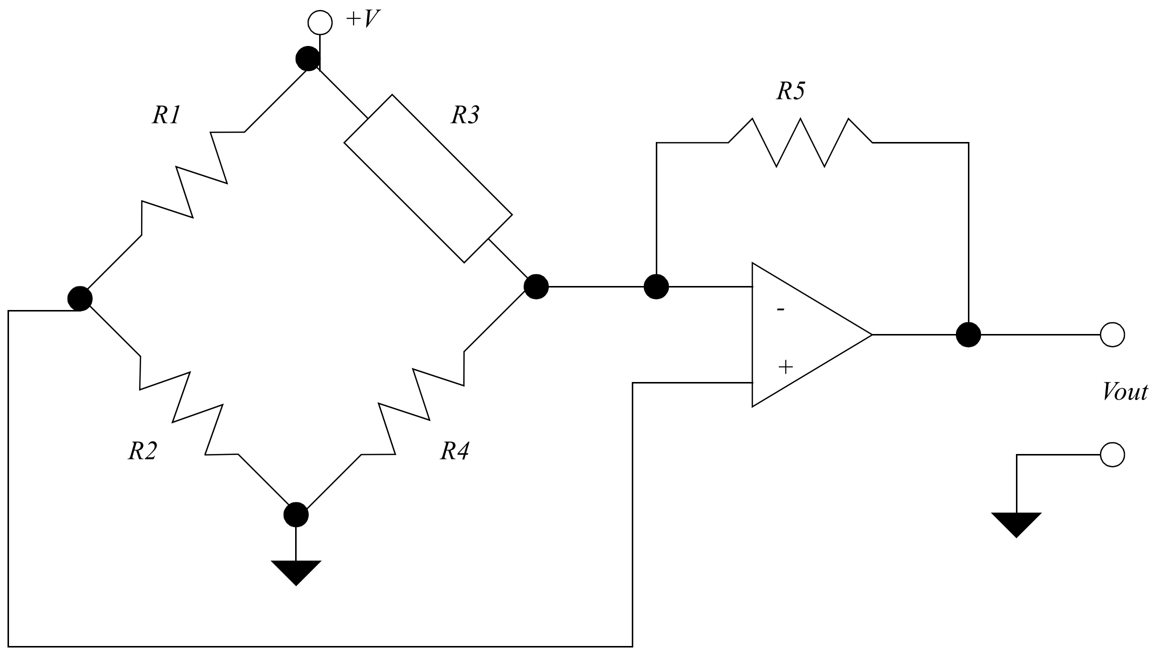


Figure 12.30 A Simple thermistor Signal Conditioning Circuit

Thermistors are small, inexpensive devices that are often made as beads, or metallized surfaces. The devices respond quickly to temperature changes, and they have a higher resistance, so junction effects are not an issue. Typical accuracies are 1%, but the devices are not linear, have a limited temperature/resistance range and can be self heating.

Other Sensors

IC sensors are becoming more popular. They output a digital reading and can have accuracies better than 0.01%. But, they have limited temperature ranges, and require some knowledge of interfacing methods for serial or parallel data.

Pyrometers are non-contact temperature measuring devices that use radiated heat. These are normally used for high temperature applications, or for production lines where it is not possible to mount other sensors to the material.

12.4 Light

Light Dependant Resistors (LDR)

Light dependant resistors (LDRs) change from high resistance (>Mohms) in bright light to low resistance (<Kohms) in the dark. The change in resistance is non-linear, and is also relatively slow (ms).

Aside: an LDR can be used in a voltage divider to convert the change in resistance to a measurable voltage.

These are common in low cost night lights.

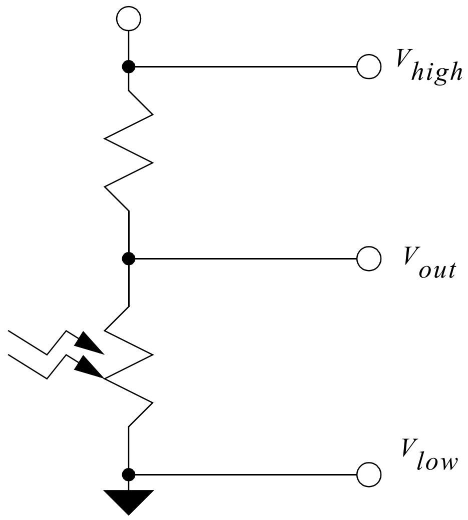


Figure 12.31 A Light Level Detector Circuit

12.5 Other Sensor Types

A number of other detectors/sensors are listed below,

- Combustion - gases such as CO2 can be an indicator of combustion
- Humidity - normally in gases
- Dew Point - to determine when condensation will form

Chemical - pH

The pH of an ionic fluid can be measured over the range from a strong base (alkaline) with pH=14, to a neutral value, pH=7, to a strong acid, pH=0. These measurements are normally made with electrodes that are in direct contact with the fluids.

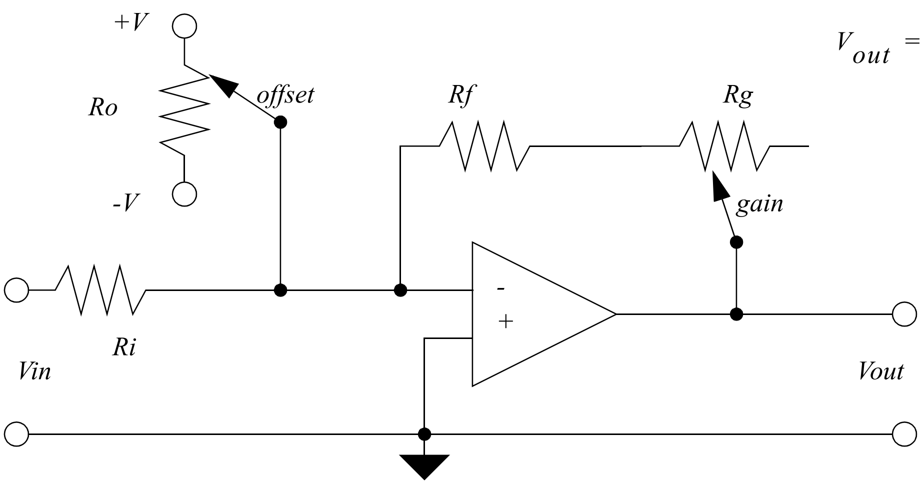
Materials - Conductivity

Conductivity of a material, often a liquid is often used to detect impurities. This can be measured directly be applying a voltage across two plates submerged in the liquid and measuring the current. High frequency inductive fields is another alternative.

12.6 Input Issues

Signals from transducers are typically too small to be read by a normal analog input card. Amplifiers are used to increase the magnitude of these signals. An example of a single ended signal amplifier is shown in Figure 12.32. The amplifier is in an inverting configuration, so the output will have an opposite sign from the input. Adjustments are provided for *gain* and *offset* adjustments.

> Note: op-amps are used in this section to implement the amplifiers because they are inexpensive, common, and well suited to simple design and construction projects. When purchasing a commercial signal conditioner, the circuitry will be more complex, and include other circuitry for other factors such as temperature compensation.



$$V_{out} = \left(\frac{R_f + R_g}{R_i}\right) V_{in} + offset$$

Figure 12.32 A Single Ended Signal Amplifier

A differential amplifier with a current input is shown in Figure 12.33. Note that Rc converts a current to a voltage. The voltage is then amplified to a larger voltage.

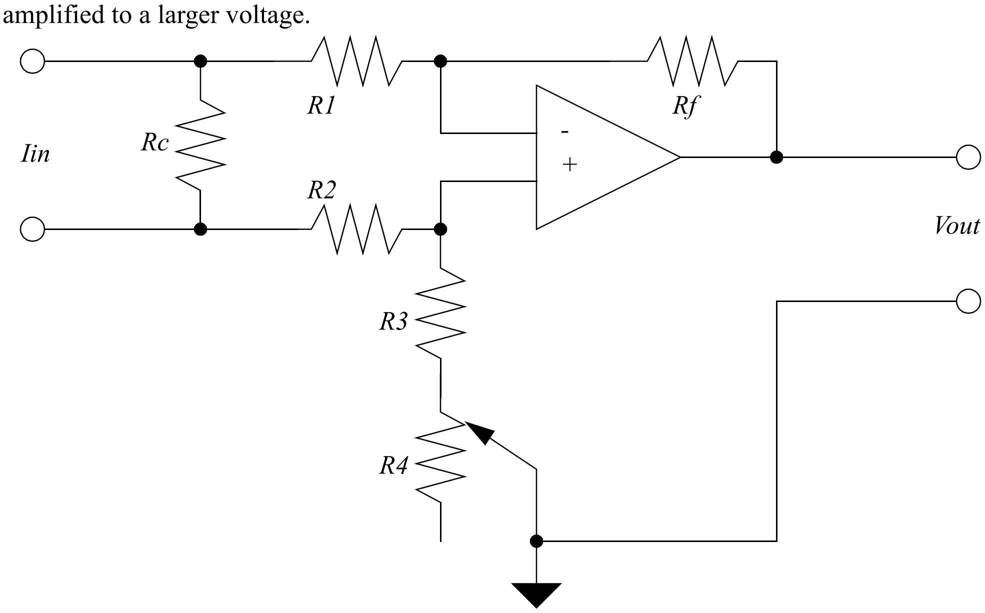


Figure 12.33 A Current Amplifier

The circuit in Figure 12.34 will convert a differential (double ended) signal to a single ended signal. The two input op-amps are used as unity gain followers, to create a high input impedance. The following amplifier amplifies the voltage difference.

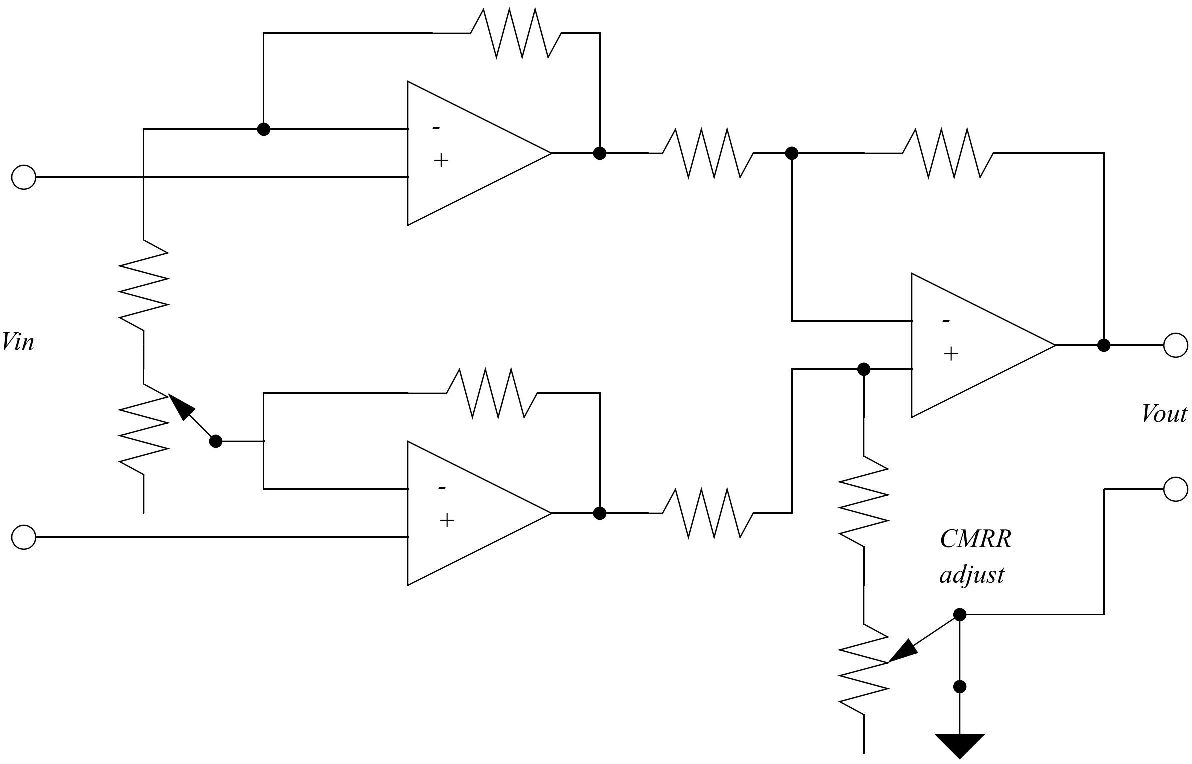
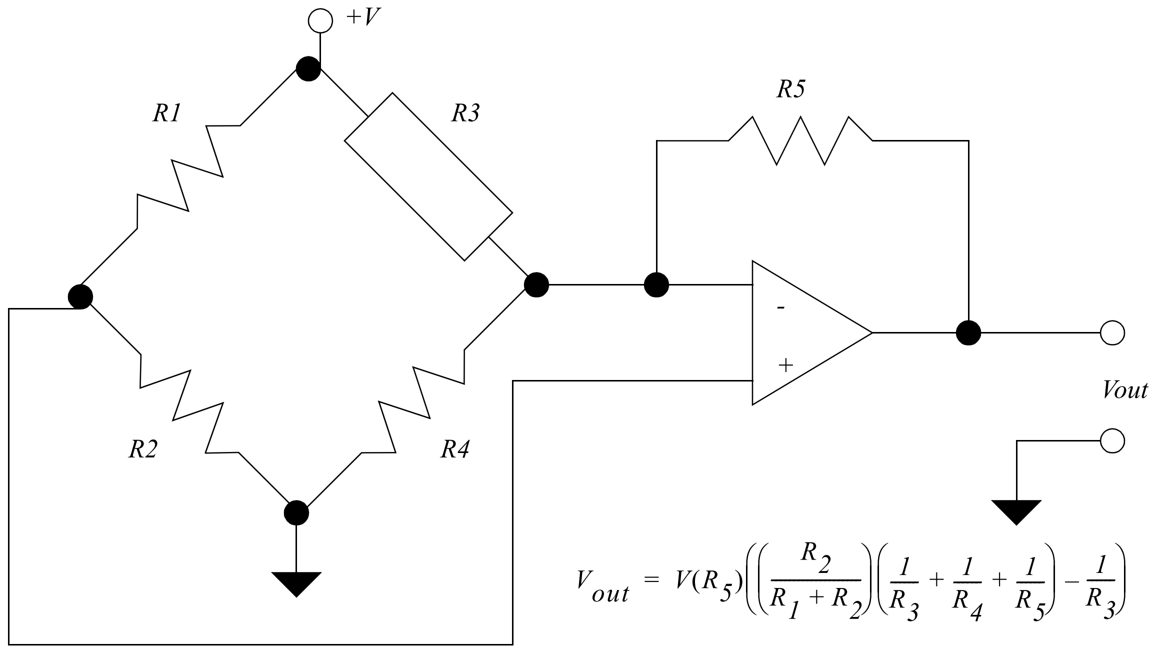


Figure 12.34 A Differential Input to Single Ended Output Amplifier

The Wheatstone bridge can be used to convert a resistance to a voltage output, as shown in Figure 12.35. If the resistor values are all made the same (and close to the value of R3) then the equation can be simplified.



$$V_{out} = V(R_5)\left(\left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right) - \frac{1}{R_3}\right)$$

or if $R = R_1 = R_2 = R_4 = R_5$

$$V_{out} = V\left(\frac{R}{2R_3}\right)$$

Figure 12.35 A Resistance to Voltage Amplifier

12.7 Sensor Glossary

- Ammeter - A meter to indicate electrical current. It is normally part of a DMM
- Bellows - This is a flexible volumed that will expand or contract with a pressure change. This often looks like a cylinder with a large radius (typically 2") but it is very thin (type 1/4"). It can be set up so that when pressure changes, the displacement of one side can be measured to determine pressure.
- Bourdon tube - Widely used industrial gage to measure pressure and vacuum. It resembles a crescent moon. When the pressure inside changes the moon shape will tend to straighten out. By measuring the displacement of the tip the pressure can be measured.
- Chromatographic instruments - laboratory-type instruments used to analyze chemical compounds and gases.
- Inductance-coil pulse generator - transducer used to measure rotational speed. Output is pulse train.
- Interferometers - These use the interference of light waves 180 degrees out of phase to determine distances. Typical sources of the monochromatic light required are lasers.
- Linear-Variable-Differential transformer (LVDT) electromechanical transducer used to measure angular or linear displacement. Output is Voltage
- Manometer - liquid column gage used widely in industry to measure pressure.
- Ohmmeter - meter to indicate electrical resistance
- Optical Pyrometer - device to measure temperature of an object at high temperatures by sensing the brightness of an objects surface.
- Orifice Plate - widely used flowmeter to indicate fluid flow rates
- Photometric Transducers - a class of transducers used to sense light, including phototubes, photodiodes, photo transistors, and photo conductors.
- Piezoelectric Accelerometer - Transducer used to measure vibration. Output is emf.
- Pitot Tube - Laboratory device used to measure flow.
- Positive displacement Flowmeter - Variety of transducers used to measure flow. Typical output is pulse train.
- Potentiometer - instrument used to measure voltage
- Pressure Transducers - A class of transducers used to measure pressure. Typical output is voltage. Operation of the transducer can be based on strain gages or other devices.
- Radiation pyrometer - device to measure temperature by sensing the thermal radiation emitted from the object.
- Resolver - this device is similar to an incremental encoder, except that it uses coils to generate magnetic fields. This is like a rotary transformer.
- Strain Gage - Widely used to indicate torque, force, pressure, and other variables. Output is change in resistance due to strain, which can be converted into voltage.
- Thermistor - Also called a resistance thermometer; an instrument used to measure temperature. Operation is based on change in resistance as a function of temperature.
- Thermocouple - widely used temperature transducer based on the Seebeck effect, in which a junction of two dissimilar metals emits emf related to temperature.
- Turbine Flowmeter - transducer to measure flow rate. Output is pulse train.
- Venturi Tube - device used to measure flow rates.

12.8 Summary

- Selection of continuous sensors must include issues such as accuracy and resolution.
- Angular positions can be measured with potentiometers and encoders (more accurate).
- Tachometers are useful for measuring angular velocity.
- Linear positions can be measured with potentiometers (limited accuracy), LVDTs (limited range), moire fringes (high accuracy).
- Accelerometers measure acceleration of masses.
- Strain gages and piezoelectric elements measure force.
- Pressure can be measured indirectly with bellows and Bourdon tubes.
- Flow rates can be measured with Venturi valves and pitot tubes.
- Temperatures can be measured with RTDs, thermocouples, and thermistors.
- Input signals can be single ended for more inputs or double ended for more accuracy.

12.9 References

12.1 Bryan, L.A. and Bryan, E.A., Programmable Controllers; Theory and Implementation, Industrial Text Co., 1988.
12.2 Swainston, F., A Systems Approach to Programmable Controllers, Delmar Publishers Inc., 1992.

12.10 Problems With Solutions

Problem 12.1 A potentiometer is to be used to measure the position of a rotating robot link (as a voltage divider). The power supply connected across the potentiometer is 5.0 V, and the total wiper travel is 300 degrees. The wiper arm is directly connected to the rotational joint so that a given rotation of the joint corresponds to an equal rotation of the wiper arm.

a) If the joint is at 42 degrees, what voltage will be output from the potentiometer?
b) If the joint has been moved, and the potentiometer output is 2.765V, what is the position of the potentiometer?

Problem 12.2 What is the resolution of an absolute optical encoder that has six binary tracks? nine tracks? twelve tracks?

Problem 12.3 If a thermocouple generates a voltage of 30mV at 800F and 40mV at 1000F, what voltage will be generated at 1200F?

Problem 12.4 Name two types of inputs that would be analog input values (versus a digital value).

Problem 12.5 Search the web for common sensor manufacturers for 5 different types of continuous sensors. If possible identify prices for the units. Sensor manufacturers include (Hyde Park, Banner, Allen Bradley, Omron, etc.)

Problem 12.6 Suggest a couple of methods for collecting data on the factory floor.

Problem 12.7 A motor has an encoder mounted on it. The motor is driving a reducing gear box with a 50:1 ratio. If the position of the geared down shaft needs to be positioned to 0.1 degrees, what is the minimum resolution of the incremental encoder?

Problem 12.8 What is the difference between a strain gauge and an accelerometer? How do they work?

Problem 12.9 Use the equations for a permanent magnet DC motor to explain how it can be used as a tachometer.

Problem 12.10 What are the trade-offs between encoders and potentiometers?

Problem 12.11 The table of position and voltage values below were measured for an inexpensive potentiometer. Write a C subroutine that will accept a voltage value and interpolate the position value.

theta (deg)	V
0	0.1
67	0.6
145	1.6
195	2.4
213	3.4
296	4.2
315	5.0

12.11 Problem Solutions

Answer 12.1

a) $$V_{out} = (V_2 - V_1)\left(\frac{\theta_w}{\theta_{max}}\right) + V_1 = (5V - 0V)\left(\frac{42deg}{300deg}\right) + 0V = 0.7V$$

b) $$2.765V = (5V - 0V)\left(\frac{\theta_w}{300deg}\right) + 0V$$

$$2.765V = (5V - 0V)\left(\frac{\theta_w}{300deg}\right) + 0V$$

$$\theta_w = 165.9deg$$

Answer 12.2 360°/64steps, 360°/512steps, 360°/4096steps.

Answer 12.3

$$V_{out} = \alpha(T - T_{ref}) \qquad 0.030 = \alpha(800 - T_{ref}) \qquad 0.040 = \alpha(1000 - T_{ref})$$

$$\frac{1}{\alpha} = \frac{800 - T_{ref}}{0.030} = \frac{1000 - T_{ref}}{0.040}$$

$$800 - T_{ref} = 750 - 0.75T_{ref}$$

$$50 = 0.25T_{ref} \qquad T_{ref} = 200F \qquad \alpha = \frac{0.040}{1000 - 200} = \frac{50\mu V}{F}$$

$$V_{out} = 0.00005(1200 - 200) = 0.050V$$

Answer 12.4 Temperature and displacement.

Answer 12.5 Sensors can be found at www.ab.com, www.omron.com, etc.

Answer 12.6 Examples include ‘data buckets’, smart machines, PLCs with analog inputs, and network connections.

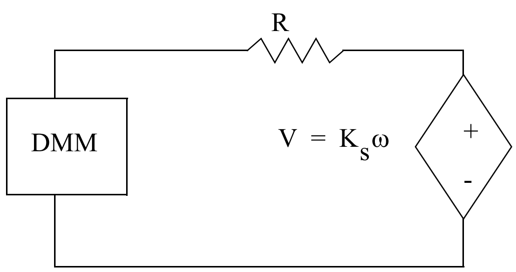
Answer 12.7

$$\theta_{output} = 0.1\frac{deg}{count} \qquad \frac{\theta_{input}}{\theta_{output}} = \frac{50}{1} \qquad \theta_{input} = 50\left(0.1\frac{deg}{count}\right) = 5\frac{deg}{count}$$

$$R = \frac{360\frac{deg}{rot}}{5\frac{deg}{count}} = 72\frac{count}{rot}$$

Answer 12.8 Strain gauge measures strain in a material using a stretching wire that increases resistance - accelerometers measure acceleration with a cantilevered mass on a piezoelectric element.

Answer 12.9 When the motor shaft is turned by another torque source a voltage is generated that is proportional to the angular velocity. This is the reverse emf. A DMM, or other high impedance instrument can be used to measure this, thus minimizing the loses in resistor R.



$$\dot{\omega} + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right)$$

$$V_s = \omega(K) + \dot{\omega}\left(\frac{JR}{K}\right)$$

Answer 12.10 Encoders cost more but can have higher resolutions. Potentiometers have limited ranges of motion.

Answer 12.11

```
#define         SIZE    7;
double          data[SIZE][2] = {{0.0, 0.1},
                               {67.0,  0.6},
                               {145.0, 1.6},
                               {195.0, 2.4},
                               {213.0, 3.4},
                               {296.0, 4.2},
                               {315.0, 5.0}};

double theta(double V){
        int i;
        for(i = 0; i < SIZE-1; i++){
                if((V >= data[i][0]) && (V <= data[i+1][0]){
                        return  (data[i+1][1] - data[i][1])
                                * (V - data[i][0]) / (data[i+1][0]
                                - data[i][0]) + data[i][1];
                }
        }
        printf("ERROR: rate out of range\n");
        exit(1);
}
```

12.12 Problems Without Solutions

Problem 12.12 Write a simple C program to read incremental encoder inputs (A and B) to determine the current position of the encoder. Note: use the quadrature encoding to determine the position of the motor.

Problem 12.13 A high precision potentiometer has an accuracy of +/- 0.1% and can rotate 300degrees and is used as a voltage divider with a of 0V and 5V. The output voltage is being read by an A/D converter with a 0V to 10V input range. How many bits does the A/D converter need to accommodate the accuracy of the potentiometer?

Problem 12.14 A potentiometer is connected to a microcontroller analog input. The potentiometer can rotate 300 degrees, and the voltage supply for the potentiometer is +/-10V. Write a program to read the voltage from the potentiometer and convert it to an angle in radians stored in ‘angle’.

13. Continuous Actuators

Topic 13.1 *Servo motors; AC and DC*
Topic 13.2 *Stepper motors.*
Topic 13.3 *Single axis motion control.*
Topic 13.4 *Hydraulic actuators.*
Objective 13.1 *To understand the main differences between continuous actuators.*
Objective 13.2 *Be able to select a continuous actuator.*
Objective 13.3 *To be able to plan a motion for a single servo actuator.*

Continuous actuators allow a system to position or adjust outputs over a wide range of values. Even in their simplest form, continuous actuators tend to be mechanically complex devices. For example, a linear slide system might be composed of a motor with an electronic controller driving a mechanical slide with a ball screw. The cost for such actuators can easily be higher than for the control system itself. These actuators also require sophisticated control techniques that will be discussed in later chapters. In general, when there is a choice, it is better to use discrete actuators to reduce costs and complexity.

13.1 Electric Motors

An electric motor is composed of a rotating center, called the rotor, and a stationary outside, called the stator. These motors use the attraction and repulsion of magnetic fields to induce forces, and hence motion. Typical electric motors use at least one electromagnetic coil, and sometimes permanent magnets to set up opposing fields. When a voltage is applied to these coils the result is a torque and rotation of an output shaft. There are a variety of motor configuration the yields motors suitable for different applications. Most notably, as the voltages supplied to the motors will vary the speeds and torques that they will provide.

- Motor Categories
    - AC motors - rotate with relatively constant speeds proportional to the frequency of the supply power.
        - induction motors - squirrel cage, wound rotor - inexpensive, efficient.
        - synchronous - fixed speed, efficient.
    - DC motors - have large torque and speed ranges.
        - permanent magnet - variable speed.
        - wound rotor and stator - series, shunt and compound (universal).
    - Hybrid.
        - brushless permanent magnet - higher cost but high torques over a wide speed range.
        - stepper motors - Very low speed and torque, medium cost, stable position control.
- Contactors are used to switch motor power on/off.
- Drives can be used to vary motor speeds electrically. This can also be done with mechanical or hydraulic machines.
- Popular drive categories.
    - Variable Frequency Drives (VFD) - vary the frequency of the power delivered to the motor to vary speed.
    - DC motor controllers - variable voltage or current to vary the motor speed.
    - Eddy Current Clutches for AC motors - low efficiency, uses a moving iron drum and windings.
    - Wound rotor AC motor controllers - low efficiency, uses variable resistors to adjust the winding currents.

A control system is required when a motor is used for an application that requires continuous position or velocity. A typical controller is shown in Figure 13.1. In any controlled system a command generator is required to specify a desired position. The controller will compare the feedback from the encoder to the desired position or velocity to determine the system error. The controller will then generate an output, based on the system error. The output is then passed through a power amplifier, which in turn drives the motor. The encoder is connected directly to the motor shaft to provide feedback of position.

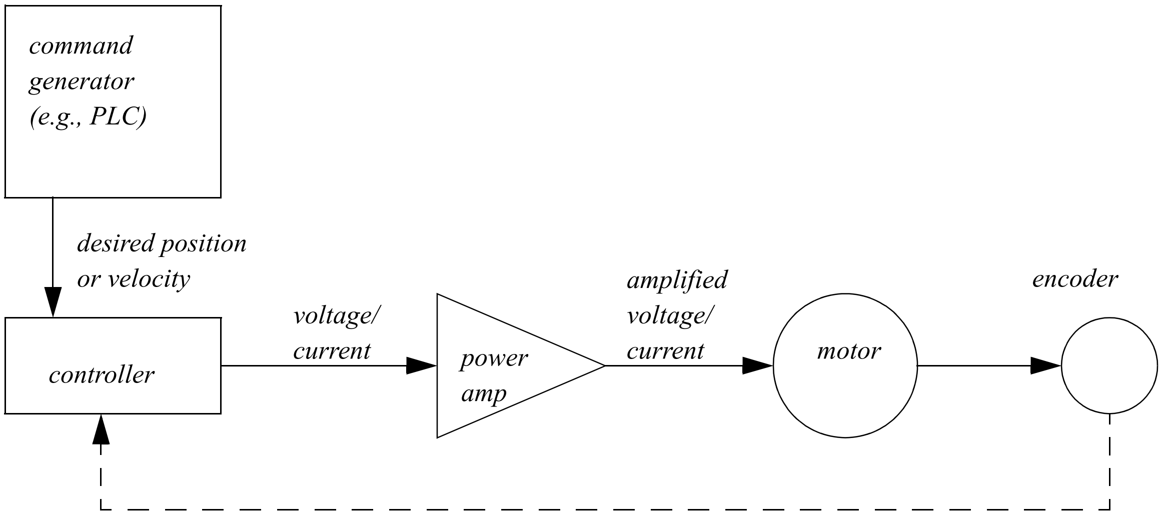


Figure 13.1 A Typical Feedback Motor Controller

Basic Brushed DC Motors

In a DC motor there is normally a set of coils on the rotor that turn inside a stator populated with permanent magnets. Figure 13.2 shows a simplified model of a motor. The magnets provide a permanent magnetic field for the rotor to push against. When current is run through the wire loop it creates a magnetic field.

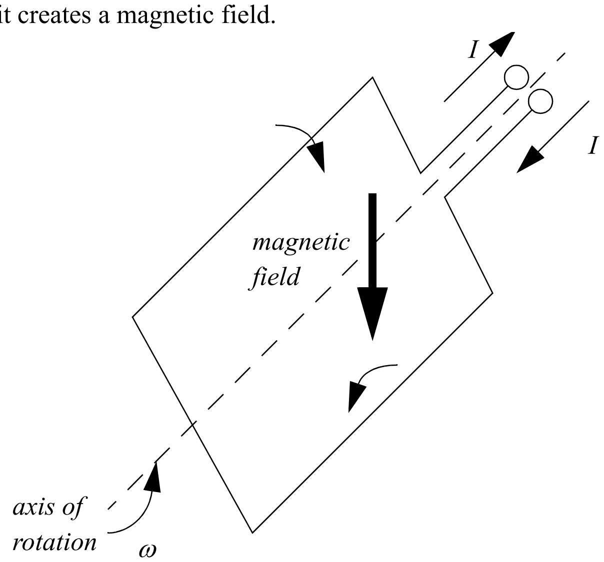


Figure 13.2 A Simplified Rotor

The power is delivered to the rotor using a commutator and brushes, as shown in Figure 13.3. In the figure the power is supplied to the rotor through graphite brushes rubbing against the commutator. The commutator is split so that every half revolution the polarity of the voltage on the rotor, and the induced magnetic field reverses to push against the permanent magnets.

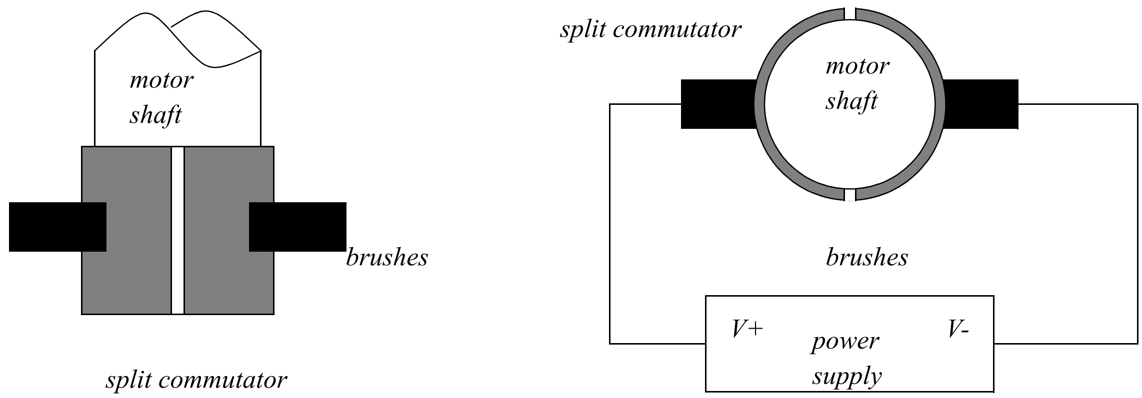


Figure 13.3 A Split Ring Commutator

The direction of rotation will be determined by the polarity of the applied voltage, and the speed is proportional to the voltage. A feedback controller is used with these motors to provide motor positioning and velocity control.

These motors are losing popularity to brushless motors. The brushes are subject to wear, which increases maintenance costs. In addition, the use of brushes increases resistance, and lowers the motors efficiency.

ASIDE: The controller to drive a servo motor normally uses a Pulse Width Modulated (PWM) signal. As shown below the signal produces an effective voltage that is relative to the time that the signal is on. The percentage of time that the signal is on is called the duty cycle. When the voltage is on all the time the effective voltage delivered is the maximum voltage. So, if the voltage is only on half the time, the effective voltage is half the maximum voltage. This method is popular because it can produce a variable effective voltage efficiently. The frequency of these waves is normally above 20KHz, above the range of human hearing.

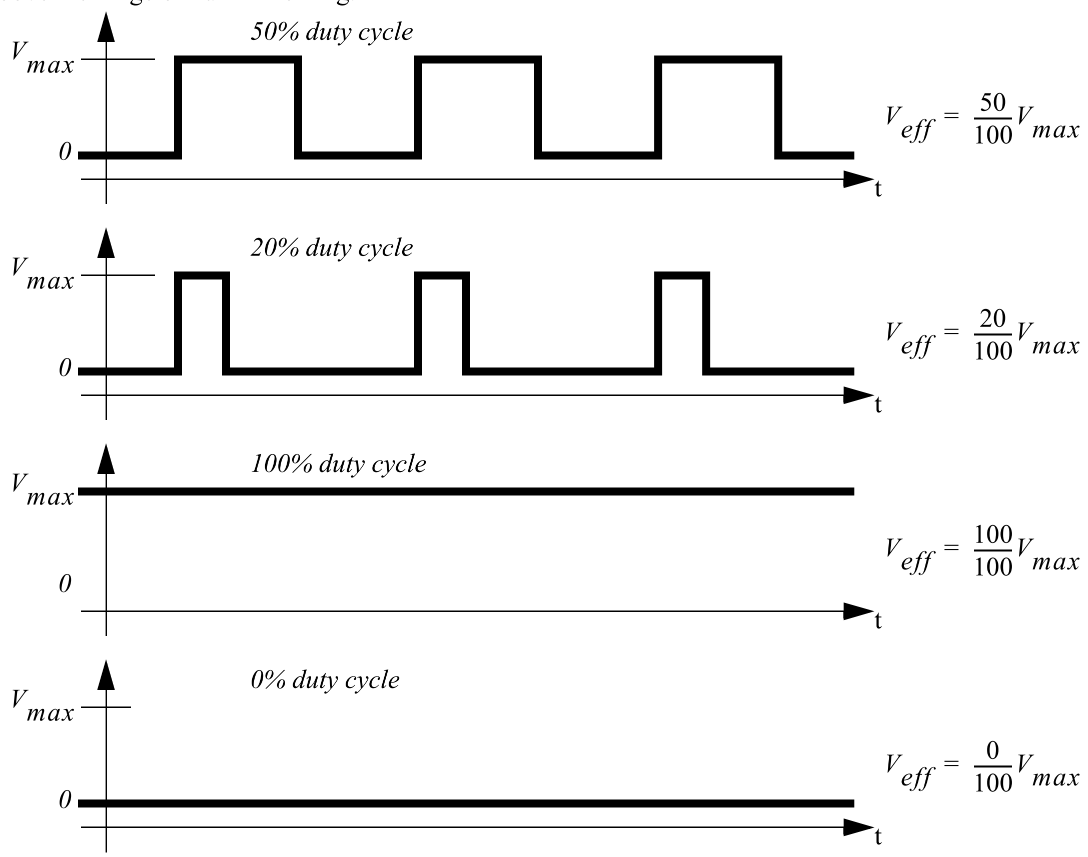


Figure 13.4 Pulse Width Modulation (PWM) For Control

ASIDE: A PWM signal can be used to drive a motor with the circuit shown below. The PWM signal switches the NPN transistor, thus switching power to the motor. In this case the voltage polarity on the motor will always be the same direction, so the motor may only turn in one direction.

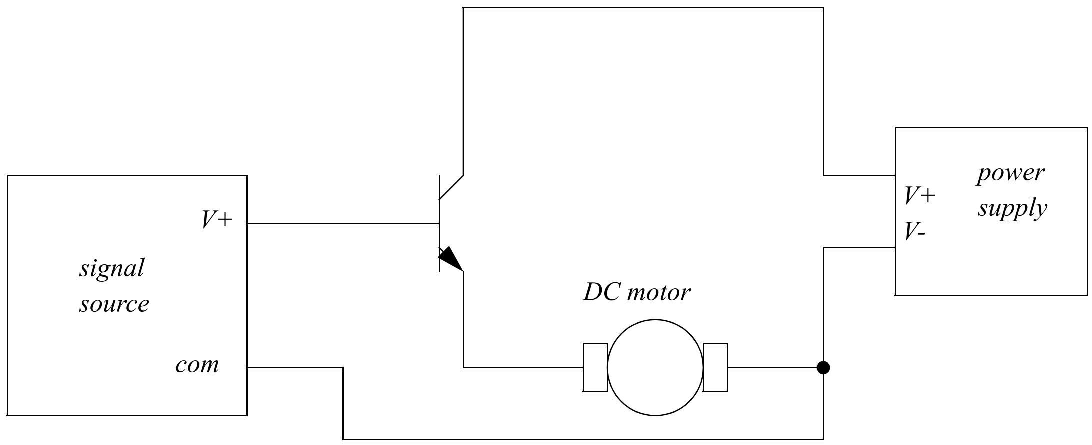


Figure 13.5 PWM Unidirectional Motor Control Circuit

Aside: When a motor is to be controlled with PWM in two directions the H-bridge circuit (shown below) is a popular choice. These can be built with individual components, or purchased as integrated circuits for smaller motors. To turn the motor in one direction the PWM signal is applied to the Va inputs, while the Vb inputs are held low. In this arrangement the positive voltage is at the left side of the motor. To reverse the direction the PWM signal is applied to the Vb inputs, while the Va inputs are held low. This applies the positive voltage to the right side of the motor.

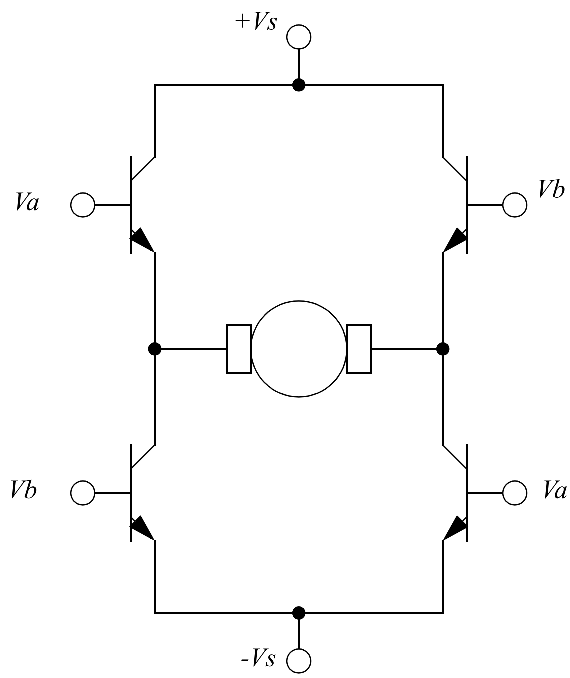


Figure 13.6 PWM Bidirectional Motor Control Circuit

AC Induction Motors

Induction motors are relatively low cost and high efficiency when used around a rated speed. These motors do not use magnets or brushes, thus lowering the construction costs. The only wear problems on the motor occur on the bearings, providing for a long running life. The key to these motors is the construction, an induction motor has the windings on the stator, Figure 13.7. The rotor is normally a squirrel cage design. The squirrel cage is a cast aluminum core that when exposed to a changing magnetic field will set up an opposing field. When an AC voltage is applied to the stator coils an AC magnetic field is created, the squirrel cage sets up an opposing magnetic field and the resulting torque causes the motor to turn.

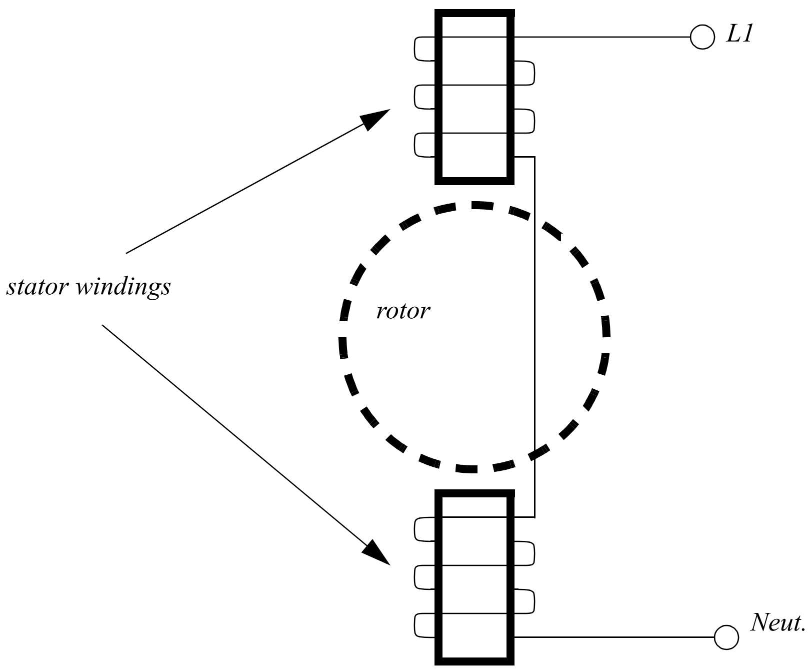


Figure 13.7 A 2 Pole Single Phase AC Motor

In a three phase AC motor there are six poles. The AC current and voltage on each phases is 120 degrees out of phase. Over each cycle of the current, from positive to negative to positive, the magnetic field will reverse on each of the poles. Figure 13.8 shows a three phases motor arrangement. The sum of the three magnetic fields would appear to be like one large magnetic field rotating once for every cycle of the current. It is possible to build motors with additional poles to slow the overall speed.

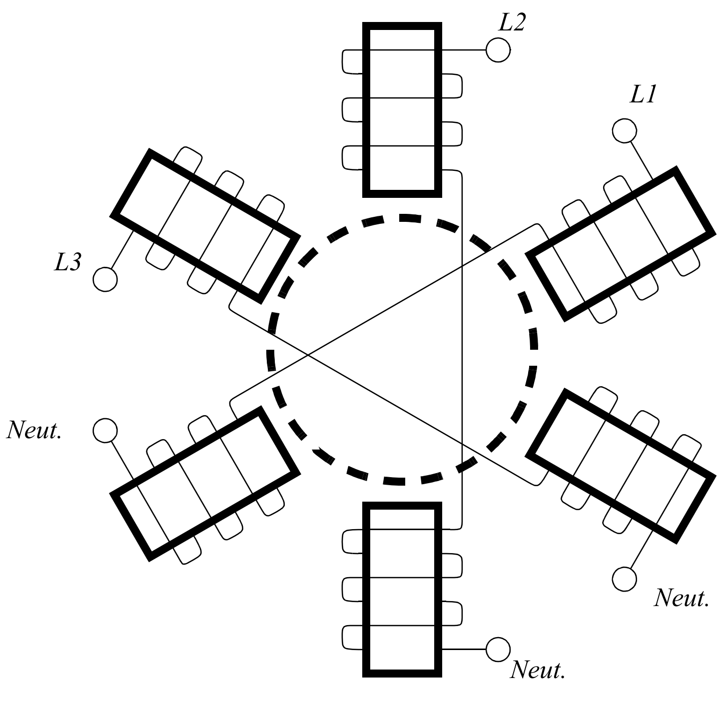


Figure 13.8 A 6 Pole 3-Phase AC Motor

There are many possible variations on the basic induction motor design. The number of windings (poles) can be an integer multiple of the number of phases of power. More poles results in a lower rotational speed of the motor. Alternate rotor types for induction motors are listed below. Regardless of the construction the function of the rotor is to intersect changing magnetic fields from the stator. The changing field induces currents in the rotor. These currents in turn set up magnetic fields that oppose fields from the stator, generating a torque.

- Squirrel cage - The conductors on the rotor have the shape of a wheel with end caps and bars.
- Wound Rotor - The rotor has coils wound. These are sometimes connected to external contacts with a commutator.

Induction motors require ‘slip’. If the motor turns at the precise speed of the stator field, it will not see a changing magnetic field. The result would be a collapse of the rotor magnetic field. As a result an induction motor always turns slightly slower than the stator field. The difference is called the slip. This is typically a few percent. As the motor is loaded the slip will increase until the motor stalls.

In general these motors should be used for applications that only require a single rotational direction. The torque speed curve for a typical induction motor is shown in Figure 13.9. When the motor is used with a fixed frequency AC source the synchronous speed of the motor will be the frequency of AC voltage divided by the number of poles in the motor. The motor actually has the maximum torque below the synchronous speed. For example a 2 pole motor might have a synchronous speed of (2*60*60/2) 3600 RPM, but be rated for 3520 RPM. When a feedback controller is used the issue of slip becomes insignificant.

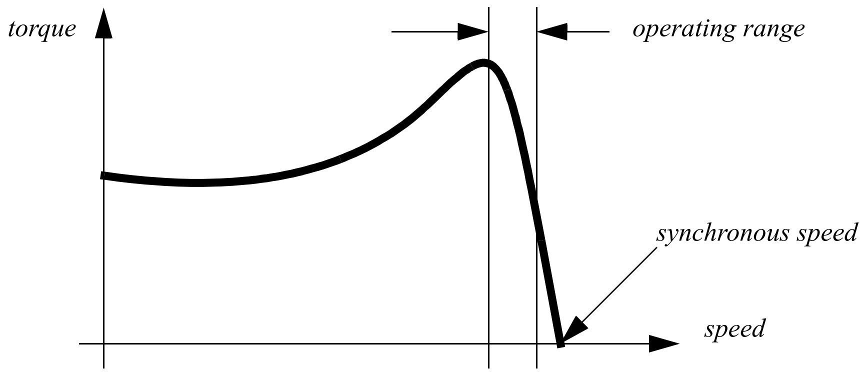


Figure 13.9 Torque Speed Curve for an Induction Motor

The torque-speed curve of an induction will vary, based on the construction. NEMA defines five basic induction motor designs designated by a letter A to E. These different constructions results in different costs and efficiencies, Figure 13.10. Regardless of the motor type the torque will become zero at the maximum speed. Past that the motor will act as a brake.

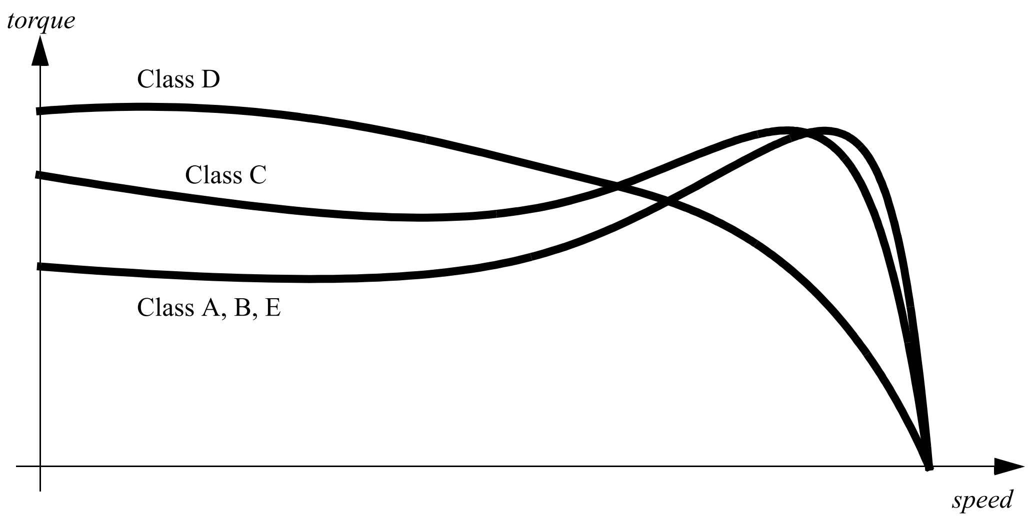


Figure 13.10 NEMA Squirrel Cage Torque Speed Curves

The rotational speed of an induction motor is a function of the AC power frequency, Figure 13.11. For example, a motor powered with 60Hz AC could rotate as fast as 60Hz or 3600RPM. Adding poles divides this number. The simplest motor is 2 pole and will only have one north and south pole at anytime. A four pole motor adds a second, duplicated but shifted, stator winding set, but runs under 1800RPM. The slip is the difference between the ideal and actual motor speed. In an ideal motor with no loses or load the slip is 0, practically the slip is normally around 5%. For example a common speed rating for an AC motor is 1720RPM. The rated speed is the steady state speed for the motor with the rated torque.

$$RPM = \frac{f120}{p}\left(1 - \frac{S}{100\%}\right)$$

where,

f = power frequency (60Hz typ.)

p = number of poles (2, 4, 6, etc...)

RPM = motor speed in rotations per minute

S = motor slip

Figure 13.11 Induction Motor Speed Calculation

Given that the speed of the motor is a function of the frequency, the speed of the motor is changed using the frequency, not the voltage. Variable frequency motor drives control the speed of the motors by synthesizing a variable frequency AC waveform, as shown in Figure 13.12. Variable Frequency Drives (VFDs) are very common for large motors in the Horsepower, or KW, power ranges.

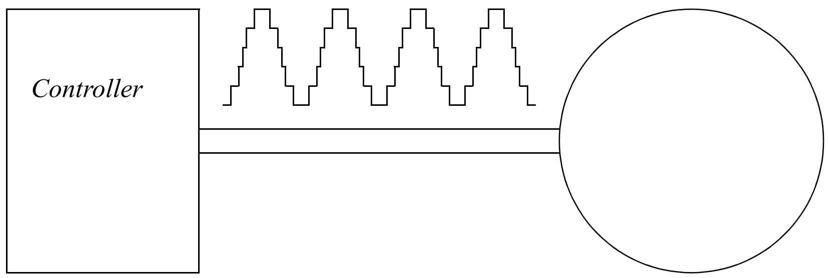


Figure 13.12 AC Motor Speed Control

- Single phase induction motors are typically used for loads under 1HP. Various types (based upon their starting and running modes) are,
  - Split phase - there are two windings on the motor. A starting winding is used to provide torque at lower speeds.
  - Capacitor run -
  - Capacitor start -
  - Capacitor start and run -
  - Shaded pole - these motors use a small offset coil (such as a single copper winding) to encourage the field buildup to occur asymmetrically. These motors are for low torque applications much less than 1HP.
  - Universal motors (also used with DC) have a wound rotor and stator that are connected in series.

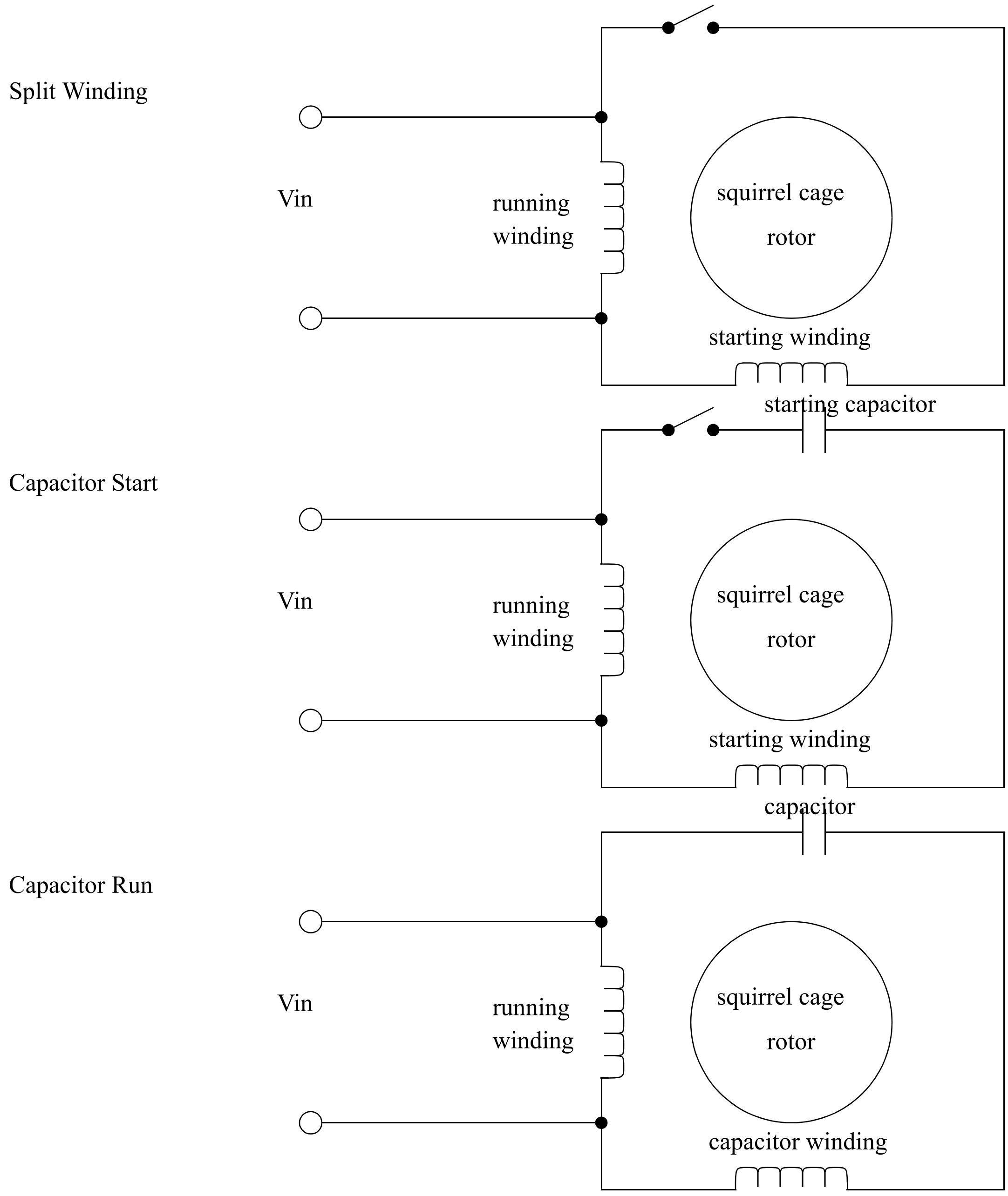


Figure 13.13 Single Phase Motor Configurations

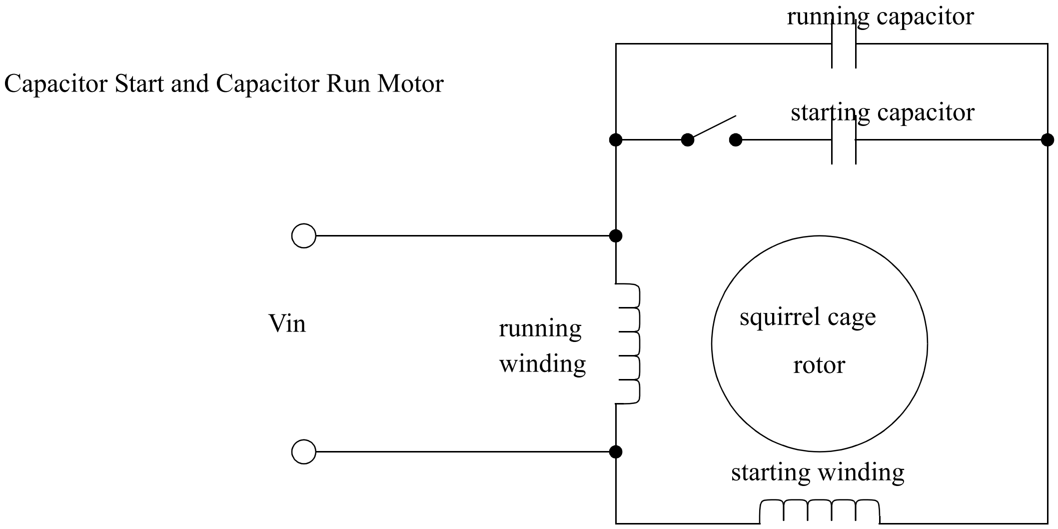


Figure 13.14 Single Phase Motor Configurations

Other items of note when designing with AC motors include:

- Single phase AC motors can run in either direction. To compensate for this a shading pole is used on the stator windings. It basically acts as an inductor to one side of the field which slows the field buildup and collapse. The result is that the field strength seems to naturally rotate.
- Thermal protection is normally used in motors to prevent overheating.
- Starting AC motors can be hard because of the low torque at low speeds. To deal with this a switching arrangement is often used. At low speeds other coils or capacitors are connected into the circuits. At higher speeds centrifugal switches disconnect these and the motor behavior switches.

Brushless DC Motors

Brushless motors use a permanent magnet on the rotor, and use windings on the stator. Therefore there is no need to use brushes and a commutator to switch the polarity of the voltage on the coil. The lack of brushes means that these motors require less maintenance than the brushed DC motors.

A typical Brushless DC motor could have three poles, each corresponding to one power input, as shown in Figure 13.15.

Each of coils is separately controlled. The coils are switched on to attract or repel the permanent magnet rotor.

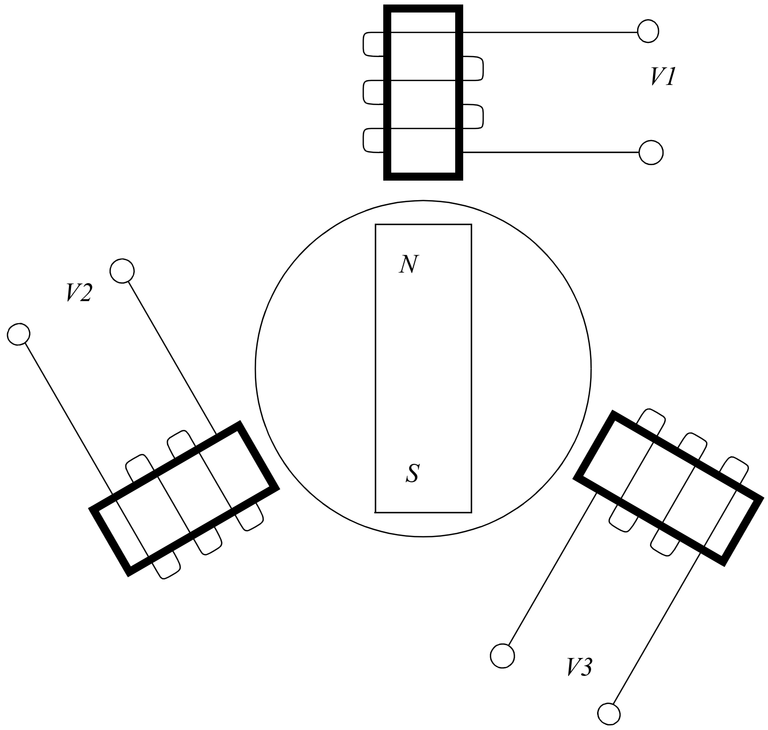


Figure 13.15 A Brushless DC Motor

To continuously rotate these motors the current in the stator coils must alternate continuously. If the power supplied to the coils was a 3-phase AC sinusoidal waveform, the motor will rotate continuously. The applied voltage can also be trapezoidal, which will give a similar effect. The changing waveforms are controller using position feedback from the motor to select switching times. The speed of the motor is proportional to the frequency of the signal.

A typical torque speed curve for a brushless motor is shown in Figure 13.16.

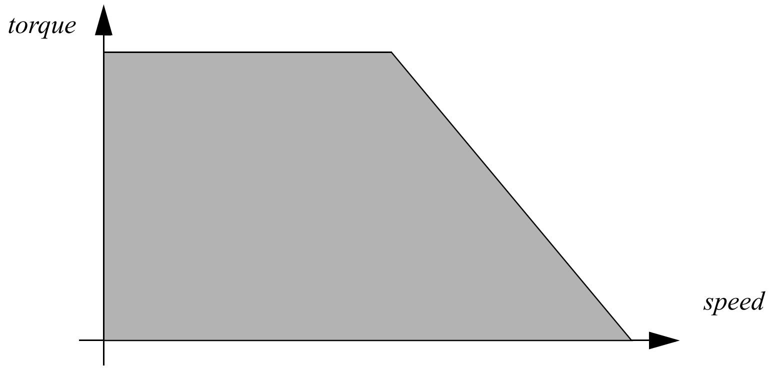


Figure 13.16 Torque Speed Curve for a Brushless DC Motor

Stepper Motors

Stepper motors are designed for positioning. They move one step at a time with a typical step size of 1.8 degrees giving 200 steps per revolution. Other motors are designed for step sizes of 1.8, 2.0, 2.5, 5, 15 and 30 degrees.

There are two basic types of stepper motors, unipolar and bipolar, as shown in Figure 13.17. The unipolar uses center tapped windings and can use a single power supply. The bipolar motor is simpler but requires a positive and negative supply and more complex switching circuitry.

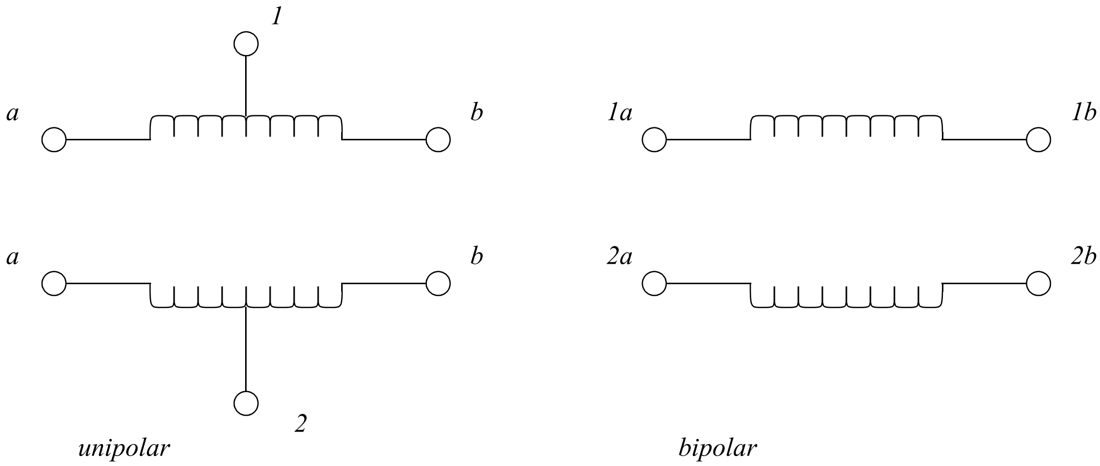
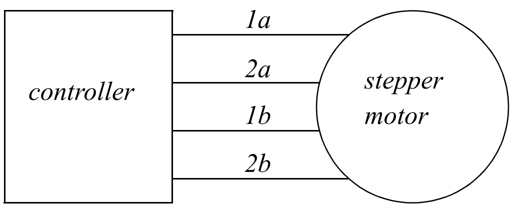


Figure 13.17 Unipolar and Bipolar Stepper Motor Windings

The motors are turned by applying different voltages at the motor terminals. The voltage change patterns for a unipolar motor are shown in Figure 13.18. For example, when the motor is turned on we might apply the voltages as shown in line 1. To rotate the motor we would then output the voltages on line 2, then 3, then 4, then 1, etc. Reversing the sequence causes the motor to turn in the opposite direction. The dynamics of the motor and load limit the maximum speed of switching, this is normally a few thousand steps per second. When not turning the output voltages are held to keep the motor in position.



Step	1a	2a	1b	2b
1	1	0	1	0
2	0	1	1	0
3	0	1	0	1
4	1	0	0	1

To turn the motor the phases are stepped through 1, 2, 3, 4, and then back to 1. To reverse the direction of the motor the sequence of steps can be reversed, eg. 4, 3, 2, 1, 4, ... If a set of outputs is kept on constantly the motor will be held in position.

Figure 13.18 Stepper Motor Control Sequence for a Unipolar Motor

Stepper motors do not require feedback except when used in high reliability applications and when the dynamic conditions could lead to slip. A stepper motor slips when the holding torque is overcome, or it is accelerated too fast. When the motor slips it will move a number of degrees from the current position. The slip cannot be detected without position feedback.

Stepper motors are relatively weak compared to other motor types. The torque speed curve for the motors is shown in Figure 13.19. In addition they have different static and dynamic holding torques. These motors are also prone to resonant conditions because of the stepped motion control.

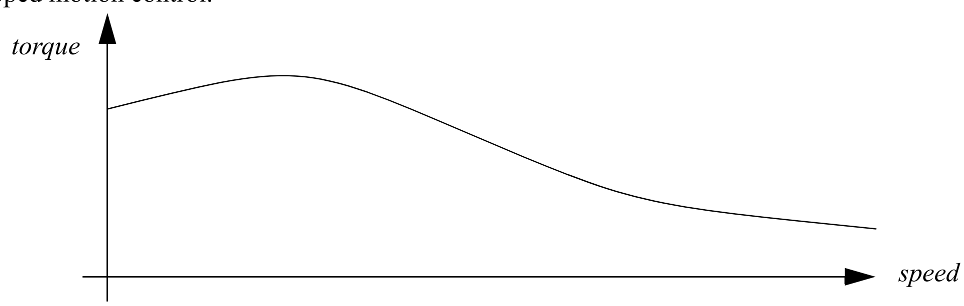


Figure 13.19 Stepper Motor Torque Speed Curve

The motors are used with controllers that perform many of the basic control functions. At the minimum a *translator* controller will take care of switching the coil voltages. A more sophisticated *indexing* controller will accept motion parameters, such as distance, and convert them to individual steps. Other types of controllers also provide finer step resolutions with a process known as *microstepping*. This effectively divides the logical steps described in Figure 13.18 and converts them to sinusoidal steps.

- Translators - the user indicates maximum velocity and acceleration and a distance to move
- Indexer - the user indicates direction and number of steps to take
- Microstepping - each step is subdivided into smaller steps to give more resolution

Wound Field Motors

Basic induction motors do not have a commutator, however it is possible to use a commutator to connect the wound rotor to outside circuitry. (Note: wound rotors are used instead of a squirrel cage.) Wound rotor induction motors use varying external resistances to change the motor torque speed curve. As the resistance value is increased the motor torque speed curve shifts from the Class A to Class D.

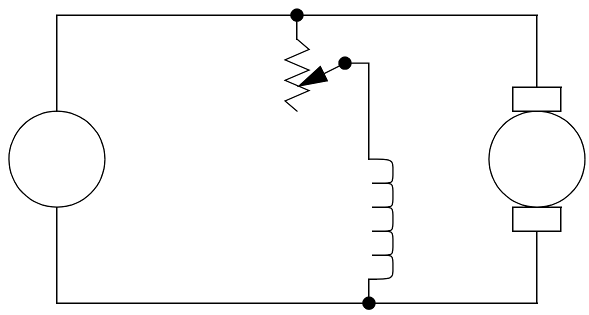
- Universal motors were presented earlier for DC applications, but they can also be used for AC power sources. This is because the field polarity in the rotor and stator both reverse as the AC current reverses.
- Synchronous motors are different from induction motors in that they are designed to rotate at the frequency of the fields, in other words there is no slip.
- Synchronous motors use generated fields in the rotor to oppose the stators field.

• Uses DC power on the rotor and stator to generate the magnetic field (i.e., no permanent magnets)

• Shunt motors

- Have the rotor and stator coils connected in parallel.
- When the load on these motors is reduced the current flow increases slightly, increasing the field, and slowing the motor.
- These motors have a relatively small variation in speed as they are varied, and are considered to have a relatively constant speed.
- The speed of the motor can be controlled by changing the supply voltage, or by putting a rheostat/resis-

tor in series with the stator windings.



$$I_a = \frac{V_a}{R_a}$$

$$T = K_t I_a \phi$$

where,

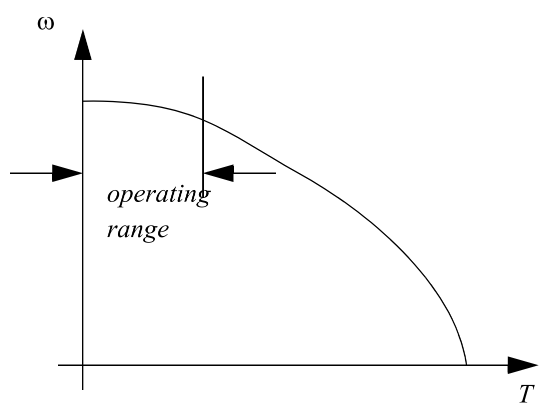
I_a, V_a, R_a = *Armature current, voltage and resistance*

T = *Torque on motor shaft*

K_t = *Motor speed constant*

ϕ = *motor field flux*

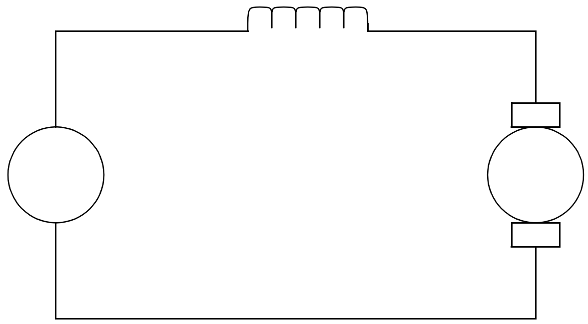
•



• Series motors

- Have the rotor and stator coils connected in series.
- As the motor speed increases the current increases, the motor can theoretically accelerate to infinite speeds if unloaded. This makes the dangerous when used in applications where they are potentially unloaded.
- These motors typically have greater starting torques that shunt motors

•



$$I_a = \frac{V_a}{R_a + R_f}$$

$$T = K_t I_a \phi = K_t I_a^2$$

where,

I_a, V_a = *Armature current, voltage*

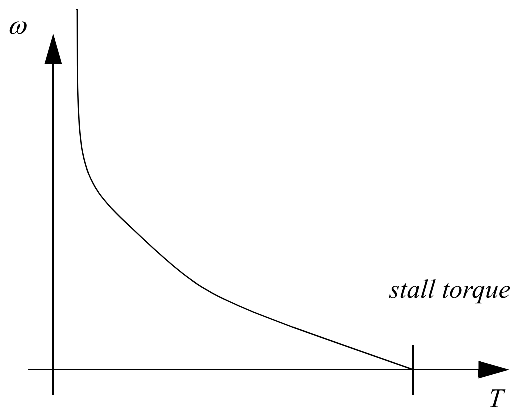
R_a, R_f = *Armature and field coil resistance*

T = *Torque on motor shaft*

K_t = *Motor speed constant*

ϕ = *motor field flux*

•



[[Note: In basic form, but to be continued with additional text later]]

$$e_f = r_a i_a + D l_a i_a + e_m$$

$$e_m = K_e \theta D$$

$$T = K_T i_a$$

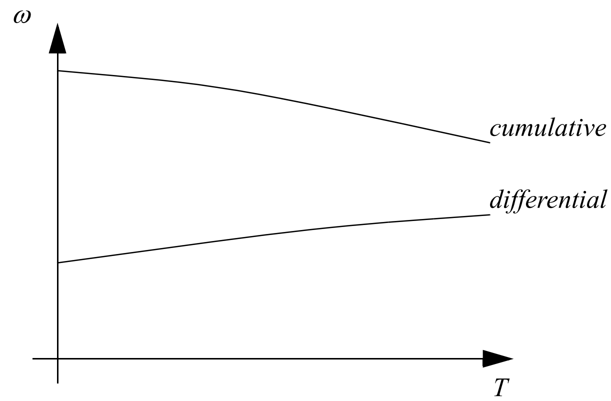
$$e_a = (r_a + l_a D) i_a + K_e D\theta$$

$$e_a = (r_a + l_a D)\left(\frac{T}{K_T}\right) + K_e D\theta$$

Figure 13.20 Equations for an armature controlled DC motor

• Compound motors\

- Have the rotor and stator coils connected in series.
- Differential compound motors have the shunt and series winding field aligned so that they oppose each other.
- Cumulative compound motors have the shunt and series winding fields aligned so that they add



$$e_f = r_f i_f + l_f i_f D$$

$$T = K_T i_f$$

$$\frac{T}{\theta} = JD^2 + BD$$

$$\frac{\theta}{T} = \frac{1}{JD^2 + BD}$$

$$\frac{\theta}{i_f} = \frac{\theta}{T}\frac{T}{i_f} = \frac{K_T}{JD^2 + BD}$$

$$\frac{\theta}{e_f} = \frac{\theta}{i_f}\frac{i_f}{e_f} = \left(\frac{K_T}{JD^2 + BD}\right)\left(\frac{1}{r_f + l_f D}\right)$$

$$\frac{T}{e_f} = \frac{T}{i_f}\frac{i_f}{e_f} = K_T\left(\frac{1}{r_f + l_f D}\right)$$

Figure 13.21 Equations for a controlled field motor

13.2 Hydraulics

Hydraulic systems are used in applications requiring a large amount of force and slow speeds. When used for continuous actuation they are mainly used with position feedback. An example system is shown in Figure 13.22. The controller examines the position of the hydraulic system, and drivers a servo valve. This controls the flow of fluid to the actuator. The remainder of the provides the hydraulic power to drive the system.

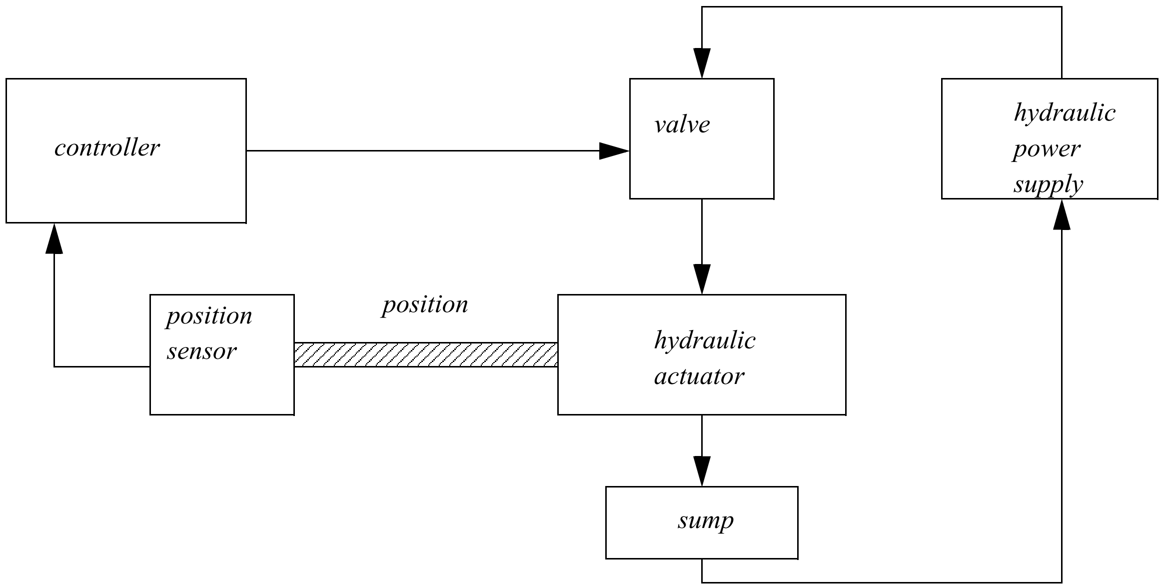


Figure 13.22 Hydraulic Servo System

The valve used in a hydraulic system is typically a solenoid controlled valve that is simply opened or closed. Newer, more expensive, valve designs use a scheme like pulse with modulation (PWM) which open/close the valve quickly to adjust the flow rate.

13.3 Other Systems

The continuous actuators discussed earlier in the chapter are the more common types. For the purposes of completeness additional actuators are listed and described briefly below.

- Heaters - to control a heater with a continuous temperature a PWM scheme can be used to limit a DC voltage, or an SCR can be used to supply part of an AC waveform.
- Pneumatics - air controlled systems can be used for positioning with suitable feedback. Velocities can also be controlled using fast acting valves.
- Linear Motors - a linear motor works on the same principles as a normal rotary motor. The primary difference is that they have a limited travel and their cost is typically much higher than other linear actuators.
- Ball Screws - rotation is converted to linear motion using balls screws. These are low friction screws that drive nuts filled with ball bearings. These are normally used with slides to bear mechanical loads.

13.4 Summary

- AC motors are low cost and work in a relatively narrow speed range (e.g., around 1700RPM).
- DC motors work over a range of speeds and tend to be more controllable. Costs are higher for controls or maintenance.
- Motion control introduces velocity and acceleration objectives to servo control.

13.5 Problems With Solutions

Problem 13.1 A stepping motor is to be used to drive each of the three linear axes of a Cartesian coordinate robot. The motor output shaft will be connected to a screw thread with a screw pitch of 0.125”. It is desired that the control resolution of each of the axes be 0.025”

a) to achieve this control resolution how many step angles are required on the stepper motor?

b) What is the corresponding step angle?

c) Determine the pulse rate that will be required to drive a given joint at a velocity of 3.0"/sec.

Problem 13.2 For the stepper motor in the previous question, a pulse train is to be generated by the robot controller.
a) How many pulses are required to rotate the motor through three complete revolutions?
b) If it is desired to rotate the motor at a speed of 25 rev/min, what pulse rate must be generated by the robot controller?

Problem 13.3 Explain the differences between stepper motors, variable frequency induction motors and DC motors using tables.

Problem 13.4 Short answer,
a) Compare the various types of motors discussed in the class using a detailed table.
b) When using a motor there are the static and kinetic friction limits. Will deadband correction allow the motor to move slower than both, one, or neither? Explain your answer.
c) What is the purpose of a calibration curve?

13.6 Problem Solutions

Answer 13.1 a)

$$P = 0.125\left(\frac{in}{rot}\right) \qquad R = 0.025\frac{in}{step}$$

$$\theta = \frac{R}{P} = \frac{0.025\frac{in}{step}}{0.125\left(\frac{in}{rot}\right)} = 0.2\frac{rot}{step} \qquad \text{Thus} \qquad \frac{1}{0.2\frac{rot}{step}} = 5\frac{step}{rot}$$

b)

$$\theta = 0.2\frac{rot}{step} = 72\frac{deg}{step}$$

c)

$$PPS = \frac{3\frac{in}{s}}{0.025\frac{in}{step}} = 120\frac{steps}{s}$$

Answer 13.2 a)

$$pulses = (3rot)\left(5\frac{step}{rot}\right) = 15steps$$

b)

$$\frac{pulses}{s} = \left(25\frac{rot}{min}\right)\left(5\frac{step}{rot}\right) = 125\frac{steps}{min} = 125\left(\frac{1min}{60s}\right)\frac{steps}{min} = 2.08\frac{step}{s}$$

Answer 13.3

	speed	torque
stepper motor	very low speeds	low torque
vfd	limited speed range	good at rated speed
dc motor	wide range	decreases at higher speeds

Answer 13.4 a)

Motor Type	Cost	Torque	Speed	Applications
AC/Induction	low	med	limited	consumer applications/large power
DC Brushed	low/med	med	variable	short life
DC Brushless	high	med	variable	high precision
Stepper	low/med	low	low	positioning
Shunt	med	med	varies	
Series	med	high	varies	large break away torques

b) Deadband correction allows the motor to break free of the static friction. Once moving freely the torque required to 'stick' the motor is determined by the lower kinetic friction. Generally this means that the motor can move slightly slower than the static friction minimum speed, but not the kinetic friction minimum speed.
c) Calibration is a process where instrumentation outputs are related to inputs. These results are then used later to relate measurement equipment outputs with actual phenomenon. For example, in the laboratory, tachometers are calibrated by turning them at a steady speed. The speed is measured with a strobe tachometer and the voltage output is also recorded. These are then used to make a graph relating voltage and speed. Later the strobe tachometer is not used and the voltage output of the tach. is used to calculate the speed.

13.7 Problems Without Solutions

Problem 13.5 A stepper motor is to be used to actuate one joint of a robot arm in a light duty pick and place application. The step angle of the motor is 10 degrees. For each pulse received from the pulse train source the motor rotates through a distance of one step angle.
a) What is the resolution of the stepper motor?
b) Relate this value to the definitions of control resolution, spatial resolution, and accuracy, as discussed in class.
c) For the stepper motor, a pulse train is to be generated by a motion controller. How many pulses are required to rotate the motor through three complete revolutions? If it is desired to rotate the motor at a speed of 25 rev/min, what pulse rate must be generated by the robot controller?

Problem 13.6 Describe the voltage ripple that would occur when using a permanent magnet DC motor as a tachometer. Hint: consider the use of the commutator to switch the polarity of the coil.

Problem 13.7 Compare the advantages/disadvantages of DC permanent magnet motors and AC induction motors.

14. Bode Plots

Topic 14.1 Using Bode plots to relate system response to frequency.
Objective 14.1 To be able to describe the response of a system using Bode plots.

14.1 Introduction

When a phasor transform is applied to a transfer function the result can be expressed as a magnitude and angle that are functions of frequency. The magnitude is the gain, and the angle is the phase shift. In the previous chapter these values were calculated for a single frequency and then multiplied by the input values to get an output value. At different frequencies the transfer function value will change. The transfer function gain and phase angle can be plotted as a function of frequency to give an overall picture of system response.

Aside: Consider a 'graphic equalizer' commonly found on home stereo equipment. The spectrum can be adjusted so that high or low tones are emphasized or muted. The position of the sliders adjusts the envelope that the audio signal is filtered through. The sliders trace out a Bode gain plot. In theoretical terms the equalizer can be described with a transfer function. As the slides are moved the transfer function is changed, and the bode plot shifts. In the example below the slides are positioned to pass more of the lower frequencies. The high frequencies would not be passed clearly, and might sound somewhat muffled.

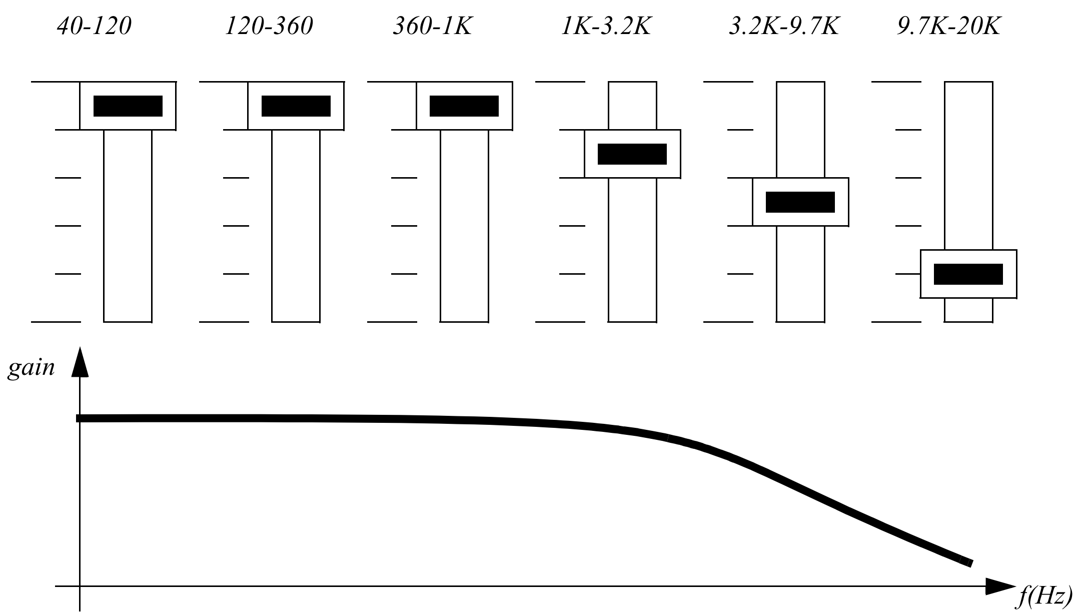


Figure 14.1 Commonly seen Bode plot

The mass-spring-damper transfer function from the previous chapter is expanded in Figure 14.2. In this example the transfer function is multiplied by the complex conjugate to eliminate the complex number in the denominator. The magnitude of the resulting transfer function is the gain, and the phase shift is the angle. Note that to correct for the quadrant of the phase shift angles pi radians is subtracted for certain frequency values.

The results in Figure 14.2 are normally left in variable form so that they may be analyzed for a range of frequencies. An example of this type of analysis is done in Figure 14.3. A set of frequencies is used for calculations. These need to be converted from Hz to rad/s before use. For each one of these the gain and phase angle is calculated. The gain gives a ratio between the input sine wave and output sine wave of the system. The magnitude of the output wave can be calculated by multiplying the input wave magnitude by the gain. (Note: recall this example was used in the previous chapter) The phase angle can be added to the input wave to get the phase of the output wave. Gain is normally converted to 'dB' so that it may cover a larger range of values while still remaining similar numerically. Also note that the frequencies are changed in multiples of tens, or magnitudes.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}$$

$$\frac{x(\omega)}{F(\omega)} = \left[\frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}\right]\left[\frac{(-j)(3000\omega) + (2000 - 1000\omega^2)}{(-j)(3000\omega) + (2000 - 1000\omega^2)}\right]$$

$$\frac{x(\omega)}{F(\omega)} = \frac{(2000 - 1000\omega^2) - j(3000\omega)}{(2000 - 1000\omega^2)^2 + (3000\omega)^2}$$

$$\left|\frac{x(\omega)}{F(\omega)}\right| = \frac{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}{(2000 - 1000\omega^2)^2 + (3000\omega)^2} = \frac{1}{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}$$

$$\theta = \operatorname{atan}\left(\frac{-3000\omega}{-1000\omega^2 + 2000}\right) = \operatorname{atan}\left(\frac{-3\omega}{2-\omega^2}\right) \qquad for\ \omega \le 2$$

$$\theta = \operatorname{atan}\left(\frac{-3\omega}{2-\omega^2}\right) - \pi \qquad for\ \omega > 2$$

Figure 14.2 A phasor transform example

f(Hz)	(rad/sec)	Gain	Gain (dB)	θ(rad.)	θ(deg.)
0	0				
0.001	0.006283				
0.01	0.06283				
0.1	0.6283				
1	6.283				
10	62.83				
100	628.3				
1000	6283				

Note: the gain values will cover many magnitudes of values. To help keep the graphs rational we will "compress" the values by converting them to dB (decibels) using the following formula.

$$gain_{db} = 20\log(gain)$$

Note: negative phase angles mean that the mass motion lags the force.

Note: The frequencies chosen should be chosen to cover the points with the greatest amount of change.

Figure 14.3 A phasor transform example (continued)

In this example gain is defined as x/F. Therefore F is the input to the system, and x is the resulting output. The gain means that for each unit of F input to the system, there will be gain*F=x output. If the input and output are sinusoidal, there is a difference in phase between the input and output wave of θ (the phase angle). This is shown in Figure 14.4, where an input waveform is supplied with three sinusoidal components. For each of the frequencies a gain and phase shift are calculated. These are then used to calculate the resulting output wave, relative to the input wave. The resulting output represents the steady-state response to the sinusoidal output.

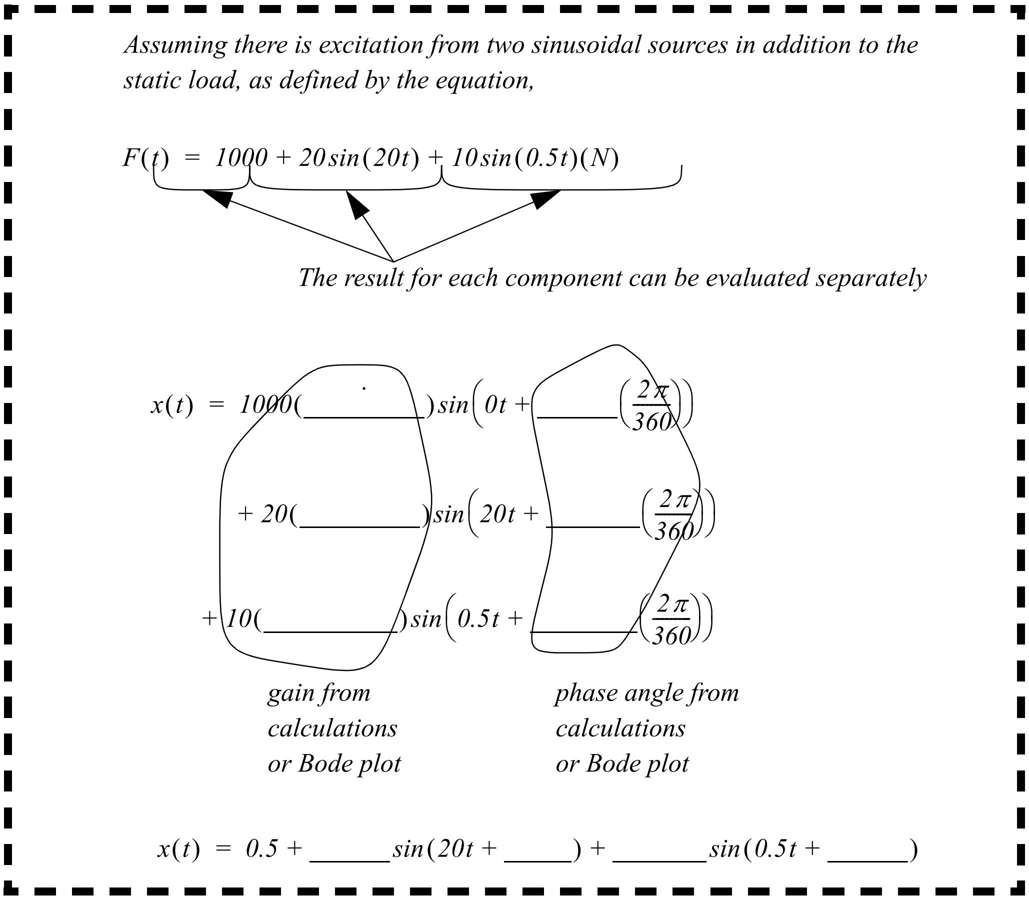


Figure 14.4 A phasor transform example (continued)

14.2 Bode Plots

In the previous section we calculated a table of gains and phase angles over a range of frequencies. Graphs of these values are called Bode plots. These plots are normally done on semi-logarithm scaled graph paper, such as that seen in Figure 14.5. Along the longer axis of this paper the scale is logarithmic (base 10). This means that if the paper started at 0.1 on one side, the next major division would be 1, then 10, then 100, and finally 1000 on the other side of the paper. The basic nature of logarithmic scales prevents the frequency from being zero. Along the linear axis (the short one) the gains and phase angles are plotted, normally with two graphs side-by-side on a single sheet of paper.

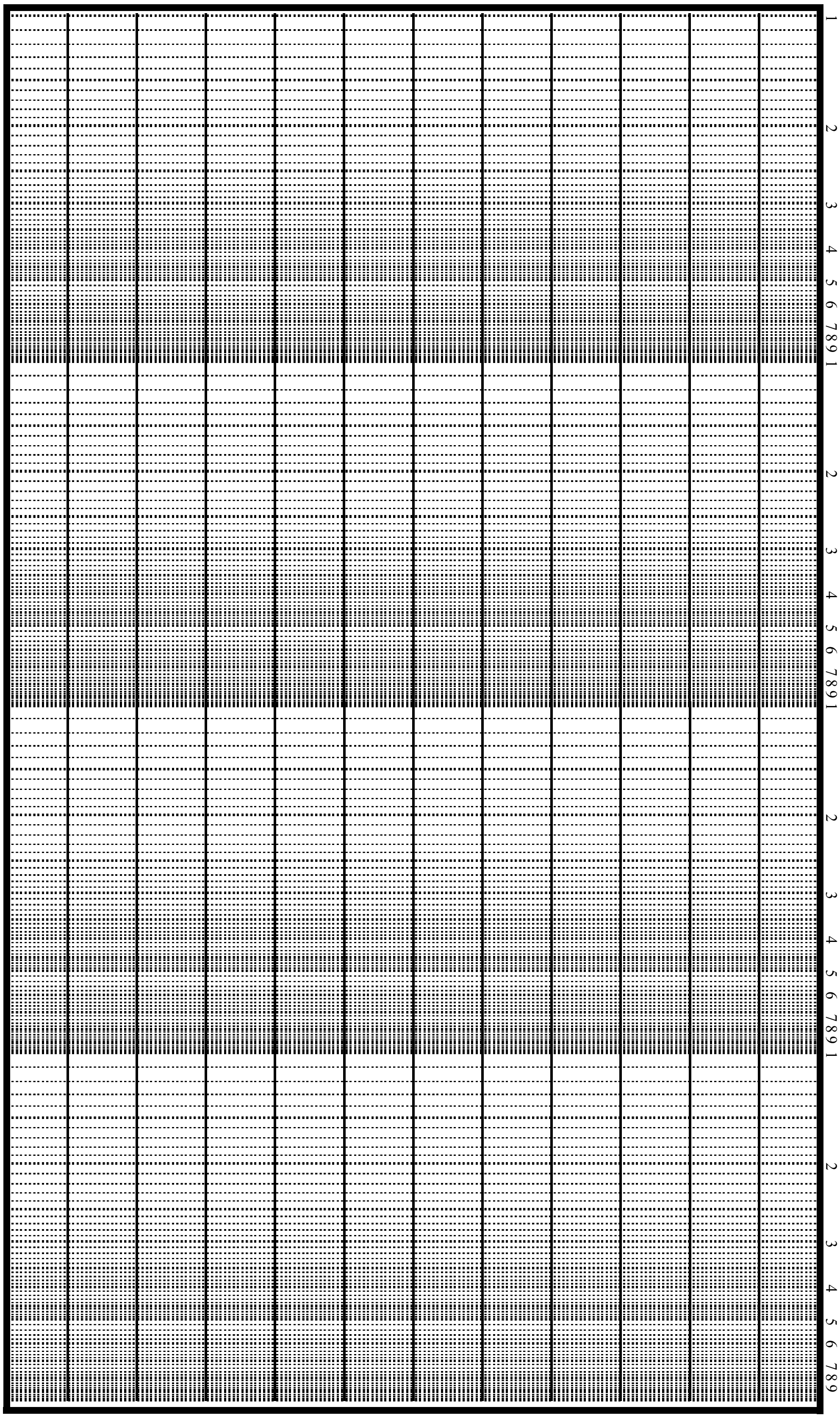


Figure 14.5 Four cycle semi-logarithmic scale graph paper

```
steps_per_dec = 6;
decades = 6;
start_freq = 0.1;

// the transfer function
function foo=G(w)
        D = %i * w;
        foo = (D + 5) / (D^2 + 100*D + 10000);
endfunction

// this section writes the values to a datafile that may be graphed in a spreadsheet
fd = mopen("data.txt", "w");
for step = 0:(steps_per_dec * decades),
        f = start_freq * 10 ^ (step / steps_per_dec); // calculate the next frequency
        w = f * 2 * %pi);// convert the frequency to radians
        [gain, phase] = polar(G(w));// find the gain and convert it to mag and angle
        gaindb = 20 * log10(gain);// convert magnitude to dB
        phasedeg = 180 * phase / %pi;// convert to degrees
        mfprint(fd, "%f, %f, %f \n", f, gaindb, phasedeg);
end
mclose(fd);

// to graph it directly the following is used
D = poly(0, 'D');
h = syslin('c', (D + 5) / (D^2 + 100*D + 10000) ) ;
bode(h, 0.1, 1000, 'Sample Transfer Function');
```

Figure 14.6 Aside: Bode Plot Example with Scilab

Aside: This can also be done entirely with phasors in Cartesian notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236-0.425j$$

$$x = 20(\cos(0.3\text{rad}) + j\sin(0.3\text{rad})) = 19.1 + 5.91j$$

$$\frac{y}{19.1+5.91j} = (1.236-0.425j)(19.1+5.91j) = 26.1-0.813j = 26.1\angle -0.031$$

$$y(t) = 26.11\sin(9t-0.031)$$

Aside: This can also be done entirely with phasors in polar notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right) = \frac{13.45\angle 0.733}{10.30\angle 1.064} = \frac{13.45}{10.30}\angle 0.733-1.064 = 1.31\angle -0.331$$

$$x = 20\angle 0.3$$

$$\frac{y}{20\angle 0.3} = 1.31\angle -0.331$$

$$y = 1.31(20)\angle(-0.331+0.3) = 26.2\angle -0.031$$

$$y(t) = 26.2\sin(9t-0.031)$$

14.3 Straight Line Approximations

An approximate technique for constructing a gain Bode plot is shown in Figure 14.7. This method involves looking at the transfer function and reducing it to roots in the numerator and denominator. Once in that form, a straight line approximation for each term can be drawn on the graph. An initial gain is also calculated to shift the results up or down. When done, the straight line segments are added to produce a more complex straight line curve. A smooth curve is then drawn over top of this curve.

Bode plots for transfer functions can be approximated with the following steps.

1. Plot the straight line pieces.
    a) The gain at 0rad/sec is calculated and used to find an initial offset. For example this transfer function starts at 10(D+1)/(D+1000)=10(0+1)/(0+1000)=0.01=-40dB.
    b) Put the transfer function in root form to identify corner frequencies. For example (D+1)/(D+1000) will have corner frequencies at 1 and 1000 rad/sec.
    c) Curves that turn up or down are drawn for each corner frequency. At each corner frequency a numerator term causes the graph to turn up, each term in the dominator causes the graph to turn down. The slope up or down is generally +/- 20dB/decade for each term. Also note that squared (second-order) terms would have a slope of +/- 40dB/decade.
2. The effect of each term is added up to give the resulting straight line approximation.
3. When the smooth curve is drawn, there should commonly be a 3dB difference at the corner frequencies. In second-order systems the damping factor may make the corner flatter or peaked.

Figure 14.7 The method for Bode graph straight line gain approximation

Note: Some of the straight line approximation issues are discussed below.

Why is there 3dB between a first order corner and the smooth plot, and the phase angle is 45 degrees of the way to +/- 90 degrees.

$$G(j\omega) = \frac{1}{\omega_c + j\omega}$$

the initial gain is

$$G(0) = \frac{1}{\omega_c + j0} = \frac{1}{\omega_c}$$

at the corner frequency $\omega_c = \omega$

$$G(j\omega) = \frac{1}{\omega_c + j\omega_c} = \frac{1}{\omega_c\sqrt{2}\angle\frac{\pi}{4}} = \frac{1}{\omega_c\sqrt{2}}\angle -\frac{\pi}{4}$$

Therefore the difference is

$$diff = \frac{G(j\omega)}{G(0)} = \frac{1}{\sqrt{2}}\angle -\frac{\pi}{4} = 20log\left(\frac{1}{\sqrt{2}}\right)\angle -\frac{\pi}{4} = -3.01dB\angle -\frac{\pi}{4}$$

Why does a first order pole go down at 20dB/dec?

$$|G(j\omega)| = \left|\frac{1}{\omega_c + j\omega}\right|$$

before the corner frequency, $\omega_c > j\omega$

$$|G(j\omega)| = \left|\frac{1}{\omega_c}\right| = 20log\left(\omega_c^{-1}\right) = -20log(\omega_c)$$

after the corner frequency, $\omega_c < j\omega$

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = 20log(\omega^{-1}) = -20log(\omega)$$

each time the frequency increases by a multiple of 10, the logarithm value becomes 1 larger, thus resulting in a gain change of -20 dB.

Why does a pole make the phase angle move by -90deg after the corner frequency?

$$G(j\omega) = \frac{1}{\omega_c + j\omega} = \frac{1\angle 0}{\sqrt{\omega_c^2 + \omega^2}\angle atan\left(\frac{\omega}{\omega_c}\right)} = \frac{1}{\sqrt{\omega_c^2 + \omega^2}}\angle -atan\left(\frac{\omega}{\omega_c}\right)$$

before the corner frequency, $\omega_c > j\omega$

$$angle(G(j\omega)) = -atan\left(\frac{\omega}{\omega_c}\right) = -atan\left(\frac{0}{\omega_c}\right) = 0$$

after the corner frequency, $\omega_c < j\omega$

$$angle(G(j\omega)) = -atan\left(\frac{\omega}{\omega_c}\right) = -atan(\infty) = -\frac{\pi}{2}$$

Figure 14.8 Why the straight line method works

An example of the straight line plotting technique is shown in Figure 14.9. In this example the transfer function is first put into a root form. In total there are three roots, 1, 10 and 100 rad/sec. The single root in the numerator will cause the curve to start upward with a slope of 20dB/dec after 1rad/sec. The two roots will cause two curves downwards at -20dB/dec starting at 10 and 100 rad/sec. The initial gain of the transfer function is also calculated, and converted to decibels. The frequency axis is rad/sec by default, but if Hz are used then it is necessary to convert the values.

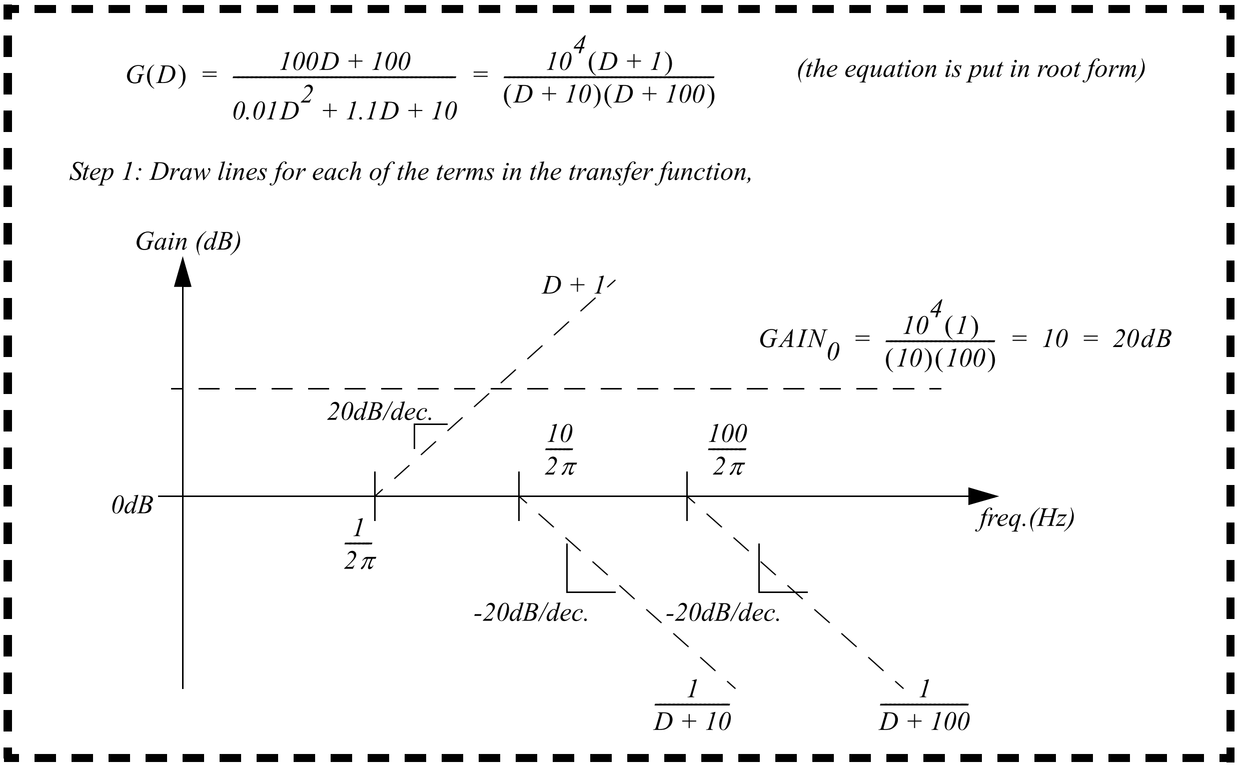


Figure 14.9 An approximate gain plot example

The example is continued in Figure 14.10 where the straight line segments are added to produce a combined straight line curve.

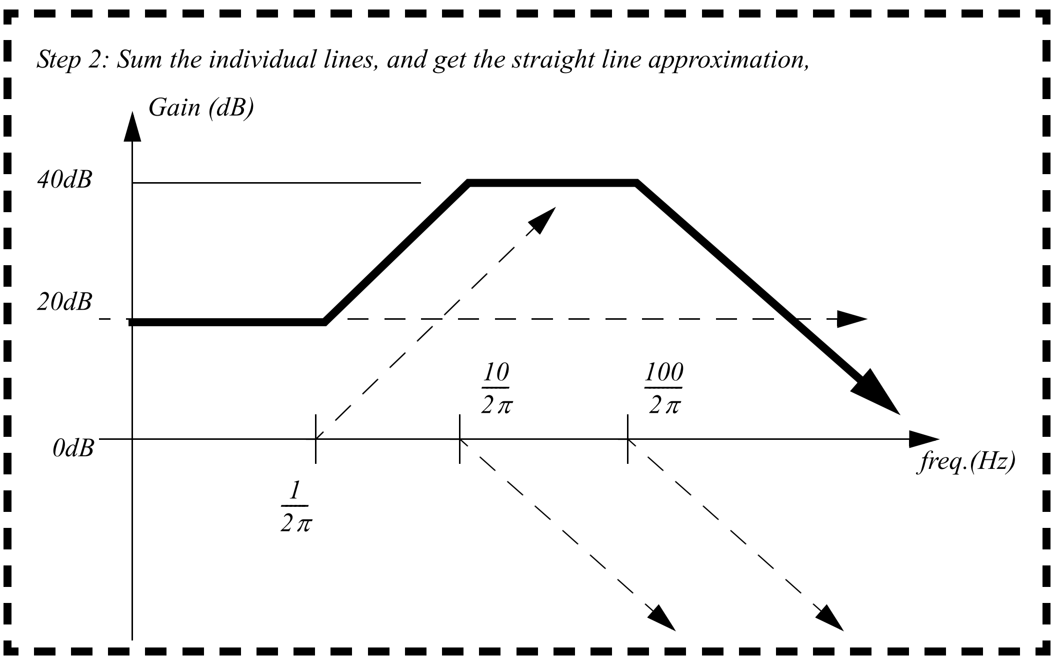


Figure 14.10 An approximate gain plot example (continued)

Finally a smooth curve is fitted to the straight line approximation. When drawing the curve imagine that there are rubber bands at the corners that pull slightly and smooth out. For a simple first-order term there is a 3dB gap between the smooth corner and the function. Higher order functions will be discussed later.

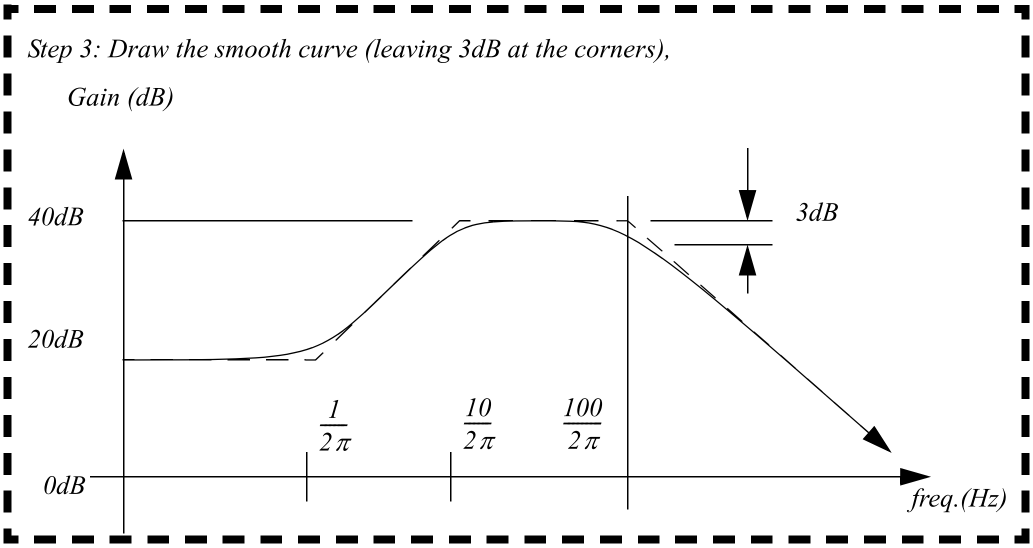


Figure 14.11 An approximate gain plot example (continued)

The process for constructing phase plots is similar to that of gain plots, as seen in Figure 14.13. The transfer function is put into root form, and then straight line phase shifts are drawn for each of the terms. Each term in the numerator will cause a positive shift of 90 degrees, while terms in the denominator cause negative shifts of 90 degrees. The phase shift occurs over two decades, meaning that for a center frequency of 100, the shift would start at 10 and end at 1000. If there are any lone 'D' terms on the top or bottom, they will each shift the initial value by 90 degrees, otherwise the phase should start at 0degrees.

Phase plots for transfer functions can be approximated with the following steps.

1. Plot the straight line segments.
   a) Put the transfer function in root form to identify center frequencies. For example (D+1)/(D+1000) will have center frequencies at 1 and 1000 rad/sec. This should have already been done for the gain plot.
   b) The phase at 0rad/sec is determined by looking for any individual D terms. Effectively they have a root of 0rad/sec. Each of these in the numerator will shift the starting phase angle up by 90deg. Each in the denominator will shift the start down by 90 deg. For example the transfer function 10(D+1)/(D+1000) would start at 0 deg while 10D(D+1)/(D+1000) would start at +90deg.
   c) Curves that turn up or down are drawn around each center frequency. Again terms in the numerator cause the curve to go up 90 deg, terms in the denominator cause the curves to go down 90 deg. Curves begin to shift one decade before the center frequency, and finish one decade after.
2. The effect of each term is added up to give the resulting straight line approximation.
3. The smooth curve is drawn.

Figure 14.12 The method for Bode graph straight line gain approximation

The previous example started in Figure 14.9 is continued in Figure 14.13 to develop a phase plot using the approximate technique. There are three roots for the transfer function. None of these are zero, so the phase plot starts at zero degrees. The root in the numerator causes a shift of positive 90 deg, starting one decade before 1rad/sec and ending one decade later. The two roots in the denominator cause a shift downward.

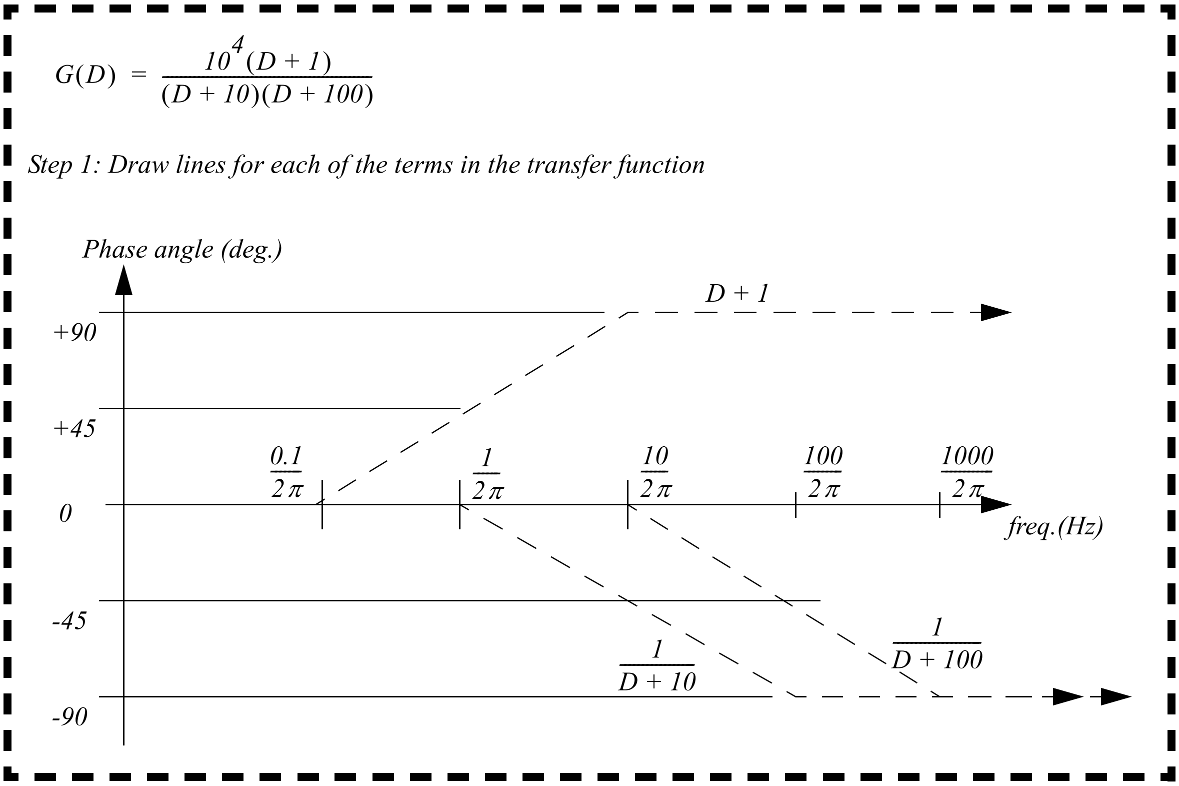


Figure 14.13 An approximate phase plot example

The straight line segments for the phase plot are added in Figure 14.14 to produce a straight line approximation of the final plot. A smooth line approximation is drawn using the straight line as a guide. Again, the concept of a rubber band will smooth the curve.

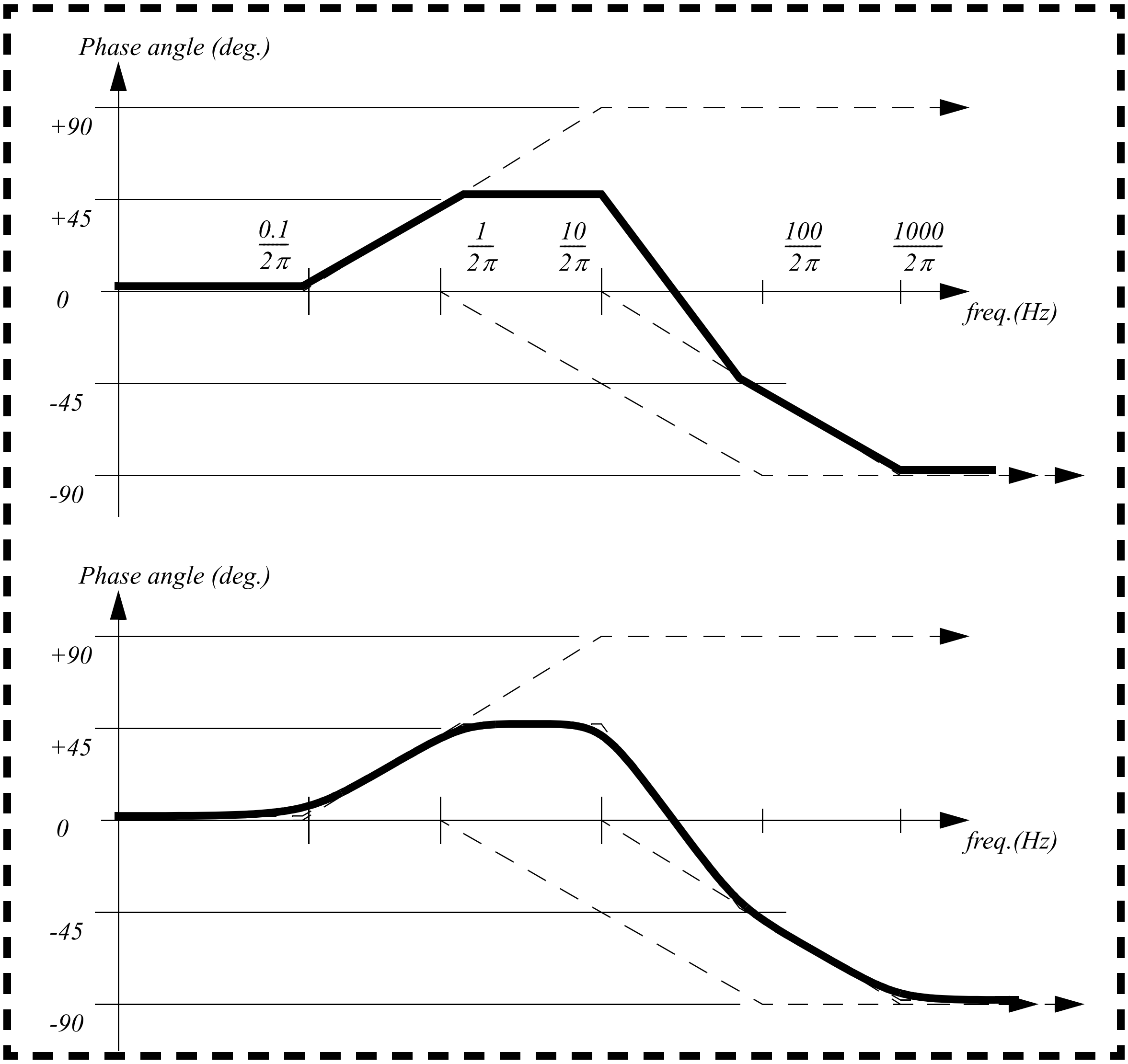


Figure 14.14 An approximate phase plot example (continued)

Second Order Underdamped Terms

The previous example used a transfer function with real roots. In a second-order system with double real roots (overdamped) the curve can be drawn with two overlapping straight line approximations. If the roots for the transfer function are complex (underdamped the corner frequencies will become peaked. This can be handled by determining the damping factor and natural frequency as shown in Figure 14.15. The peak will occur at the damped frequency. The peaking effect will become more pronounced as the damping factor goes from 0.707 to 0 where the peak will be infinite.

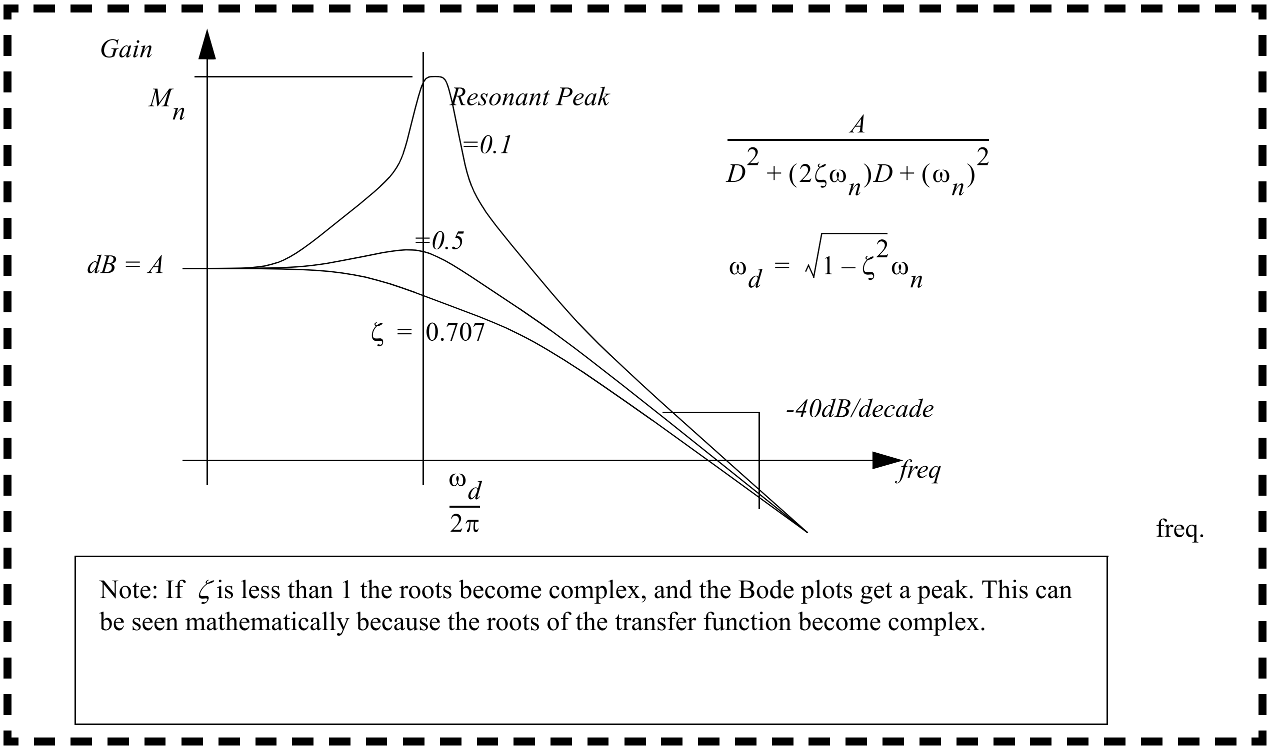


Figure 14.15 Resonant peaks

The approximate techniques do decrease the accuracy of the final solution, but they can be calculated quickly. In addition these curves provide an understanding of the system that makes design easier. For example, a designer will often describe a system with a Bode plot, and then convert this to a desired transfer function.

Lone Ds on the Top or Bottom

In some cases the transfer function will have a lone 'D' on the top or bottom. In these cases it means that the gain of the system starts (i.e., f = 0Hz) starts at infinity for a 'D' on the bottom, or at a gain of 0 (negative infinity dB) for a 'D' on the top.Or in other words, consider the Ds to have corner frequencies at zero. This means that Ds on the top start at infinity and come down. Ds on the bottom start at negative infinity and come up. If at all possible cancel the Ds on the top and bottom, if any remain then to draw the Bode plot you must work from the right side of the graph (i.e. f = infinity Hz).

Consider the example in Figure 14.16. The D squared term on the top is squared, so the system is second order and the phase angle starts at 180 degrees. The gain starts at 0 (-infinity dB) goes upwards at 40dB/dec. As the upward slope of +40dB/dec approaches the corner frequency of the bottom, with a slope of -40dB/dec, the two effects cancel and the slope becomes 0, for a vertical line. The problem solution is similar when the lone Ds are on the bottom. With these types of problems the top and bottom of the transfer function might go to zero or infinity. Normally we would say this is underdefined. In these cases, L'Hospital's rule may be used.

$$\frac{D^2}{(D + 20\pi)^2}$$

For the initial gain: $D = 0j$

$$\frac{(0j)^2}{(0j + 20\pi)^2} = \frac{0^2 j^2}{(20\pi)^2} = \frac{0\angle 180°}{400\pi^2\angle 0°} = 0\angle 180° = -\infty dB\angle 180°$$

Note: If necessary L'Hospital's rule can be used if the top and bottom are both zero or infinity.

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

For the final gain: $D = \infty j$

$$\lim_{D \to \infty j} \frac{D^2}{(D + 20\pi)^2} = \lim_{D \to \infty j} \frac{2D}{2(D + 20\pi)} = \lim_{D \to \infty j} \frac{2}{2} = 1\angle 0° = 0dB\angle 0°$$

or less formally....

$$\frac{(\infty j)^2}{(\infty j + 20\pi)^2} = \frac{\infty^2 j^2}{\infty^2 j^2} = \frac{\infty^2\angle 180°}{\infty^2\angle 180°} = 1\angle 0° = 0dB\angle 0°$$

Corner frequencies: $D = \infty j$

Zeroes: 0Hz, 0Hz

Poles: $\frac{20\pi}{2\pi}Hz = 10Hz, 10Hz$

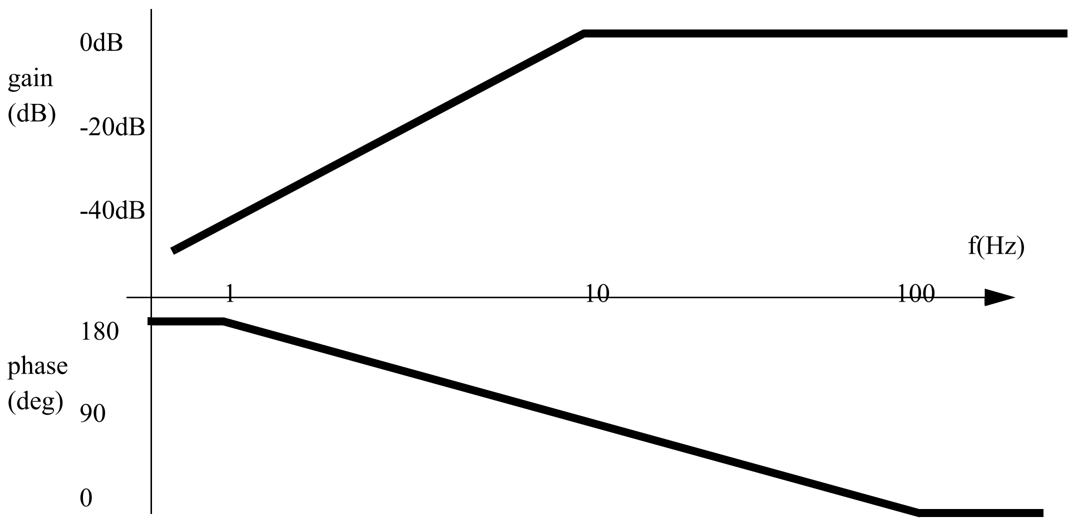


Figure 14.16 A System Gain Starting at Negative Infinity

In all of the examples presented, were we start or stop could be drawn from the left or right. This will not always be the case as shown in the Figure 14.17. In this case the term on the top results in a slope of +20dB/dec on the left. The order of the numerator and denominator are not the same so the final result will not be horizontal. The bottom is second order, compared to the first order top, so the final result will be a downward slope of -20dB/dec. This leaves the dilemma of where the graph starts and ends. The solution is to calculate one or more points on the curve using phasor transforms.

$$\frac{D}{(D + 20\pi)^2}$$

For the initial gain: $D = 0j$

$$\frac{(0j)}{(0j + 20\pi)^2} = \frac{0j}{(20\pi)^2} = \frac{0 – 90°}{400\pi^2 – 0°} = 0 – 90° = -\infty dB – 90°$$

For the final gain: $D = \infty j$

$$\frac{\infty j}{(\infty j + 20\pi)^2} = \frac{\infty j}{\infty^2 j^2} = \frac{\infty – 90°}{\infty^2 – 180°} = 0 — 90° = -\infty dB — 90°$$

Corner frequencies: $D = \infty j$

Zeroes: $0 Hz$

Poles: $\frac{20\pi}{2\pi} Hz = 10Hz, 10Hz$

Gain at 1Hz: $D = 2\pi j$

$$\frac{(2\pi j)}{(2\pi j + 20\pi)^2} = 0.0015758 – 1.371459 rad = -56dB – 78.6°$$

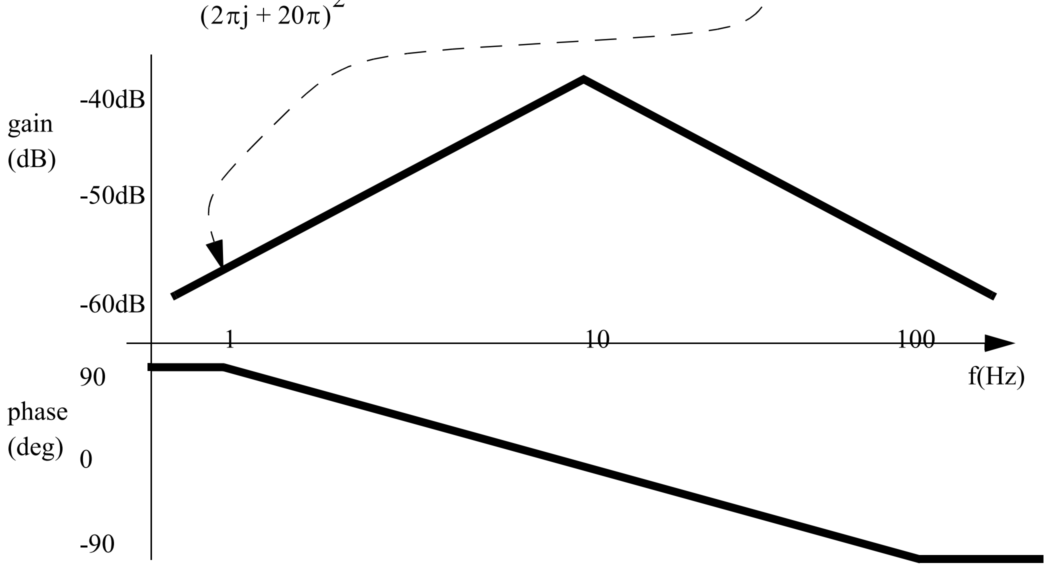


Figure 14.17 Calculating a point to set a position

14.4 Frequency Response Functions

In many cases the behavior of a system must be determined experimentally. One common method is to excite the system with a range of input frequencies, measure the output response, and plot the gain and phase. This is the experimental equivalent of the Bode plot, called the Frequency Response Functions (FRF). Practically this means that you apply an impulse function (literally a hammer for some systems). The response is measures as a power magnitude over a range of frequencies, using a spectrum analyzer. Or, an input can be set to different frequencies and the output response measured, multiple frequencies are used to construct a graph, like the one shown in Figure 14.18.

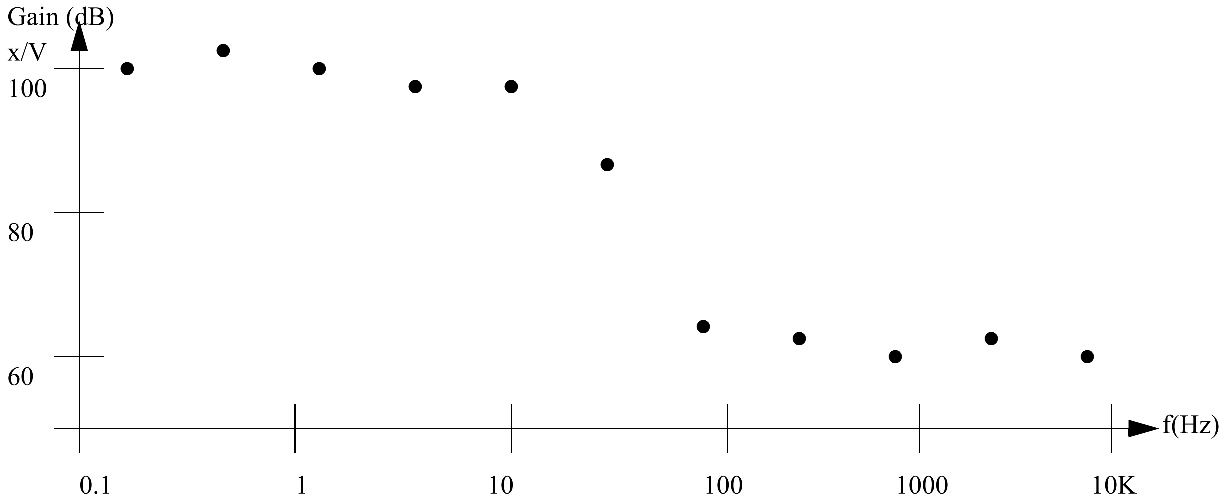
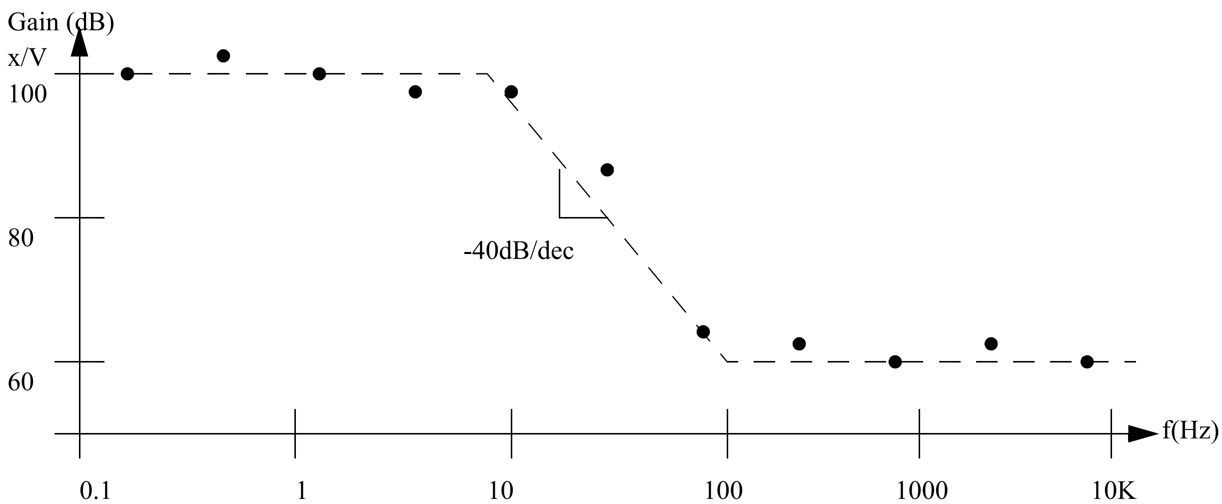


Figure 14.18 Experimentally measured Frequency Response Function (FRF)

This can be converted to a Bode plot, then transfer function by doing a straight line fit to the function. When the graph is drawn using experimental values there is some need to allow for points that are off the straight line. In other words, fit the curve using a good educated guess. Then using the form of the graph work backwards to construct a transfer function.



Corner frequencies:

zeroes: $100\text{Hz} = 200\pi\frac{\text{rad}}{\text{s}}$

poles: $10\text{Hz} = 20\pi\frac{\text{rad}}{\text{s}}$

$$= A\frac{(D + 200\pi)^2}{(D + 20\pi)^2}$$

Initial value: $D = 0j$

$$10^{\frac{100}{20}} = A\frac{(0j + 200\pi)^2}{(0j + 20\pi)^2}$$

$\backslash A = 1000$

$$= 1000\frac{(D + 200\pi)^2}{(D + 20\pi)^2}$$

Figure 14.19 Example FRF

14.5 Signal Spectrum

If a vibration signal is measured and displayed it might look like Figure 14.20. The overall sinusoidal shape is visible, along with a significant amount of 'noise'. When this is considered in greater detail it can be described with the given function. To determine the function other tools are needed to determine the frequencies, and magnitudes of the frequency components.

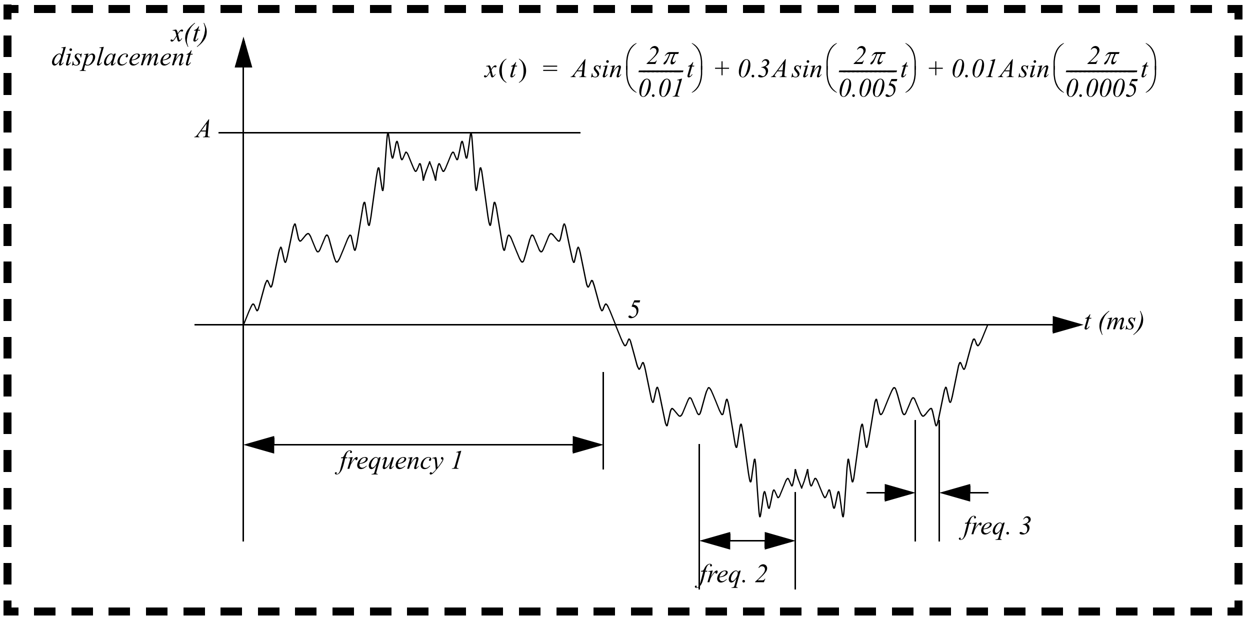


Figure 14.20 A vibration signal as a function of time

A signal spectrum displays signal magnitude as a function of frequency, instead of time. The time based signal in Figure 14.20 is shown in the spectrum in Figure 14.21. The three frequency components are clearly identifiable spikes. The height of the peaks indicates the relative signal magnitude.

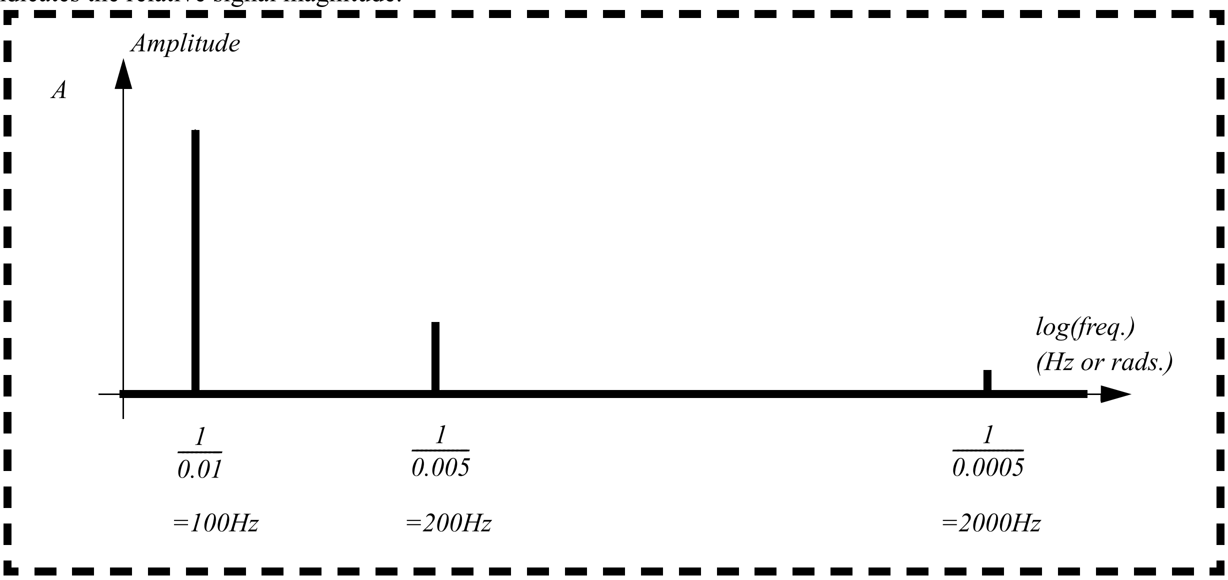


Figure 14.21 The spectrum for the signal in Figure 14.20

14.6 Summary

- Bode plots show gain and phase angle as a function of frequency.
- Bode plots can be constructed by calculating point or with straight line approximations.
- A signal spectrum shows the relative strengths of components at different frequencies.

14.7 Problems With Solutions

Problem 14.1 Given the transfer function below,

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2}$$

a) draw the straight line approximation of the Bode (gain and phase shift) plots.
b) determine the steady-state output if the input is x(t) = 20 sin(9t+0.3) using the Bode plot.

Problem 14.2 You are given the following differential equation for a spring damper pair.

$$5x + \left(\frac{d}{dt}\right)x = F \qquad F(t) = 10\sin(100t)$$

a) Write the transfer function for the differential equation if the input is F.
b) Apply the phasor transform to the transfer function to find magnitude and phase as functions of frequency.
c) Draw a Bode plot for the system using either approximate or exact techniques.
d) Use the Bode plot to find the response to;

$$F(t) = 10\sin(100t)$$

e) Put the differential equation in state variable form and use a calculator to find values in time for the given input.

$$F = 10\sin(100t)$$

t	x
0.0	
0.002	
0.004	
0.006	
0.008	
0.010	

f) Give the expected 'x' response of this first-order system to a step function input for force F = 1N for t > 0 if the system starts at rest. Hint: Use the canonical form.

Problem 14.3 Given the following transfer function perform the following operations.

$$\frac{y}{x} = \frac{D+10}{D^2 + 100D + 10000}$$

a) Draw a Bode plot using the straight line methods (using the graph paper on the next page).
b) Determine y(t) if x(t) = 20 + 3 sin(10t) using the Bode plot.
c) Determine y(t) if x(t) = 20 + 3 sin(10t) using phasors.

Problem 14.4 Draw a Bode Plot for both of the transfer functions below using the phasor transform.

$$\frac{(D+1)(D+1000)}{(D+100)^2} \quad \text{and} \quad \frac{5}{D^2}$$

Problem 14.5 Use the straight line approximation techniques to draw the Bode plot for the transfer function below.

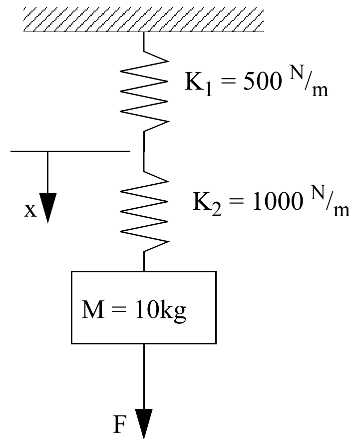
$$G = \frac{F}{x} = \frac{D+1000}{D^2 + 5D + 100}$$

Problem 14.6 Given the transfer function below,

$$\frac{V_o}{V_i} = \frac{1000}{D + 1000}$$

a) Find the steady state response of the circuit using phasors (i.e., phasor transforms) if the input is Vi=5sin(100,000t).
b) Draw an approximate Bode plot for the circuit.

Problem 14.7 The applied force 'F' is the input to the system, and the output is the displacement 'x'. Neglect the effects of gravity. a) find the transfer function.



b) What is the steady-state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if 'x' is the input.
d) Draw the bode plots for the transfer function found in a).
e) Find x(t), given F(t) = 10N for t >= 0 seconds.
f) Find x(t), given F(t) = 10N for t >= 0 seconds considering the effects of gravity.

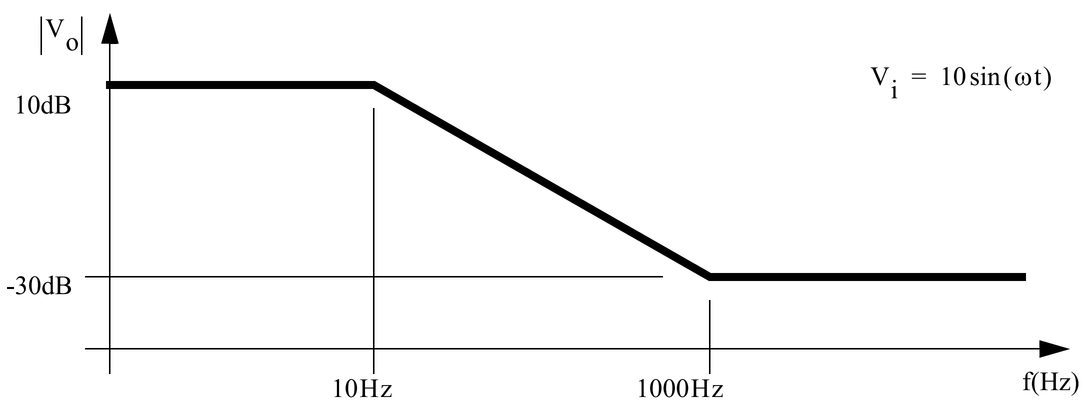
Problem 14.8 The following differential equation is supplied, with initial conditions.

$$\ddot{y} + \dot{y} + 7y = F \qquad y(0) = 1 \qquad \dot{y}(0) = 0$$

$$F(t) = 10 \qquad t > 0$$

a) Write the equation in state variable form.
b) Solve the differential equation numerically.
c) Solve the differential equation using calculus techniques.
d) Find the frequency response (gain and phase) for the transfer function using the phasor transform. Sketch the bode plots.

Problem 14.9 Given the frequency response plot below, develop a transfer function.



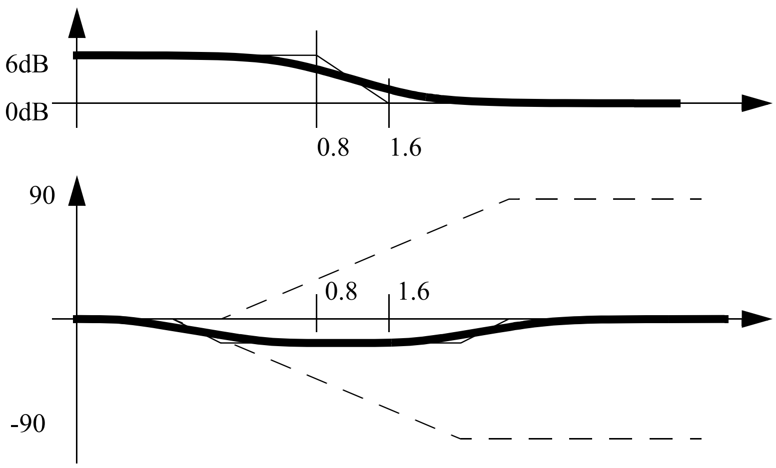
Problem 14.10 For the transfer function below draw an approximate bode plot. Use the approximate Bode plot to estimate the response to the input. Verify the results using phasors.

$$\frac{x}{f} = \frac{D(D+2\pi)}{(D+200\pi)^2} \qquad f = 5\sin(62.82t)$$

14.8 Problem Solutions

Answer 14.1

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2} = \frac{(D+10)}{(D+5)}$$



$$x(t) = 20\cos(9t+0.3)$$

Aside: the numbers should be obtained from the graphs, but I have calculated them

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236-0.425j$$

$$\frac{y}{x} = \sqrt{1.236^2+0.425^2} - \operatorname{atan}\left(\frac{-0.425}{1.236}\right) = 1.307—0.3312\text{rad}$$

$$\frac{y}{x} = 2.33\text{dB}—9.49°$$

$$y(t) = 20(1.307)\sin(9t+0.3+(-0.3312))$$

$$y(t) = 26.1\sin(9t-0.031)$$

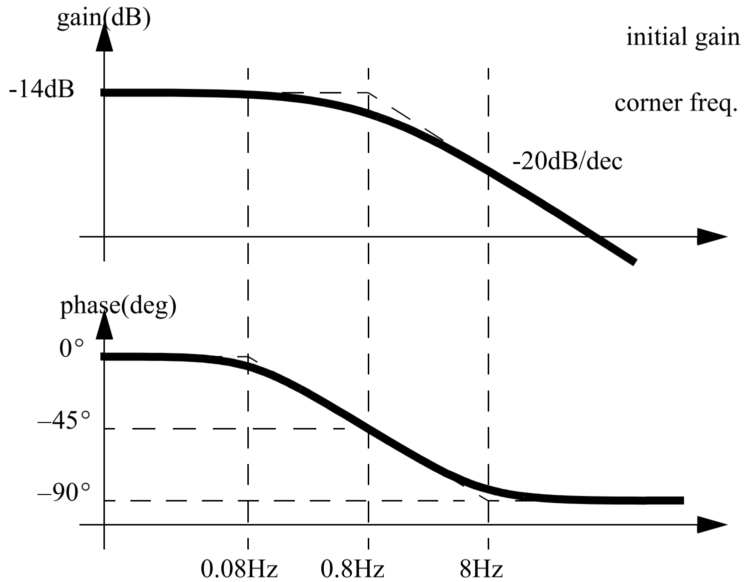
Answer 14.2 a)

$$\frac{x}{F} = \frac{1}{5 + D}$$

b)

$$\frac{x}{F} = \frac{1}{5 + D} = \frac{1}{5 + j\omega} = \frac{1\angle 0}{\sqrt{5^2 + \omega^2}\angle \operatorname{atan}\left(\frac{\omega}{5}\right)} = \frac{1}{\sqrt{5^2 + \omega^2}}\angle -\operatorname{atan}\left(\frac{\omega}{5}\right)$$

c)



$$\text{initial gain} = 20\log\left(\frac{1}{5 + 0}\right) = -14\text{dB}$$

$$\text{corner freq.} = \frac{5}{2\pi} = 0.8\text{Hz}$$

d)

$$f = \frac{100}{2\pi} = 16\text{Hz}$$

From the Bode plot,

$$\text{gain} = -40\text{dB} = 0.01$$

$$\text{phase} = -90° = -\frac{\pi}{2}\text{rad}$$

$$x(t) = 10(0.01)\sin\left(100t - \frac{\pi}{2}\right)$$

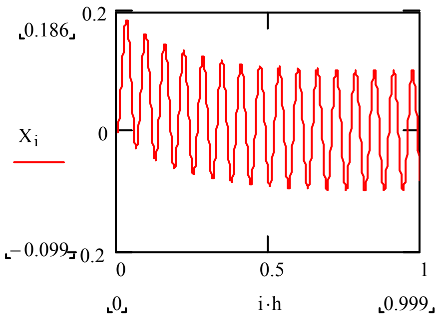
Aside: verified by calculations,

$$\frac{x}{10\angle 0} = \frac{1}{\sqrt{5^2 + 100^2}}\angle -\operatorname{atan}\left(\frac{100}{5}\right) = 0.00999\angle -1.521$$

$$x = (10\angle 0)(0.00999\angle -1.521) = 0.0999\angle -1.521$$

$$x(t) = 0.0999\sin(100t - 1.521)$$

e)



t	x
0.0	0
0.002	9.98e-4
0.004	5.92e-3
0.006	0.015
0.008	0.026
0.010	0.041

f) $$x = \frac{1}{5} - \frac{e^{-5t}}{5}$$

Answer 14.3 a)

initial:

$$\frac{0j + 10}{(0j)^2 + 100(0j) + 10000} = \frac{10–0}{10000–0} = 10^{-3}–0 = –60\text{dB}–0$$

zero: $f_c = \frac{10}{2\pi} = 1.6\text{Hz}$

poles:

$$\omega_n = \sqrt{10000} = 100 \qquad 100 = 2\zeta\omega_n \qquad \backslash\zeta = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 86.6 \qquad f_c = \frac{86.6}{2\pi} = 13.8\text{Hz}$$

(The remainder of the step is shown on the semi-logarithmic graph paper)

b) 0rad/s (from Bode plot):

$$–60\text{dB}–0 = 10^{-3}–0$$

10rad/s=1.6Hz (from Bode plot):

$$–58\text{dB}–40° = 10^{-\frac{58}{20}}–\frac{40}{180}2\pi = 1.26\times10^{-3}–1.40\text{rad}$$

$$y(t) = 20\times10^{-3} + 3.78\times10^{-3}\sin(10t + 1.40)$$

c)

for 0rad/s:

$$\frac{y}{20} = \frac{0j + 10}{(0j)^2 + 100(0j) + 10000} \qquad y = \frac{200}{10000} = 0.02–0$$

for 10rad/s:

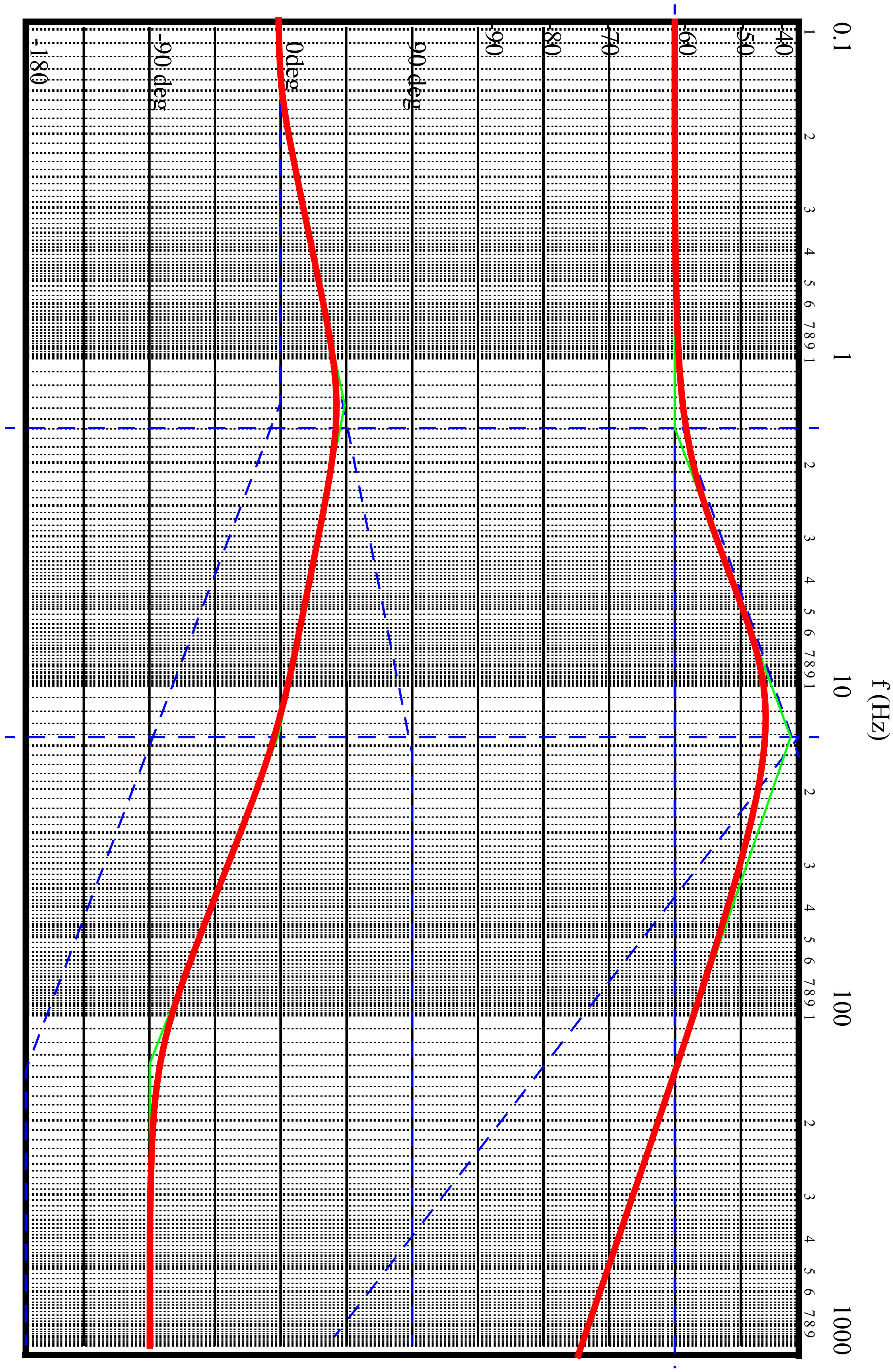
$$\frac{y}{3} = \frac{10j + 10}{(10j)^2 + 100(10j) + 10000} = \frac{\sqrt{200}–\frac{\pi}{4}}{100(10j) + (10000 - 100)}$$

$$\frac{y}{3} = \frac{14.142136–0.78539816}{9950.4–0.1007}$$

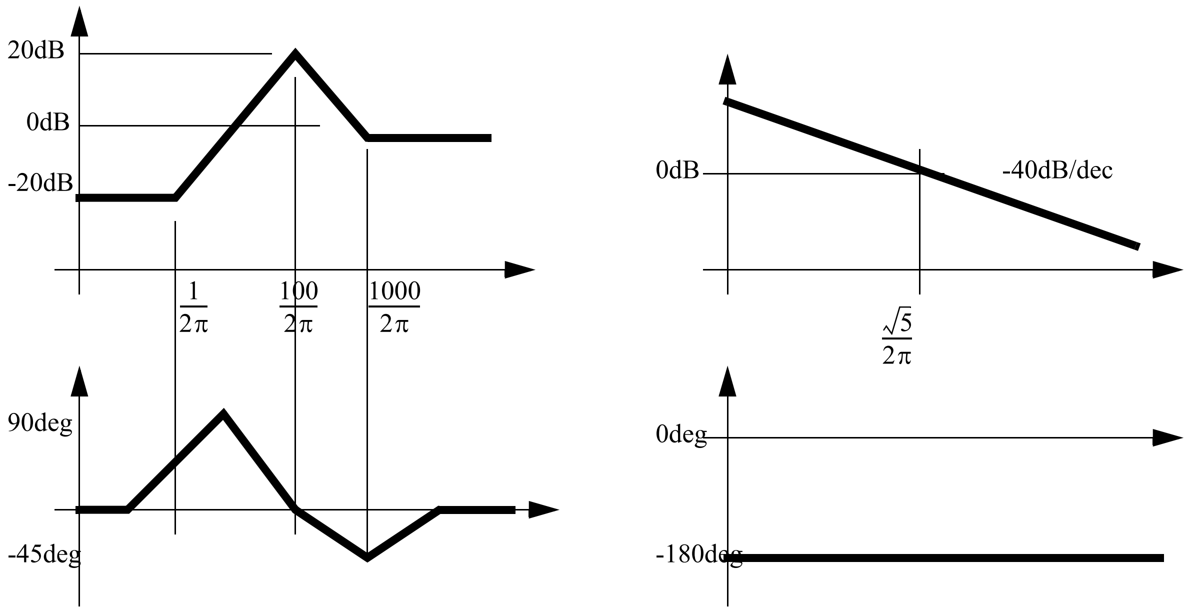
$$y = 3\left(\frac{14.142136–0.78539816}{9950.4–0.1007}\right) = \frac{3(14.142136)–(0.78539816 - 0.1007)}{9950.4}$$

$$y = 0.0042637891–0.68469816$$

$$y(t) = 0.02 + 0.00426\sin(10t + 0.685)$$


0.1
1
10
100
1000
f (Hz)
-40
-50
-60
-70
-80
-90
0deg
90 deg
-90 deg
-180

Answer 14.4


20dB
0dB
-20dB
1/2π
100/2π
1000/2π
90deg
-45deg
0dB
-40dB/dec
√5/2π
0deg
-180deg

Answer 14.5

for the numerator, (zero) $1000\frac{rad}{s} = 159Hz$

for the denominator, (poles)

$$D^2 + 2\omega_n\zeta D + \omega_n^2 = D^2 + 5D + 100$$

$$\omega_n^2 = 100$$

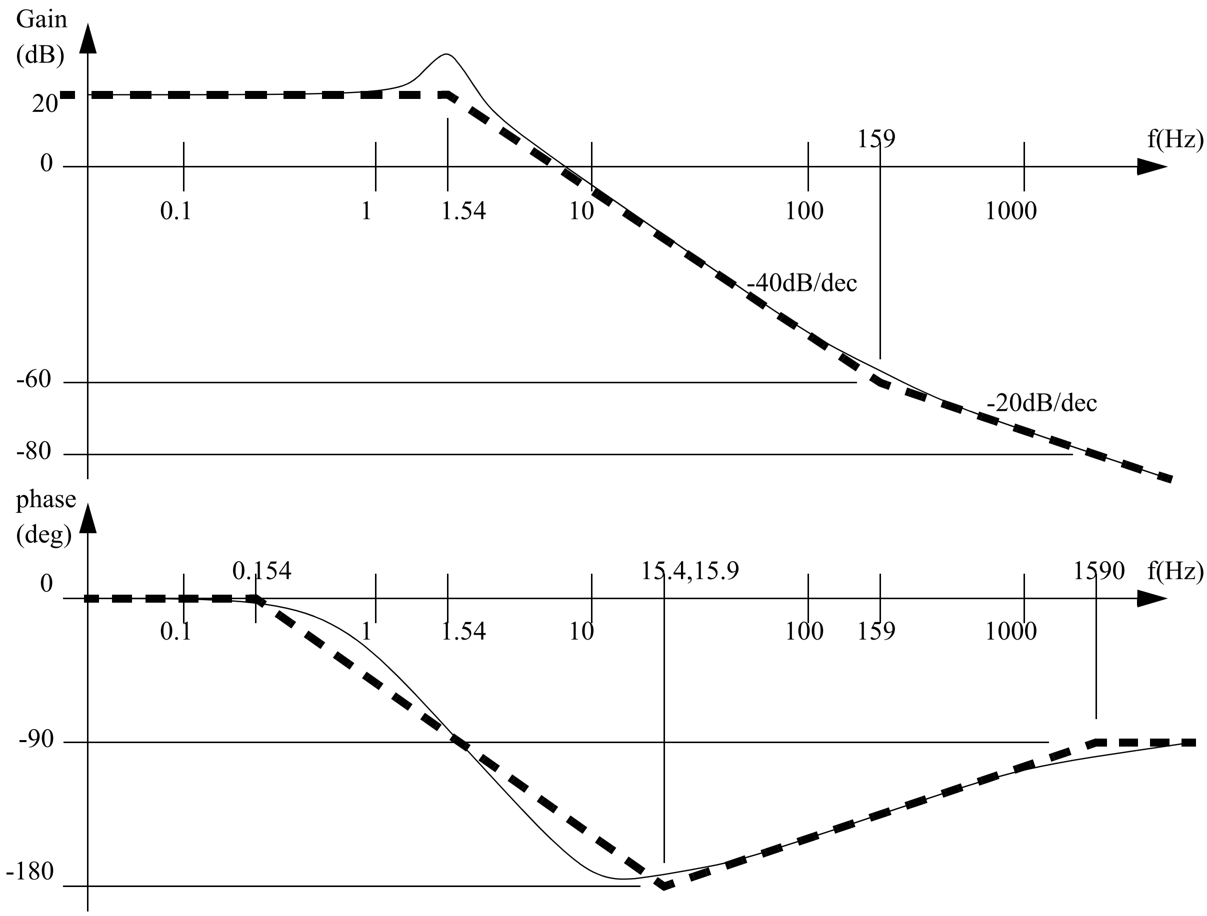
$$\backslash\omega_n = 10\frac{rad}{s} \qquad f_n = 1.59Hz$$

$$2\omega_n\zeta = 5$$

$$\backslash\zeta = \frac{5}{2(10)} = 0.25 \qquad \text{(underdamped)}$$

$$f_d = f_n\sqrt{1-\zeta^2} = 1.54Hz$$

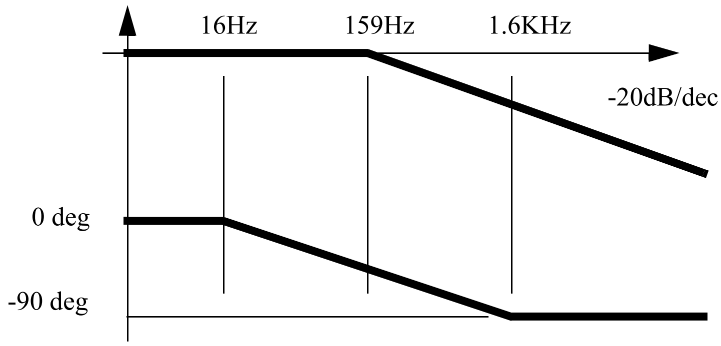
for the initial gain $G(0) = \frac{0 + 1000}{0^2 + 5(0) + 100} = 10 = 20dB$



Answer 14.6

a) $$V_o(t) = 0.050\sin(10^5 t - 1.561)$$

b)



Answer 14.7

a) $$\frac{x}{F} = \frac{0.0667}{D^2 + 33.3}\frac{m}{N}$$

b) $\omega = 1$ $\qquad \therefore D = 1j$

$$\frac{x}{F} = \frac{0.0667}{(1j)^2 + 33.3} = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}$$

$$F(t) = 10\cos(t + 1)N$$

$$\therefore F(\omega) = (10\angle 1 rad)N$$

$$\frac{x}{(10\angle 1 rad)N} = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}$$

$$\therefore x(\omega) = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}(10\angle 1 rad)N$$

$$\therefore x(\omega) = (10)2.07 \times 10^{-3} \angle (0 rad + 1 rad)m$$

$$\therefore x(\omega) = 0.0207\angle 1 rad m$$

$$\therefore x(t) = 0.0207\cos(t + 1)m$$

c) $$\frac{F}{x} = \frac{D^2 + 33.3}{0.0667}\frac{N}{m}$$

d) for the denominator, (poles)

$$D^2 + 2\omega_n \zeta D + \omega_n^2 = D^2 + 33.3$$

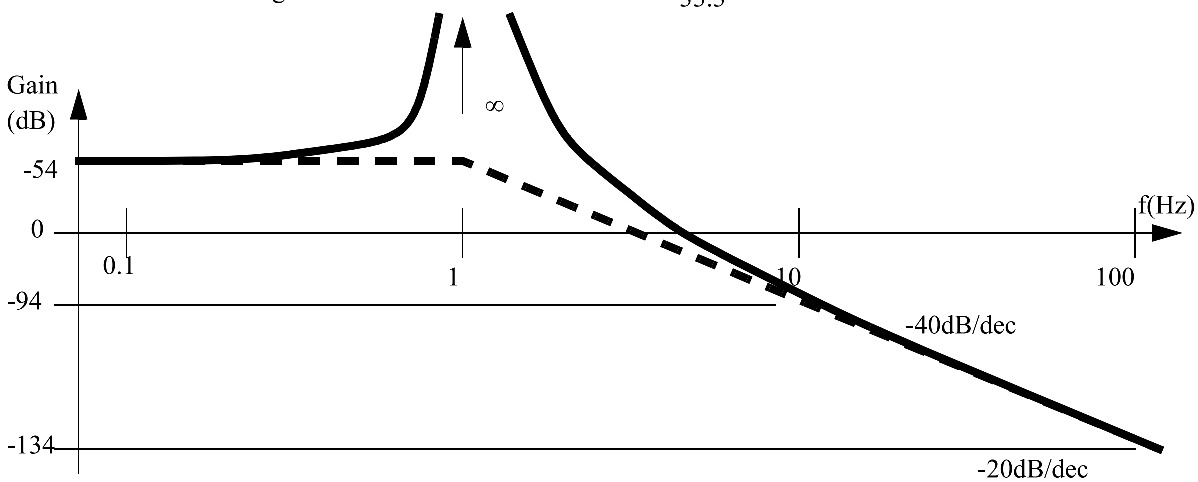
$$\omega_n = 5.77\frac{rad}{s} \qquad f_n = 0.918 Hz$$

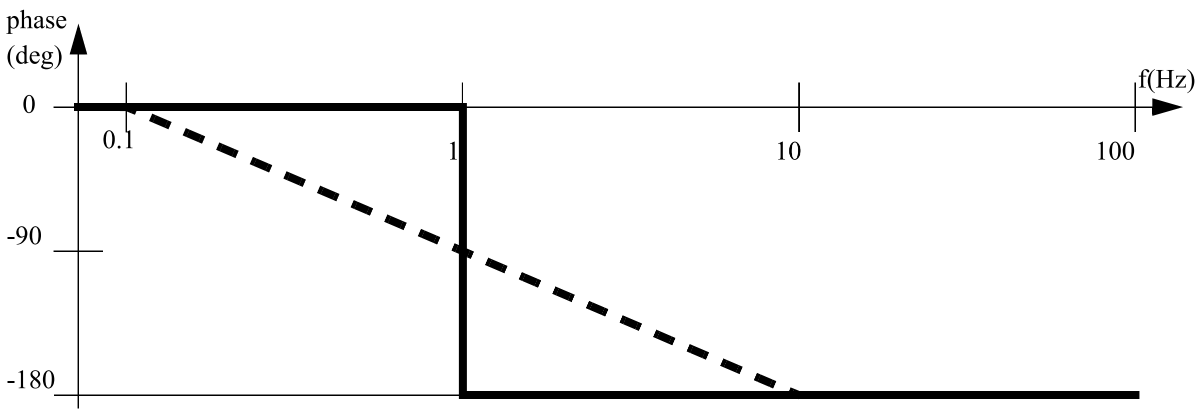
$$\zeta = 0 \qquad \text{(undamped)}$$

$$f_d = f_n\sqrt{1-\zeta^2} = 0.918 Hz$$

for the initial gain

$$G(0) = \frac{0.0667}{33.3} = 2 = -54 dB$$





e) $x(t) = (-0.020\cos(5.77t) + 0.020)m$

f) $x(t) = (-0.2162\cos(5.77t) + 0.2162)m$

Answer 14.8

a)

$$\dot{y} = v$$

$$\dot{v} = -v - 7y + F$$

b)

$$\begin{bmatrix} y_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} y_i \\ v_i \end{bmatrix} + h\left[\begin{bmatrix} 0 & 1 \\ -7 & -1 \end{bmatrix}\begin{bmatrix} y_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix}\right]$$

given $\begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad F = 10$

using h=0.001s

$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.001\left[\begin{bmatrix} 0 & 1 \\ -7 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}\right]$$

etc. until $\begin{bmatrix} y_{100} \\ v_{100} \end{bmatrix} = \begin{bmatrix} 1.428\text{m} \\ 0.000\frac{\text{m}}{\text{s}} \end{bmatrix}$

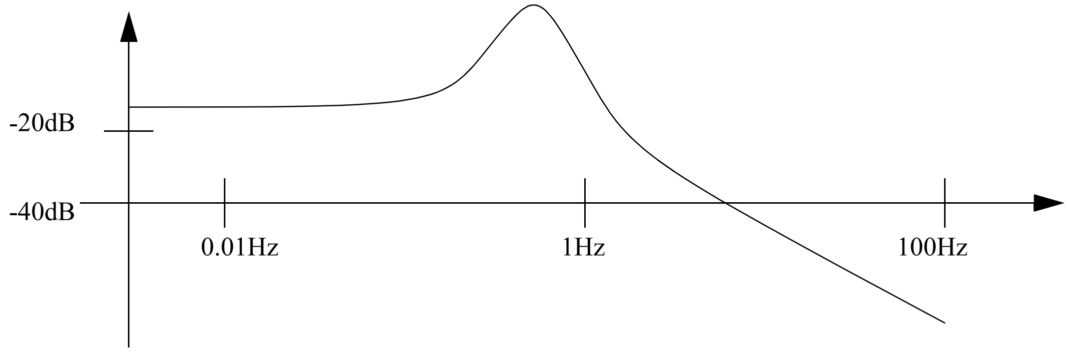
c)

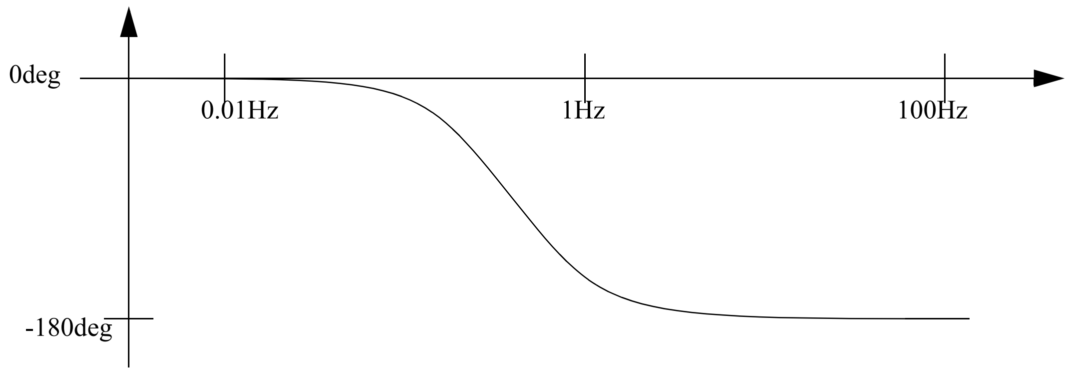
$$y(t) = (-0.437e^{-0.5t}\cos(2.598t - 0.19) + 1.429)\text{m}$$

d) $$\frac{y}{F} = \frac{1}{D^2 + D + 7} = \frac{1}{(j\omega)^2 + j\omega + 7} = \frac{1}{(7 - \omega^2) + j(\omega)}$$

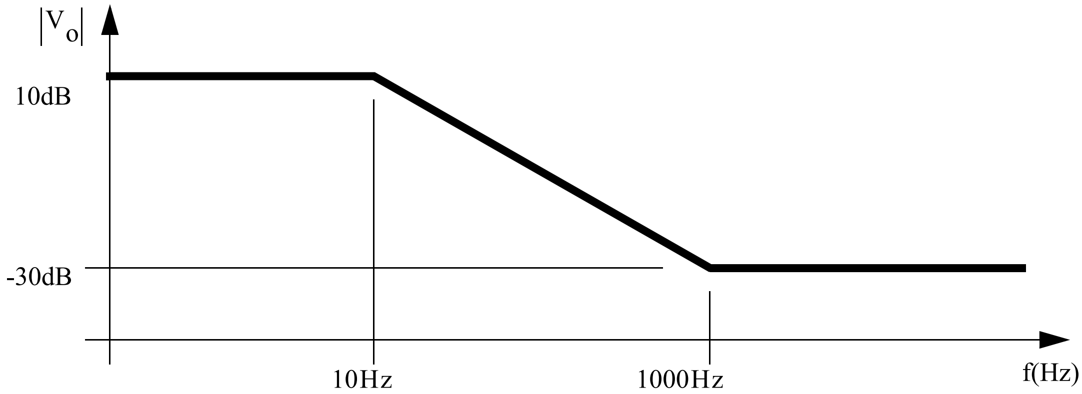
$$\left|\frac{y}{F}\right| = \frac{1}{\sqrt{(7 - \omega^2)^2 + \omega^2}}$$

$$-\theta = \frac{-0}{-\text{angle}(\omega, 7 - \omega^2)} = -(0 - \text{angle}(\omega, 7 - \omega^2)) = -\text{angle}(\omega, 7 - \omega^2)$$





Answer 14.9



$$10(2\pi) = 62.8\frac{\text{rad}}{\text{s}} \qquad 1000 = 6283\frac{\text{rad}}{\text{s}}$$

$$\frac{V_o}{V_i} = A\frac{s + 6283}{s + 62.8}$$

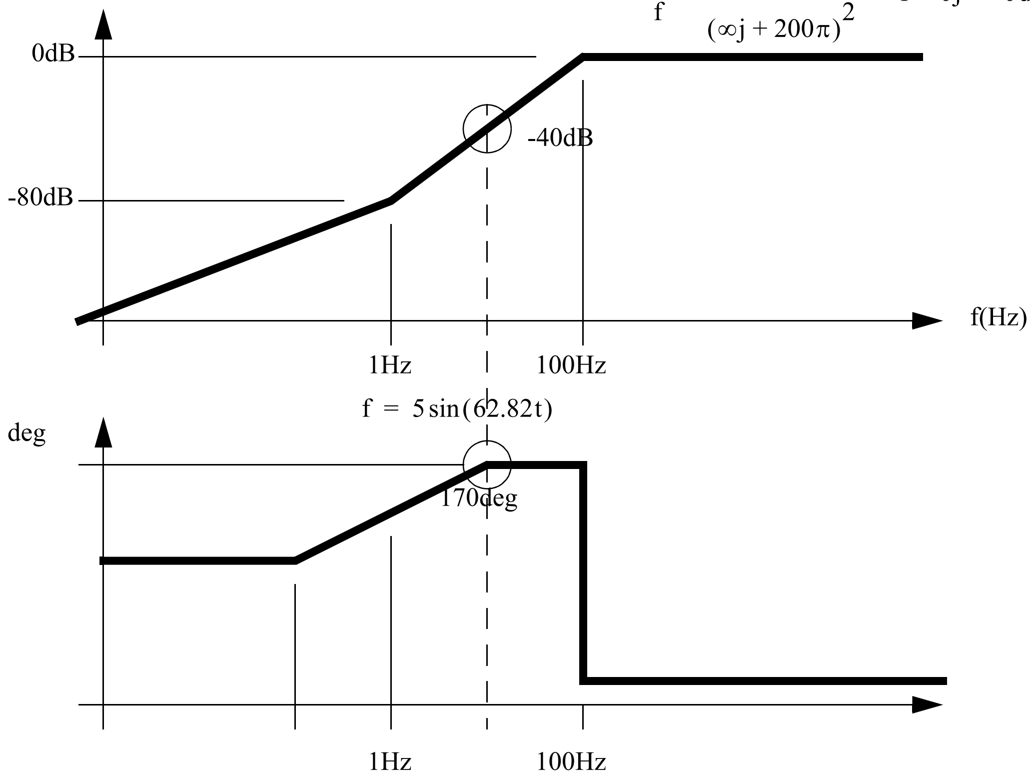
$$10^{\frac{10}{20}} = A\frac{0 + 6283}{0 + 62.8} \qquad A = 10^{\frac{10}{20}}0.01 = 31.6\times10^{-3}$$

$$\frac{V_o}{V_i} = 31.6\times10^{-3}\frac{s + 6283}{s + 62.8}$$

check: $$\frac{V_o}{V_i} = 31.6\times10^{-3}\frac{\infty j + 6283}{\infty j + 62.8} = 31.6\times10^{-3} = -30\text{dB}$$

Answer 14.10

$$\frac{x}{f} = \frac{\infty j(\infty j + 2\pi)}{(\infty j + 200\pi)^2} = 1 + 0j = 0dB\angle 0$$

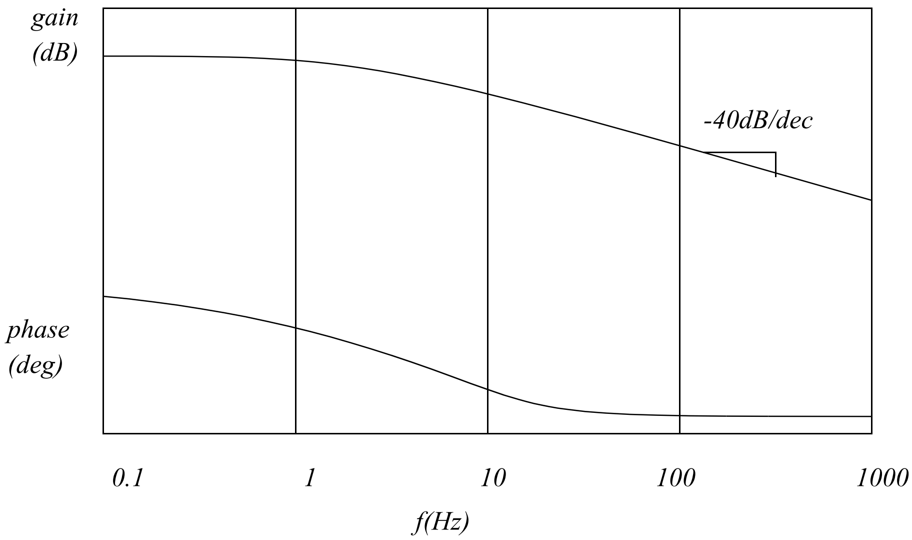


$$x = 5\left(10^{\frac{-40}{20}}\right)\sin\left(62.82t + 170\frac{\pi}{180}\right) = 0.05\sin(62.82t + 2.97)$$

$$x = \frac{j62.82(j62.82 + 2\pi)}{(j62.82 + 200\pi)^2}(5 + 0j) = 0.0497\angle 162.9° = 0.0497\sin(62.82t + 2.843)$$

14.9 Problems Without Solutions

Problem 14.11 Plot the points from Figure 14.3 in the semi-logarithmic scale graph paper in Figure 14.5. The general layout is pictured below.



Problem 14.12 Draw the straight line approximation for the transfer function.

$$\frac{D+3}{D^2 + 10000D + 10000}$$

Problem 14.13 Show that the second order transfer function below would result in a slope of +/- 40 dB/decade.

$$G(j\omega) = \frac{1}{(\omega_c + j\omega)^2}$$

Problem 14.14 For the transfer functions below, draw the bode plots using computer software.

$$\frac{(D+2\pi)D}{(D+200\pi)(D+0.02\pi)} \qquad \frac{(D+2\pi)}{D^2 + 50\pi D + 10000\pi^2} \qquad \frac{r}{c} = \frac{D^2 + 2(1)(1)D + 1^2}{D^2 + 2(0.1)(10)D + 10^2}$$

Problem 14.15 Draw Bode plots for the following functions using straight line approximations.

$$\frac{1}{D+1} \qquad \frac{1}{D^2+1} \qquad \frac{1}{(D+1)^2} \qquad \frac{1}{D^2+2D+2}$$

Problem 14.16 Use computer software, such as Mathcad or a spreadsheet, to calculate the points in Figure 14.2, and then draw Bode plots. Most software will offer options for making one axis use a base 10 logarithm scale.

Problem 14.17 Draw the Bode plot for the transfer function by hand or with computer.

$$\frac{D+3}{D^2 + 10000D + 10000}$$

Problem 14.18 Given the transfer function below,

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2}$$

a) Draw the straight line approximation of the bode plot.

b) Determine the steady state output if the input is x(s) = 20 cos(9t+.3) using the straight line plots.
c) Use an exact method to verify part b).

Problem 14.19 a) Convert the following differential equation to a transfer function.

$$5\ddot{x} + 2x = 3F$$

b) Apply a phasor (Fourier) transform to the differential equation and develop equations for the system gain and phase shift as a function of input frequency.
c) Draw a Bode plot using the equations found in part b) on the attached semi-logarithmic scale graph paper.
d) Draw a straight line approximation of the system transfer function on the attached semi-logarithmic scale graph paper.

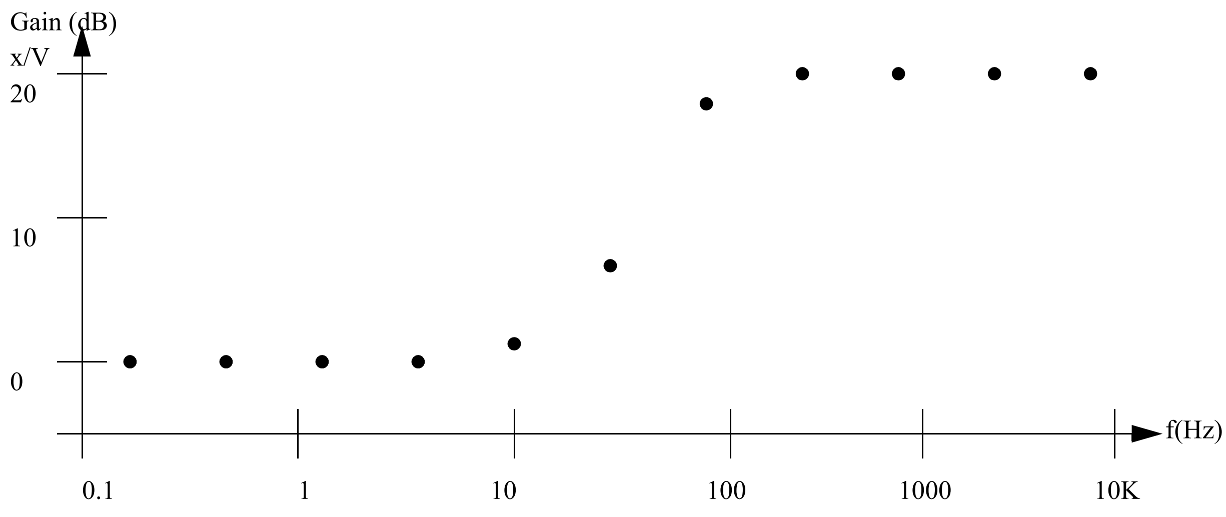
Problem 14.20 For the following transfer function,
a) Draw the Bode plot on the attached semi-logarithmic scale graph paper.
b) Given an input of F=5sin(62.82t), find the output, x, using the Bode plot.
c) Given an input of F=5sin(62.82t), find the output, x, using phasors.

$$\frac{x}{F} = \frac{D^2}{(D + 200\pi)^2}$$

Problem 14.21 For the following transfer function,
a) Draw the Bode plot on the attached semi-logarithmic scale graph paper.
b) Given an input of F=5sin(62.82t), find the output, x, using the Bode plot.
c) Given an input of F=5sin(62.82t), find the output, x, using phasors.

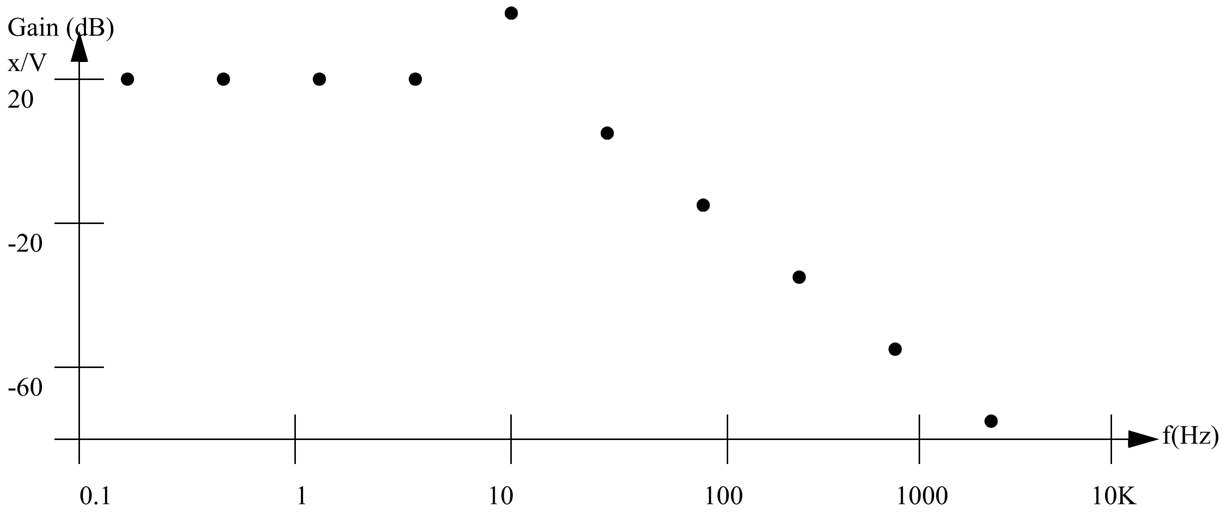
$$\frac{x}{F} = \frac{D^2(D + 2\pi)}{(D + 200\pi)^2}$$

Problem 14.22 Given the experimental Bode (Frequency Response Function) plot below, find a transfer function to model the system. The input is a voltage ‘V’ and the output is a displacement ‘x’. (Hint: after calculating the function use Scilab to verify the results.)



Problem 14.23 Given the experimental Bode (Frequency Response Function) plot below, find a transfer function to model the system. The input is a voltage ‘V’ and the output is a displacement ‘x’. (Hint: after calculating the function use

Scilab to verify the results.)



Problem 14.24 For the transfer function,

$$\frac{D(D+2\pi)}{D^2 + 300\pi D + 62500\pi^2}$$

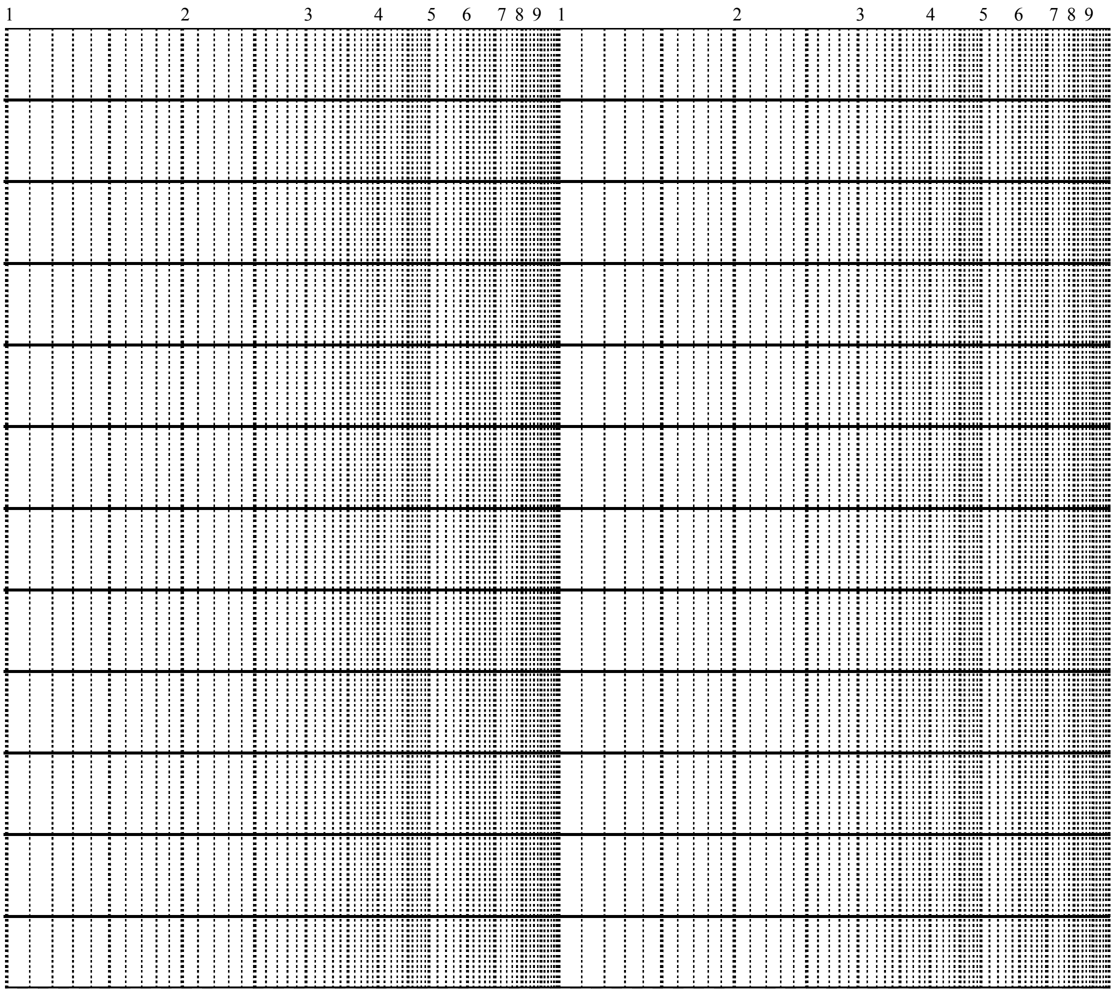
a) Use the straight line method with semi-logarithm scaled paper to draw an approximate Bode plot.
b) Verify the Bode plot by calculating values at a few points.
c) Use the Bode plot to find the response to an input of 5sin(624t) + 1sin(6.2t).

Answer 14.24 a)
b)
c)

$$8.89\sin(624t + 1.571) + 0.141\times10^{-3}\sin(6.2t + 2.356)$$

14.10 Semi-Logarithmic Scale Graph Paper

Please notice that there are a few sheets of 2 and 4 cycle semi-logarithmic scale paper attached, make additional copies if required, and if more cycles are required, sheets can be cut and pasted together. Also note that better semi-logarithmic graph paper can be purchased at technical bookstores, as well at most large office supply stores.


1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9

14.11 Exponents and Logarithms Review

• The basic properties of exponents are so important they demand some sort of mention

$$(x^n)(x^m) = x^{n+m} \qquad x^0 = 1 \text{ , if x is not 0} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\frac{(x^n)}{(x^m)} = x^{n-m} \qquad x^{-p} = \frac{1}{x^p} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

$$(x^n)^m = x^{n \cdot m} \qquad (xy)^n = (x^n)(y^n) \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Figure 14.22 Properties of exponents

• Logarithms also have a few basic properties of use,

The basic base 10 logarithm:

$$\log x = y \qquad x = 10^y$$

The basic base n logarithm:

$$\log_n x = y \qquad x = n^y$$

The basic natural logarithm (e is a constant with a value found near the start of this section:

$$\ln x = \log_e x = y \qquad x = e^y$$

Figure 14.23 Definitions of logarithms

• All logarithms observe a basic set of rules for their application,

$$\log_n(xy) = \log_n(x) + \log_n(y) \qquad \log_n(n) = 1$$

$$\log_n(1) = 0$$

$$\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$$

$$\log_n(x^y) = y\log_n(x)$$

$$\log_n(x) = \frac{\log_m(x)}{\log_m(n)}$$

$$\ln(A\angle\theta) = \ln(A) + (\theta + 2\pi k)j \qquad k \in I$$

Figure 14.24 Properties of logarithms

Problems

Problem 14.25 Are the following expressions equivalent?

a) $\log(ab) = \log(a) + \log(b)$

b) $3\log(4) = \log(16)$

c) $10^{\log(5)} = \frac{10}{5}$

Problem 14.26 Simplify the following expression.

$$\log(x^3)$$

Problem 14.27 Simplify the following expression.

$$\log(x^5) + \log(x^3)$$

Answer 14.27

$$\log(x^5) + \log(x^3) = 5\log(x) + 3\log(x) = 8\log(x)$$

Problem 14.28 Simplify the following expressions.

$$n\log(x^2) + m\log(x^3) - \log(x^4)$$

Answer 14.28

$$n\log(x^2) + m\log(x^3) - \log(x^4)$$

$$2n\log(x) + 3m\log(x) - 4\log(x)$$

$$(2n + 3m - 4)\log(x)$$

$$(2n + 3m - 4)\log(x)$$

15. Root Locus Analysis

Topic 15.1 Root-locus plots.
Objective 15.1 To be able to predict and control system stability.

The system can also be checked for general stability when controller parameters are varied using root-locus plots.

15.1 Root Locus Analysis

In a engineered system we may typically have one or more design parameters, adjustments, or user settings. It is important to determine if any of these will make the system unstable. This is generally undesirable and possibly unsafe. For example, think of a washing machine that vibrates so much that it 'walks' across a floor, or a high speed aircraft that fails due to resonant vibrations. Root-locus plots are used to plot the system roots over the range of a variable to determine if the system will become unstable, or oscillate.

Recall the general solution to a homogeneous differential equation. Complex roots will result in a sinusoidal oscillation. If the roots are real the result will be e-to-the-t terms. If the real roots are negative then the terms will tend to decay to zero and be stable, while positive roots will result in terms that grow exponentially and become unstable. Consider the roots of a second-order homogeneous differential equation, as shown in Figure 15.1 to Figure 15.7. These roots are shown on the complex planes on the left, and a time response is shown to the right. Notice that in these figures (negative real) roots on the left hand side of the complex plane cause the response to decrease while roots on the right hand side cause it to increase. The rule is that any roots on the right hand side of the plane make a system unstable. Also note that the complex roots cause some amount of oscillation.

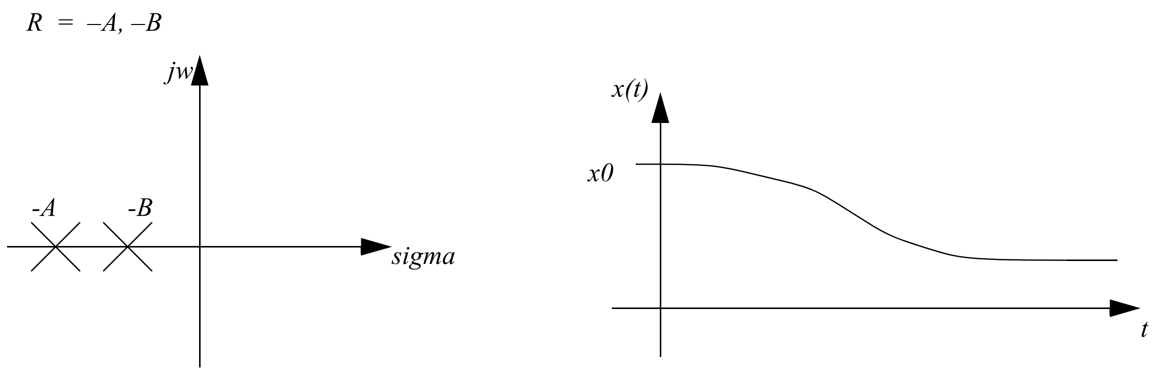


Figure 15.1 Negative real roots make a system stable

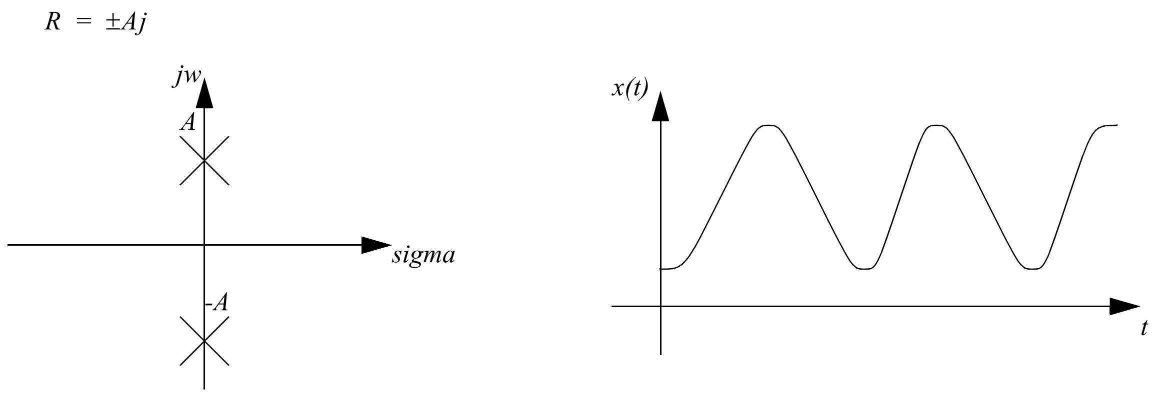


Figure 15.2 Complex roots make a system oscillate

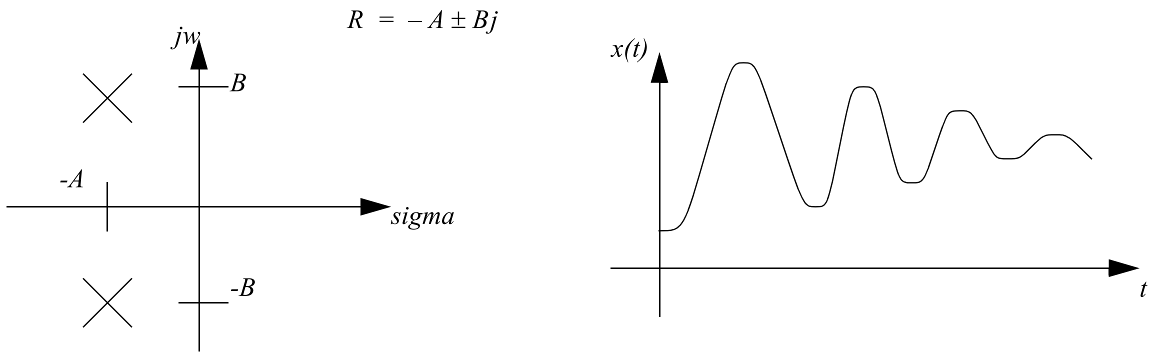


Figure 15.3 Negative real and complex roots cause decaying oscillation

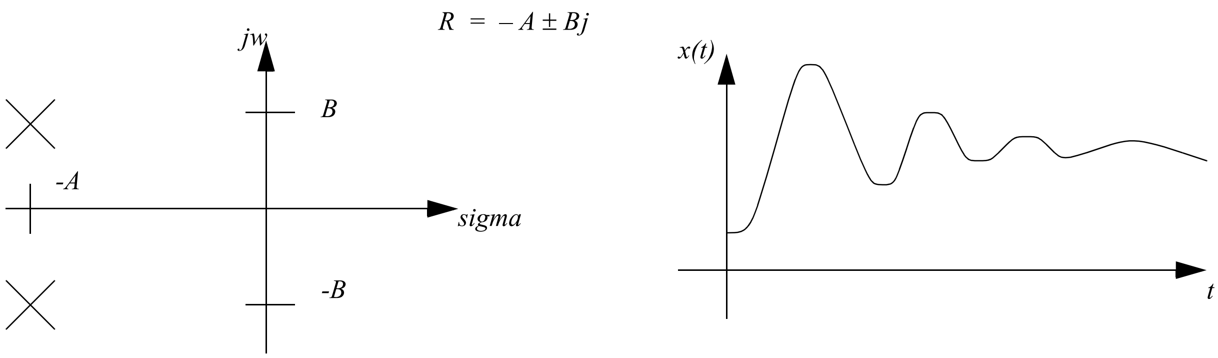


Figure 15.4 More negative real and complex roots cause a faster decaying oscillation

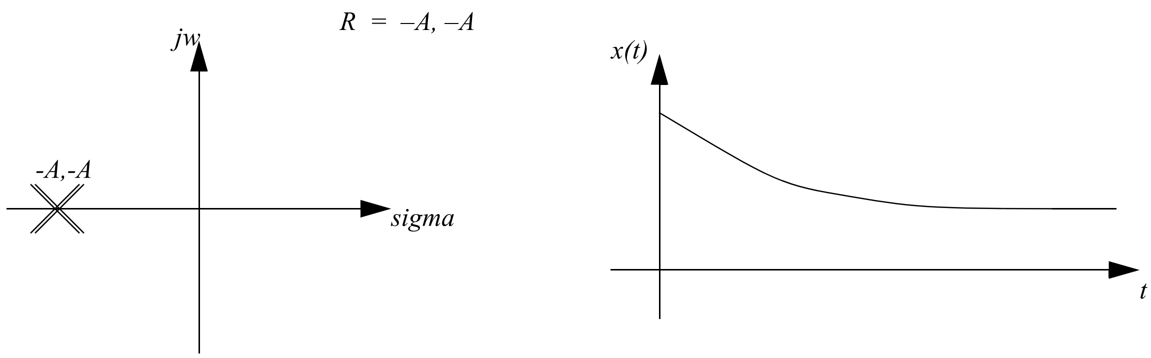


Figure 15.5 Overlapped roots are possible

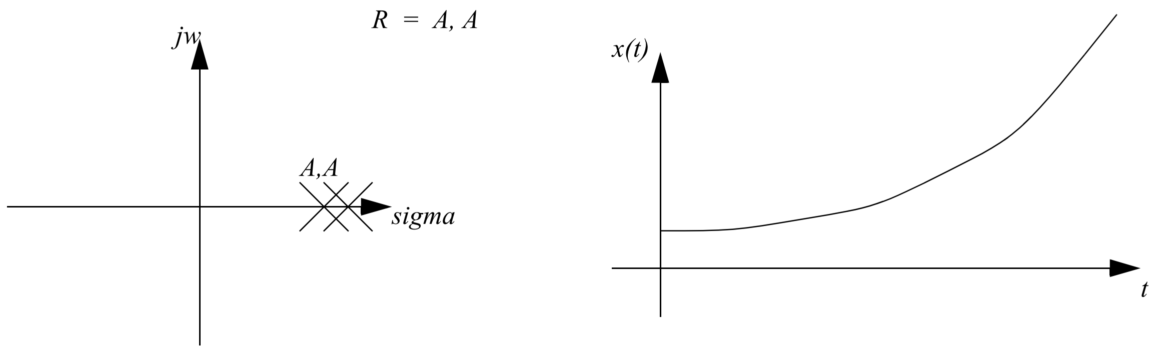


Figure 15.6 Positive real roots cause exponential growth and are unstable

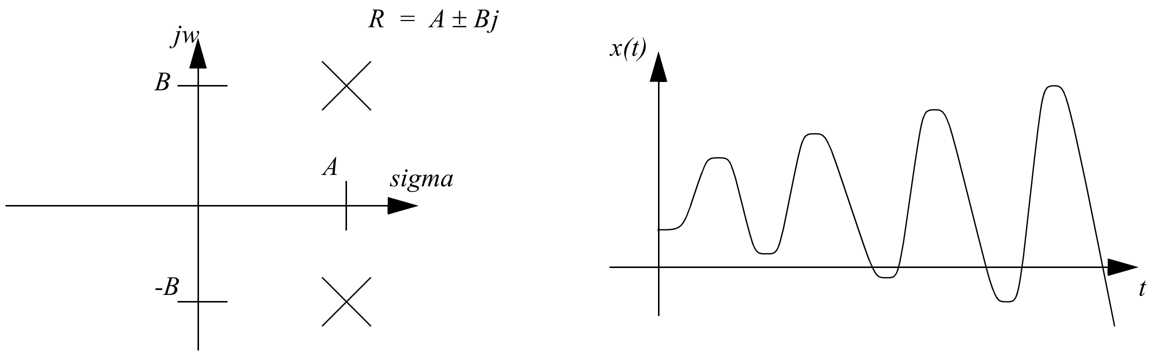


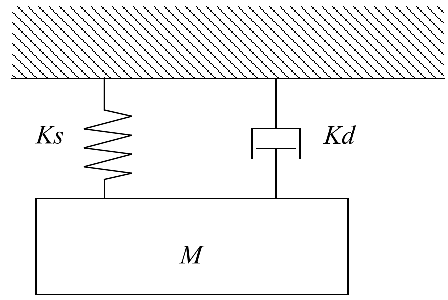
Figure 15.7 Complex roots with positive real parts have growing oscillations and are unstable

Next, recall that the denominator of a transfer function is the homogeneous equation. By analyzing the function in the denominator of a transfer function the general system response can be found. An example of root-locus analysis for a mass-spring-damper system is given in Figure 15.8. In this example the transfer function is found and the roots of the equation are written with the quadratic equation. At this point there are three unspecified values that can be manipulated to change the roots. The mass and damper values are fixed, and the spring value will be varied. The range of values for the spring coefficient should be determined by practical and design limitations. For example, the spring coefficient should not be zero or negative.

$$\frac{x(D)}{F(D)} = \frac{1}{MD^2 + K_d D + K_s}$$

Note: We want the form below

$$\frac{A}{(D+B)(D+C)}$$



$$\frac{x(D)}{F(D)} = \frac{\frac{1}{M}}{D^2 + \frac{K_d}{M}D + \frac{K_s}{M}}$$

$$B, C = \frac{\frac{-K_d}{M} \pm \sqrt{\frac{K_d^2}{M^2} - 4\frac{K_s}{M}}}{2}$$

Aside:

$$ax^2 + bx + c = (x+A)(x+B)$$

$$A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

M	Kd	Ks	B	C
0	0	0	0	0
100	100	100		
100	100	1000		
100	100	10000		

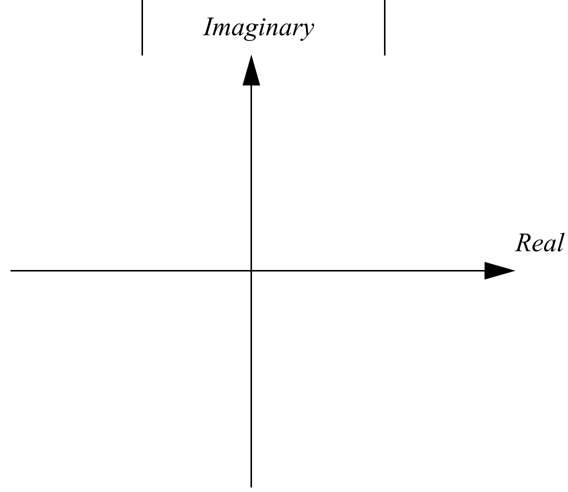
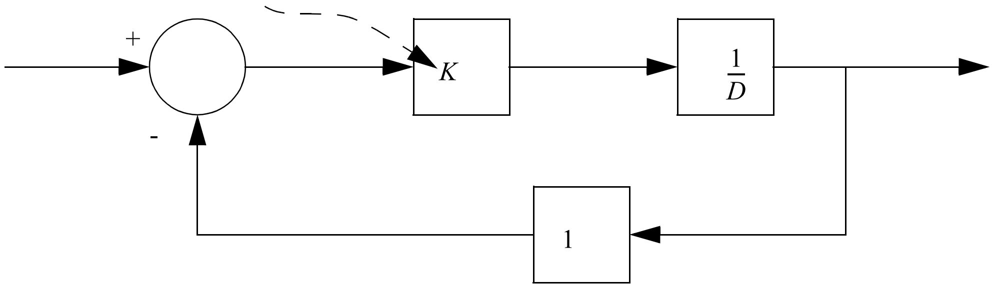


Figure 15.8 A mass-spring-damper system equation

The roots of the equation can then be plotted to provide a root locus diagram. These will show how the values of the roots change as the design parameter is varied. If any of these roots pass into the right hand plane we will know that the system is unstable. In addition complex roots will indicate oscillation.

A feedback controller with a variable control function gain is shown in Figure 15.9. The variable gain 'K' necessitates the evaluation of controller stability over the range of operating values. This analysis begins by developing a transfer function for the overall system. The root of the denominator is then calculated and plotted for a range of 'K' values. In this case all of the roots are on the left side of the plane, so the system is stable and doesn't oscillate. Keep in mind that gain values near zero put the control system close to the right hand plane. In real terms this will mean that the controller becomes unresponsive, and the system can go where it pleases. It would be advisable to keep the system gain greater than zero to avoid this region.

Note: This controller has adjustable gain. After this design is built we must anticipate that all values of K will be used. It is our responsibility to make sure that none of the possible K values will lead to instability.



$$G(D) = \frac{K}{D} \qquad H(D) = 1$$

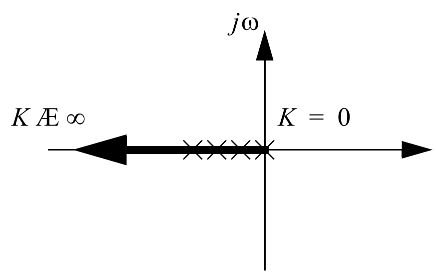
First, we must develop a transfer function for the entire control system.

$$G_S(D) = \frac{G(D)}{1 + G(D)H(D)} = \frac{\left(\frac{K}{D}\right)}{1 + \left(\frac{K}{D}\right)(1)} = \frac{K}{D + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$$D + K = 0$$

K	root
0	-0
1	-1
2	-2
3	-3
etc...	

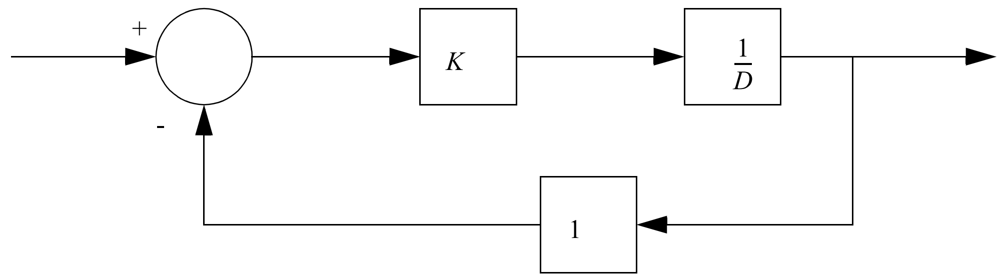


Note: This system will always be stable because all of the roots for all values of K are negative real, and it will always have a damped response. Also, larger values of K, make the system more stable.

Figure 15.9 Root-locus analysis in controller design

15.2 Computer Based Root Locus Analysis

Aside: Scilab can be used to draw root locus plots for systems of the form below, where there is a simple gain, K, multiplying the open loop gain, G(s).



$$G(D) = \frac{1}{D} \qquad H(D) = 1$$

First, we must multiply G and H

$$G(s)H(s) = \left(\frac{1}{D}\right)(1) = \frac{1}{D}$$

The numerator and denominator of this equation are then defined and plotted using the 'evans' function.

```
D = poly(0, 'D'); // define the differential operator
n = real(1.0); // define the numerator of GH
d = real(D); // define the denominator of GH
evans(n, d, 100); // plot for gains from K=0 to 100
```

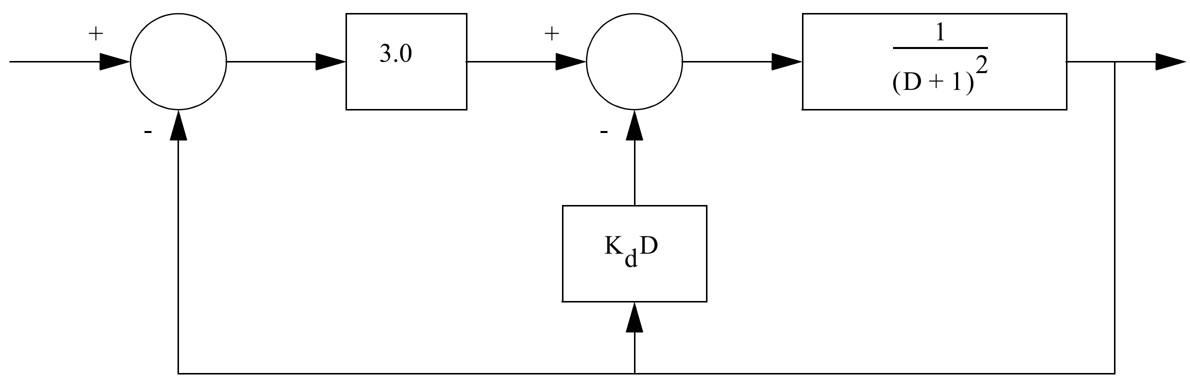
Figure 15.10 Root-locus plotting in Scilab

15.3 Summary

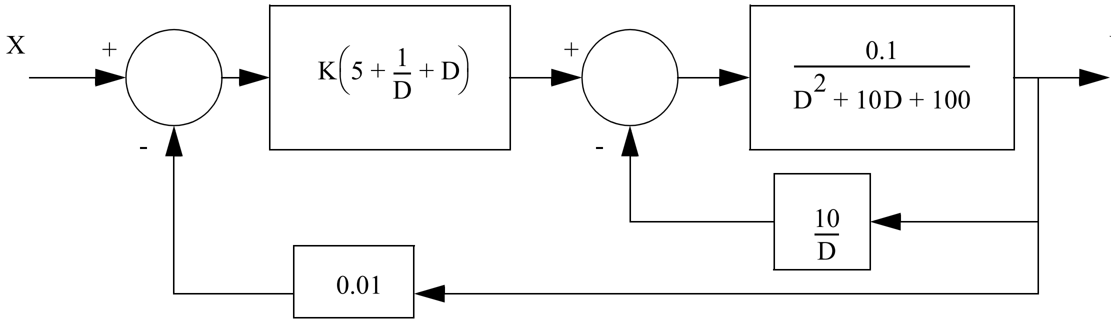
- Root-locus plots show the roots of a transfer function denominator to determine stability

15.4 Problems with Solutions

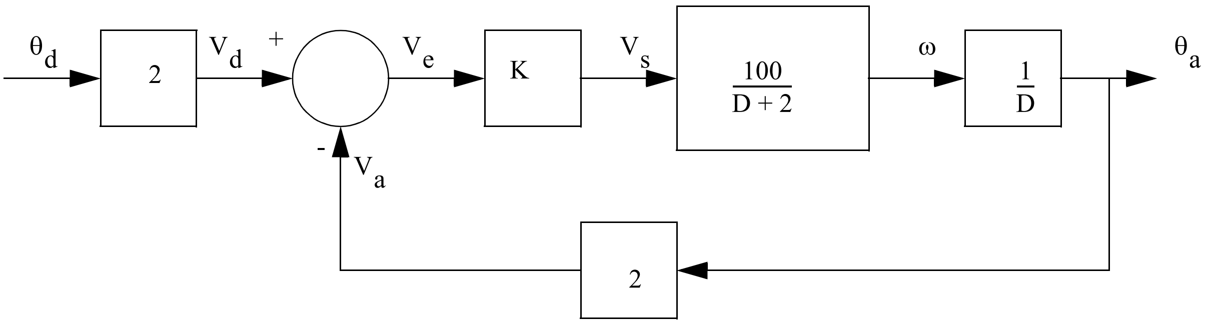
Problem 15.1 Draw the root locus diagram for the system below. specify all points and values.



Problem 15.2 Draw a root locus plot for the control system below and determine acceptable values of K, including critical points.



Problem 15.3 The block diagram below is for a motor position control system. The system has a proportional controller with a variable gain K.



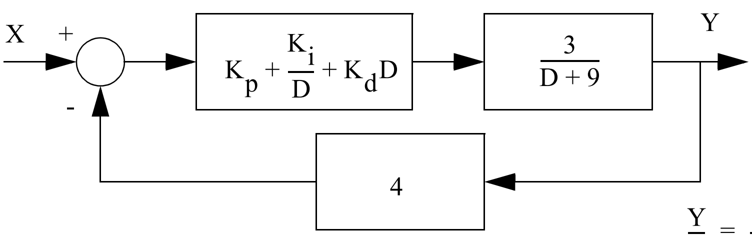
a) Simplify the block diagram to a single transfer function.
b) Draw the Root-Locus diagram for the system (as K varies). Use either the approximate or exact techniques.
c) Select a K value that will result in an overall damping factor of 1. State if the Root-Locus diagram shows that the system is stable for the chosen K.

Problem 15.4 Given the system transfer function below.

$$\frac{\theta_o}{\theta_d} = \frac{20K}{D^2 + D + 20K}$$

a) Draw the root locus diagram and state what values of K are acceptable.
b) Select a gain value for K that has either a damping factor of 0.707 or a natural frequency of 3 rad/sec.
c) Given a gain of K=10 find the steady-state response to an input step of 1 rad.
d) Given a gain of K=0.01 find the response of the system to an input step of 0.1rad.

Problem 15.5 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer function is given.



$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

a) Verify the close loop controller function given.
b) Draw a root locus plot for the controller if Kp=1 and Ki=1. Identify any values of Kd that would leave the system unstable.
c) Draw a Bode plot for the feedback system if Kd=Kp=Ki=1.
d) Select controller values that will result in a natural frequency of 2 rad/sec and damping factor of 0.5. Verify

that the controller will be stable.

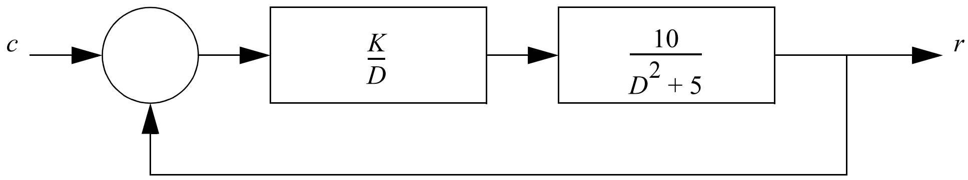
e) For the parameters found in the last step can the initial values be found?

f) If the values of Kd=1 and Ki=Kp=0, find the response to a unit ramp input as a function of time.

Problem 15.6 For the system transfer function below a) Create a root locus plot for various values of Kp from 0.0 to 2.0, if Ki = 0.1. Indicate the range of gain values where the system is stable and/or oscillates. b) Create a Bode plot for Kp = 1, and Ki = 1. c) If there is a steady state input of 10sin(10t), what is the steady state output if Ki=1 and Kp=1?

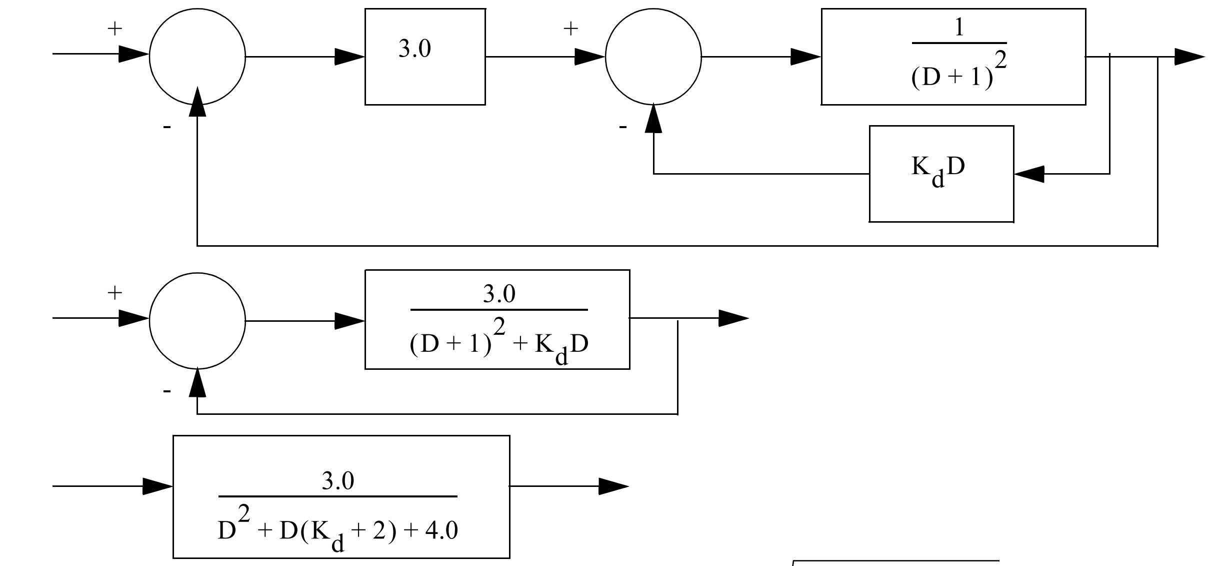
$$\frac{r}{c} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{D^3 + D^2(10K_p + 2) + D(20K_p + K_i) + (20K_i)}$$

Problem 15.7 Develop a root locus plot for various positive values of Kp. Indicate the range of Kp values that would result in a stable system.



15.5 Problem Solutions

Answer 15.1



$$D^2 + D(K_d + 2) + 4.0 = 0$$

$$D = \frac{-K_d - 2 \pm \sqrt{(K_d + 2)^2 - 4(4.0)}}{2}$$

$$D = \frac{-K_d - 2 \pm \sqrt{K_d^2 + 4K_d - 12}}{2}$$

Kd	roots
0	-1 +/- 1.732j
1	-1.5 +/- 1.323j
2	-2.000, -2.000
5	-0.628, -6.372
10	-0.343, -11.657
100	-0.039, -102.0
1000	-0.004, -1000

Critical points: (this is simple for a quadratic)

The roots becomes positive when

$$0 > -K_d - 2 \pm \sqrt{K_d^2 + 4K_d - 12}$$

$$2 + K_d > \pm\sqrt{K_d^2 + 4K_d - 12}$$

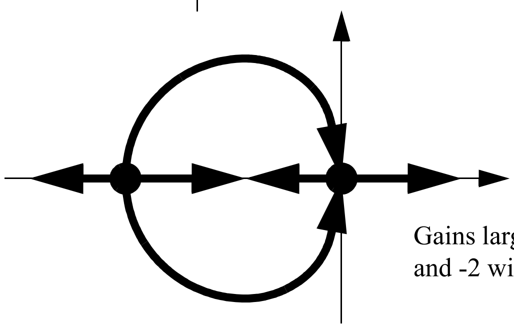
$$16 > 0$$

$$0 > -K_d - 2 \qquad K_d > -2$$

The roots becomes complex when

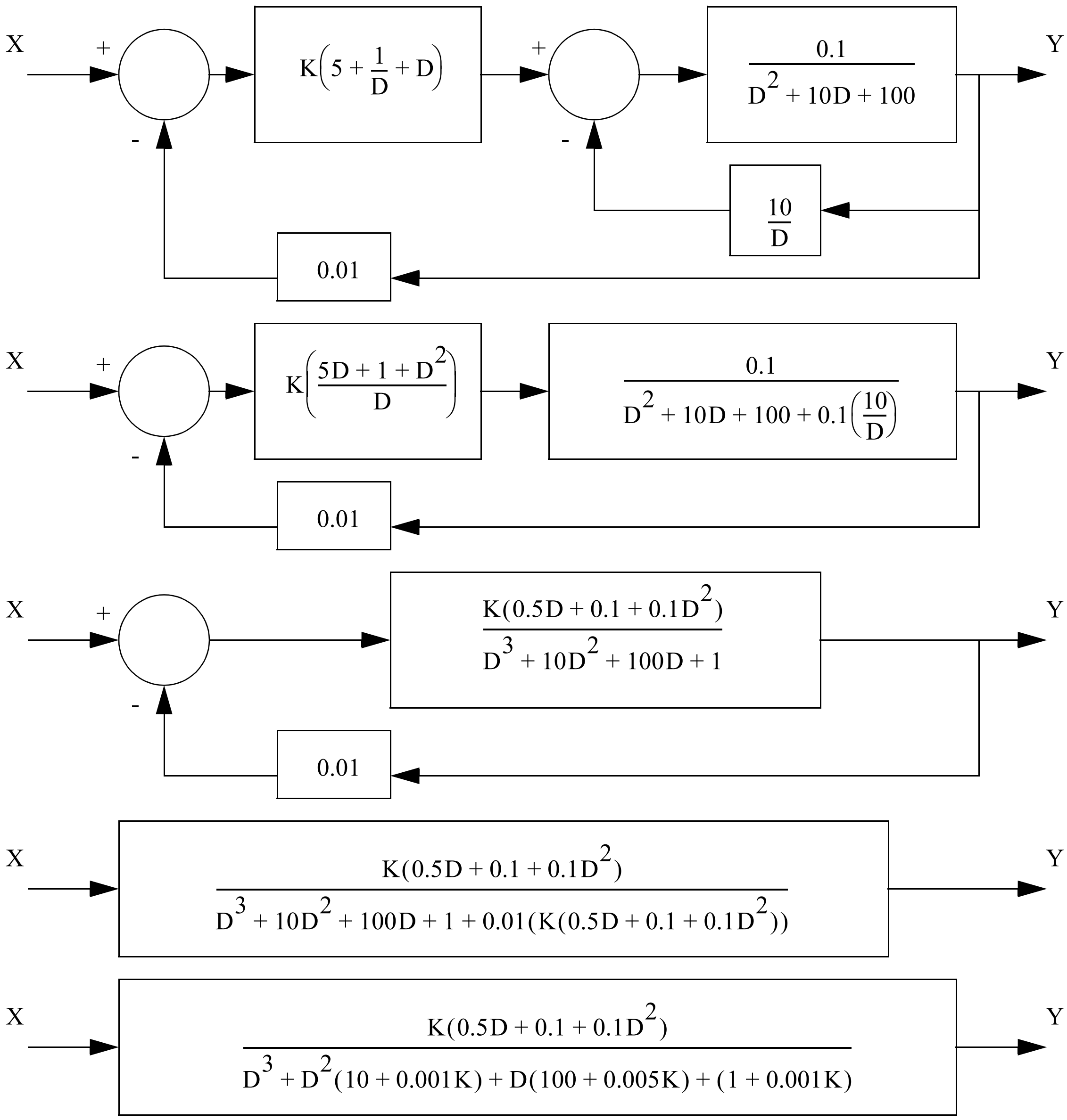
$$0 > K_d^2 + 4K_d - 12$$

$$K_d = \frac{-4 \pm \sqrt{16 - 4(-12)}}{2} \qquad K_d = -6, 2$$



Gains larger than -2 will result in a stable system. Any gains between -4 and -2 will result in oscillations.

Answer 15.2


X
+
-
K(5 + 1/D + D)
+
-
0.1 / (D^2 + 10D + 100)
10/D
0.01
Y
X
+
-
K((5D + 1 + D^2)/D)
0.1 / (D^2 + 10D + 100 + 0.1(10/D))
0.01
Y
X
+
-
K(0.5D + 0.1 + 0.1D^2) / (D^3 + 10D^2 + 100D + 1)
0.01
Y
X
K(0.5D + 0.1 + 0.1D^2) / (D^3 + 10D^2 + 100D + 1 + 0.01(K(0.5D + 0.1 + 0.1D^2)))
Y
X
K(0.5D + 0.1 + 0.1D^2) / (D^3 + D^2(10 + 0.001K) + D(100 + 0.005K) + (1 + 0.001K))
Y

Given the homogeneous equation for the system,

$$D^3 + D^2(10 + 0.001K) + D(100 + 0.005K) + (1 + 0.001K) = 0$$

The roots can be found with a calculator, Mathcad, or equivalent.

K	roots	notes
-100,000	94.3, -3.992, -0.263	
-1000	0, -4.5+/-8.65j	roots become negative
-10	-0.0099, -4.99+/-8.66j	
0	-0.01, -4.995+/-8.657j	
10	-0.01, -5+/-8.66j	
1000	-0.019, -5.49+/-8.64j	
17165.12	-0.099, -13.52, -13.546	roots become real
100,000	-0.0174, -104.3, -5.572	

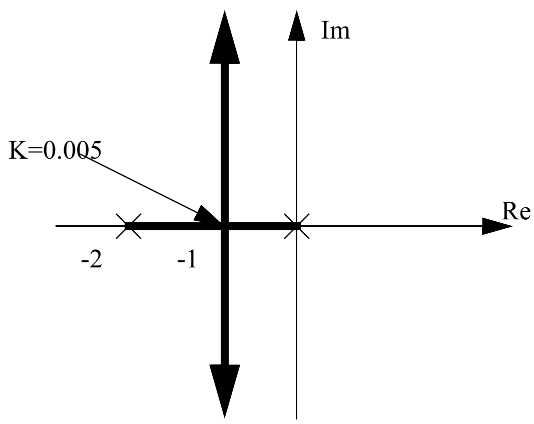
Answer 15.3 a)

$$\frac{200K}{D^2 + 2D + 200K}$$

b)

$$\text{roots} = \frac{-2 \pm \sqrt{4 - 4(200K)}}{2} = -1 \pm \sqrt{1 - 200K}$$

K	roots
0	0,-2
0.001	-0.1,-1.9
0.005	-1,-1
0.1	etc.
1	
5	
10	



c)

$$D^2 + 2D + 200K = D^2 + 2\zeta\omega_n D + \omega_n^2 \qquad \backslash\omega_n = 1 \qquad \backslash K = 0.005$$

From the root locus graph this value is critically stable.

Answer 15.4 a)

$$D^2 + D + 20K = 0$$

$$D = \frac{-1 \pm \sqrt{1 - 4(20K)}}{2}$$

For complex roots

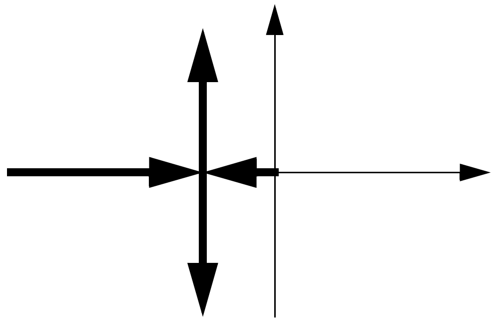
$$1 - 80K < 0 \qquad K > \frac{1}{80}$$

K	roots
0	0.000, -1.000
1/80	-0.500, -0.500
1	-0.5 +/- 4.444j
10	-0.5 +/- 14.13j
1000	-0.5 +/- 141.4j

For negative real roots (stable)

$$\frac{-1 \pm \sqrt{1 - 80K}}{2} < 0$$

$$\pm\sqrt{1 - 80K} < 1 \qquad K > 0$$



b)

Matching the second order forms,

$$2\omega_n \xi = 1 \qquad \omega_n^2 = 20K$$

The gain can only be used for the natural frequency

$$K = \frac{20}{\omega_n^2} = \frac{20}{3^2} = 2.22$$

c)

$$\frac{\theta_o}{\theta_d} = \frac{20(10)}{D^2 + D + 20(10)}$$

$$\ddot{\theta}_o + \dot{\theta}_d + \theta_d 200 = 200\theta_d$$

Homogeneous:

$$A^2 + A + 200 = 0$$

$$A = \frac{-1 \pm \sqrt{1 - 4(200)}}{2} \qquad A = -0.5 \pm 14.1j$$

$$\theta_o(t) = C_1 e^{-0.5t} \sin(14.1t + C_2)$$

Particular:

$$\theta = A$$

$$0 + 0 + A200 = 200(1\text{rad}) \qquad A = 1\text{rad}$$

$$\theta_o(t) = 1\text{rad}$$

Initial Conditions (assume at rest):

$$\theta_o(t) = C_1 e^{-0.5t} \sin(14.1t + C_2) + 1\text{rad}$$

$$\theta_o(0) = C_1(1)\sin(14.1(0) + C_2) + 1\text{rad} = 0$$

$$C_1 \sin(C_2) = -1\text{rad} \qquad \text{eqn 15.1}$$

$$\theta'_o(t) = -0.5C_1 e^{-0.5t} \sin(14.1t + C_2) - 14.1C_1 e^{-0.5t} \cos(14.1t + C_2)$$

$$0 = -0.5C_1 \sin(C_2) - 14.1C_1 \cos(C_2)$$

$$14.1\cos(C_2) = -0.5\sin(C_2)$$

$$\frac{14.1}{-0.5} = \tan(C_2) \qquad C_2 = -1.54$$

$$C_1 = \frac{-1\text{rad}}{\sin(C_2)} = \frac{-1\text{rad}}{\sin(-1.54)} = 1.000\text{rad}$$

$$\theta_o(t) = (e^{-0.5t} \sin(14.1t - 1.54) + 1)(\text{rad})$$

d)

$$\frac{\theta_o}{\theta_d} = \frac{20(0.01)}{D^2 + D + 20(0.01)}$$

$$\ddot{\theta}_o + \dot{\theta}_d + \theta_d 0.2 = 0.2\theta_d$$

Homogeneous:

$$A^2 + A + 0.2 = 0$$

$$A = \frac{-1 \pm \sqrt{1 - 4(0.2)}}{2} \qquad A = -0.7236068, -0.2763932$$

$$\theta_o(t) = C_1 e^{-0.724t} + C_2 e^{-0.276t}$$

Particular:

$$\theta = A$$

$$0 + 0 + A0.2 = 0.2(1\text{rad}) \qquad A = 1\text{rad}$$

$$\theta_o(t) = 1\text{rad}$$

Initial Conditions (assume at rest):

$$\theta_o(t) = C_1 e^{-0.724t} + C_2 e^{-0.276t} + 1\text{rad}$$

$$\theta_o(0) = C_1 e^{-0.724t} + C_2 e^{-0.276t} + 1\text{rad} = 0$$

$$C_1 + C_2 = -1\text{rad} \qquad \text{eqn 15.1}$$

$$\theta'_o(t) = -0.724\left(C_1 e^{-0.724t}\right) - 0.276\left(C_2 e^{-0.276t}\right)$$

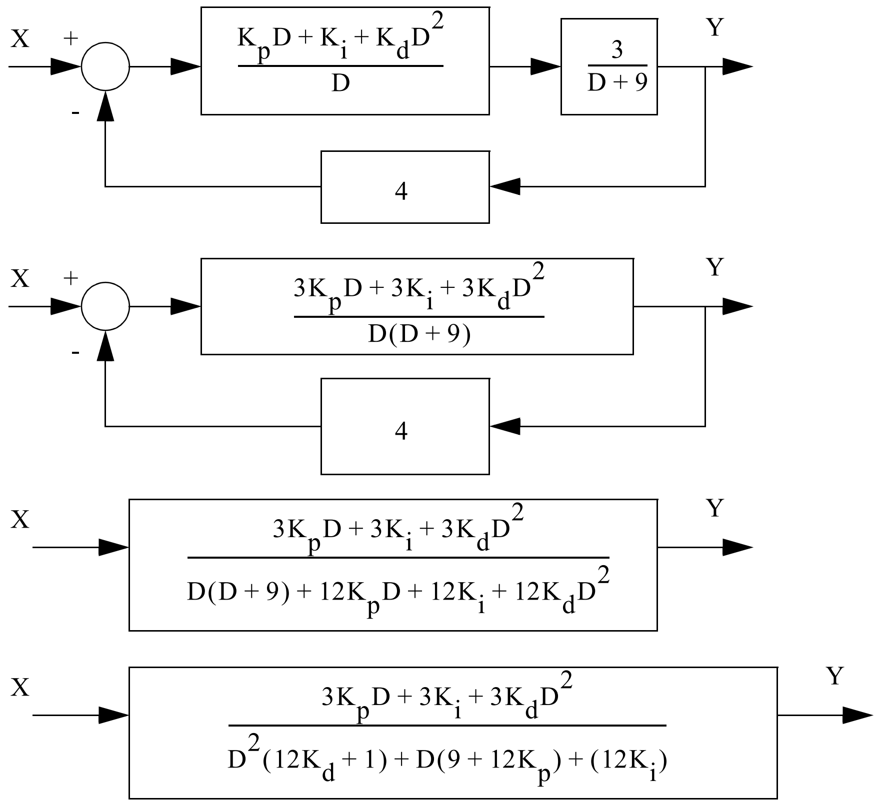
$$C_1 = -0.381 C_2$$

$$-0.381 C_2 + C_2 = -1\text{rad} \qquad C_2 = -1.616\text{rad}$$

$$C_1 = -0.381(-1.616\text{rad}) = 0.616\text{rad}$$

$$\theta_o(t) = (0.616)e^{-0.724t} + (-1.616)e^{-0.276t} + 1\text{rad}$$

Answer 15.5 a)

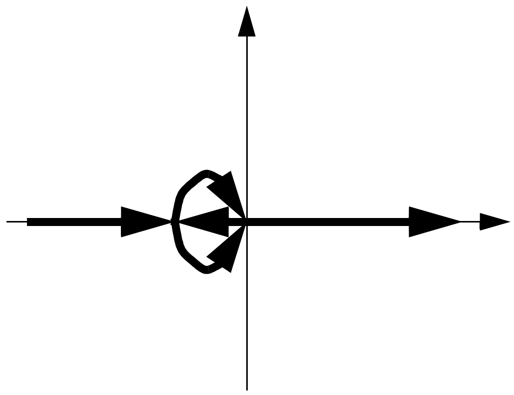

X
+
-
$\frac{K_pD + K_i + K_dD^2}{D}$
$\frac{3}{D + 9}$
Y
4
X
+
-
$\frac{3K_pD + 3K_i + 3K_dD^2}{D(D + 9)}$
Y
4
X
$\frac{3K_pD + 3K_i + 3K_dD^2}{D(D + 9) + 12K_pD + 12K_i + 12K_dD^2}$
Y
X
$\frac{3K_pD + 3K_i + 3K_dD^2}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$
Y

b)

$$D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i) = 0$$

$$D = \frac{-9 - 12K_p \pm \sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i}}{2(12K_d + 1)}$$

Kd	roots
-100	-0.092, 0.109
-10	-0.241, 0.418
-1	-0.46, 2.369
-0.1	-0.57, 105.6
0	-0.588, -20.41
1	-0.808 +/- 0.52j
10	-0.087 +/- 0.303j
100	-0.0087 +/- 0.1j



Stable for,

$$-9 - 12K_p \pm \sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i} < 0$$

$$\pm\sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i} < 9 + 12K_p$$

$$(9 + 12K_p)^2 - 4(12K_d + 1)12K_i < (9 + 12K_p)^2$$

$$-4(12K_d + 1)12K_i < 0$$

$$K_d > \frac{-1}{12}$$

Becomes complex at,

$$0 > (9 + 12K_p)^2 - 4(12K_d + 1)12K_i$$

$$576K_dK_i > (9 + 12K_p)^2 - 48K_i$$

$$K_d > \frac{(9 + 12K_p)^2 - 48K_i}{576K_dK_i} \qquad K_d > 0.682$$

c)

$$K_p = 1 \qquad K_i = 1 \qquad K_d = 1$$

$$\frac{Y}{X} = \frac{3K_pD + 3K_i + 3K_dD^2}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

$$\frac{Y}{X} = \frac{3D^2 + 3D + 3}{D^2 13 + D21 + 12} = \left(\frac{3}{13}\right)\left(\frac{D^2 + D + 1}{D^2 + D1.615 + 0.923}\right)$$

$$\text{final gain} = 20\log\left(\frac{3}{13}\right) = -12.7$$

$$\text{initial gain} = 20\log\left(\frac{3}{12}\right) = -12.0$$

for the numerator,

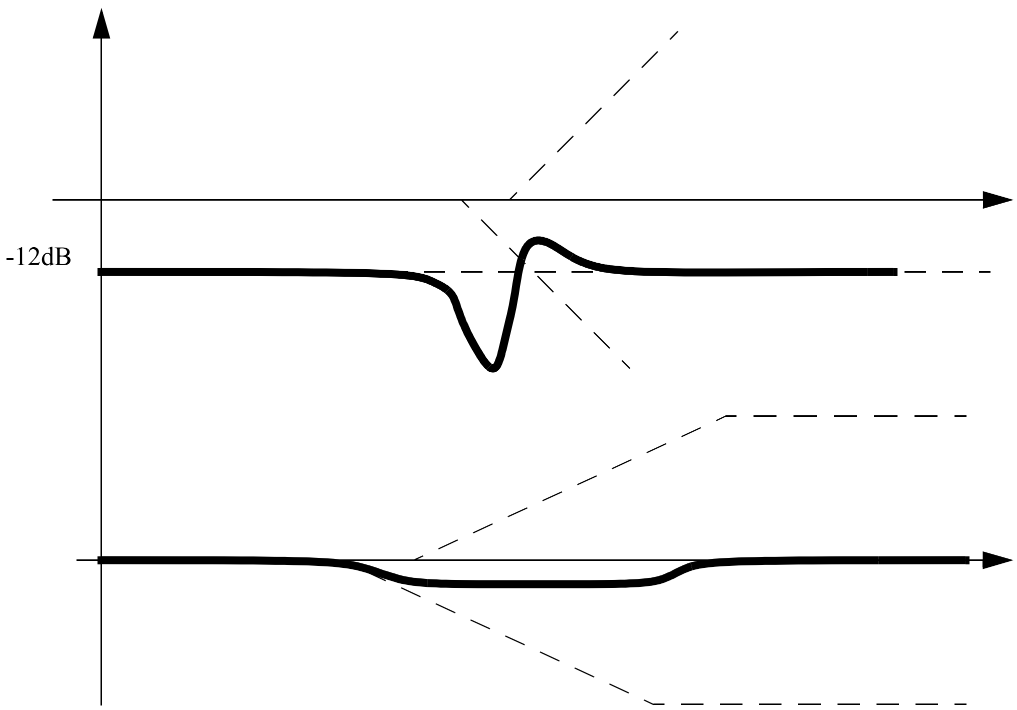
$$\omega_n = \sqrt{1} = 1 \qquad \xi = \frac{1}{2\omega_n} = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{1-0.5^2} = 0.866$$

for the denominator,

$$\omega_n = \sqrt{0.923} = 0.961 \qquad \xi = \frac{1.615}{2\omega_n} = 0.840$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 0.961\sqrt{1-0.840^2} = 0.521$$



$$\frac{Y}{X} = \frac{3K_pD + 3K_i + 3K_dD^2}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

$$\omega_n = \sqrt{\frac{12K_i}{12K_d + 1}} = 2 \qquad 12K_i = 48K_d + 4$$

$$2\xi\omega_n = \frac{9 + 12K_p}{12K_d + 1} = 20.5(2) \qquad 24K_d = 7 + 12K_p$$

At this point there are two equations and two unknowns, one value must be selected to continue, therefore,

$$K_p = 10$$

$$24K_d = 7 + 12K_p = 7 + 12(10) = 127 \qquad K_d = 5.292$$

$$12K_i = 48K_d + 4 = 48(5.292) + 4 = 258.0 \qquad K_i = 21.5$$

Now to check for stability

$$D^2(12(5.292) + 1) + D(9 + 12(10)) + (12(21.5)) = 0$$

$$64.504D^2 + 129D + 258 = 0$$

$$D = \frac{-129 \pm \sqrt{129^2 - 4(64.5)258}}{2(64.5)} = -1 \pm 1.73j$$

e) Cannot be found without an assumed input and initial conditions.

f)

$$\frac{Y}{X} = \frac{3(0)D + 3(0) + 3(1)D^2}{D^2(12(1) + 1) + D(9 + 12(0)) + (12(0))}$$

$$\frac{Y}{X} = \frac{3D^2}{13D^2 + 9D}$$

$$Y(13D^2 + 9D) = X(3D^2)$$

$$\ddot{Y}13 + \dot{Y}9 = \ddot{X}3 \qquad X = t \qquad \dot{X} = 1 \qquad \ddot{X} = 0$$

$$\ddot{Y} + \dot{Y}\frac{9}{13} = 0$$

It is a first order system,

$$Y(t) = C_1 e^{-\frac{9}{13}t} + C_2$$

$Y(0) = 0$ $\qquad$ $Y'(0) = 0$ $\qquad$ starts at rest/undeflected

$$0 = C_1 1 + C_2 \qquad C_1 = -C_2$$

$$Y'(t) = -\frac{9}{13}C_1 e^{-\frac{9}{13}t}$$

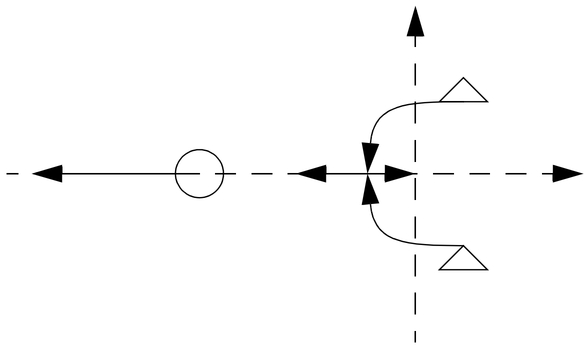
$$0 = -\frac{9}{13}C_1 1 \qquad C_1 = 0$$

$C_2 = 0$ $\qquad$ no response

Answer 15.6 a)

$$\frac{r}{c} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{D^3 + D^2(10K_p + 2) + D(20K_p + K_i) + (20K_i)}$$

$$D^3 + D^2(10K_p + 2) + D(20K_p + K_i) + (20K_i) = 0$$



K_p	Roots
0.0	–2.327, 0.1633 ± j0.9127
0.01	–2.338, 0.1188 ± j0.9173
0.03715	–2.372, 0.00006 ± j0.9183
0.05	–2.390, – 0.05505 ± j0.9131
0.1	–2.478, – 0.2609 ± j0.8596
0.27995	–3.224, – 0.7875 ± j0.00654
0.5	–5.095, –1.760, –0.2350
1.0	–10.01, –1.881, –0.1062
2.0	–20.00, –1.949, –0.05132

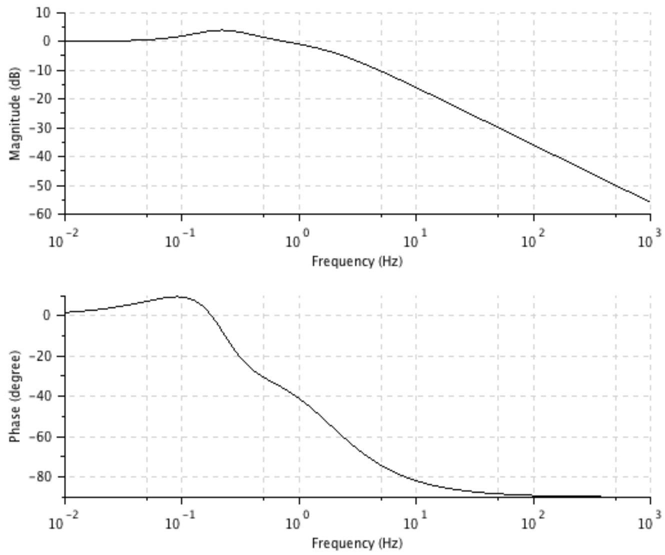
Kp < 0.038: The system is unstable and oscillates
Kp < 0.28: The system oscillates but is stable
KP > 0.28: The system does not oscillate and is stable

b)

$$\frac{r}{c} = \frac{D^2(10K_p) + D(20K_p + 10K_i) + (20K_i)}{D^3 + D^2(10K_p + 2) + D(20K_p + K_i) + (20K_i)}$$

$$\frac{r}{c} = \frac{D^2(10) + D(30) + (20)}{D^3 + D^2(12) + D(21) + (20)}$$

```
D = poly(0, 'D');
G = syslin('c', (10*D^2 + 30*D + 20) / (D^3 + 12*D^2 + 21*D + 20));
bode(G, 0.01, 1000);
```



c)

$$\frac{r}{c} = \frac{D^2(10) + D(30) + (20)}{D^3 + D^2(12) + D(21) + (20)}$$

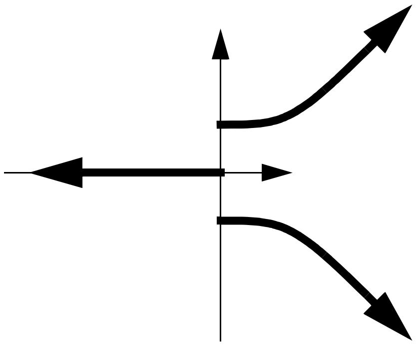
```
w = 10;
j = sqrt(-1);
D = j * w;
G = (10*D^2 + 30*D + 20) / (D^3 + 12*D^2 + 21*D + 20);
c = 10
r = c * G;
[mag, ang] = polar(r);
printf("output r = %f sin ( 10 t + %f )\n", mag, ang);
```

$$r = 7.2174\sin(10t - 0.8870)$$

Answer 15.7

$$\frac{r}{c} = \frac{\frac{10K}{D(D^2+5)}}{1+\frac{10K}{D(D^2+5)}} = \frac{10K}{D(D^2+5)+10K} = \frac{10K}{D^3(1)+D(5)+(10K)}$$

$$D^3(1)+D(5)+(10K) = 0$$



Ki	Roots
0	0, +/- 2.236j
0.01	-0.0200, 0.0100 +/- 2.24j
0.1	-0.198, 0.099 +/- 2.24j
1	-1.423, 0.712 +/- 2.55j
10	-4.28, 2.141 +/- 4.33j
100	-9.833, 4.917 +/- 8.804j
10000	-46.38, 23.19 +/- 40.23j
-0.01	0.0200, -0.0100 +/- 2.24j
-0.1	0.198, -0.0992 +/- 2.24j

There is no gain where the roots all have a negative real component, therefore this system will always be unstable.

15.1 Problems Without Solutions

Problem 15.8 Complete the Root-Locus Analysis

Given the system elements (assume a negative feedback controller),

$$G(D) = \frac{K}{D^2 + 3D + 2} \qquad H(D) = 1$$

First, find the characteristic equation,. and an equation for the roots,

$$1 + \left(\frac{K}{D^2 + 3D + 2}\right)(1) = 0$$

$$D^2 + 3D + 2 + K = 0$$

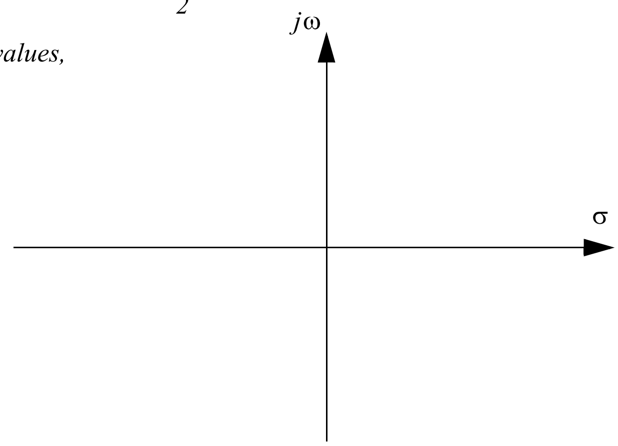
Note: For a negative feedback controller the denominator is,

$$1 + G(D)H(D)$$

$$roots = \frac{-3 \pm \sqrt{9 - 4(2 + K)}}{2} = -1.5 \pm \frac{\sqrt{1 - 4K}}{2}$$

Next, find values for the roots and plot the values,

K	roots
0	
1	
2	
3	

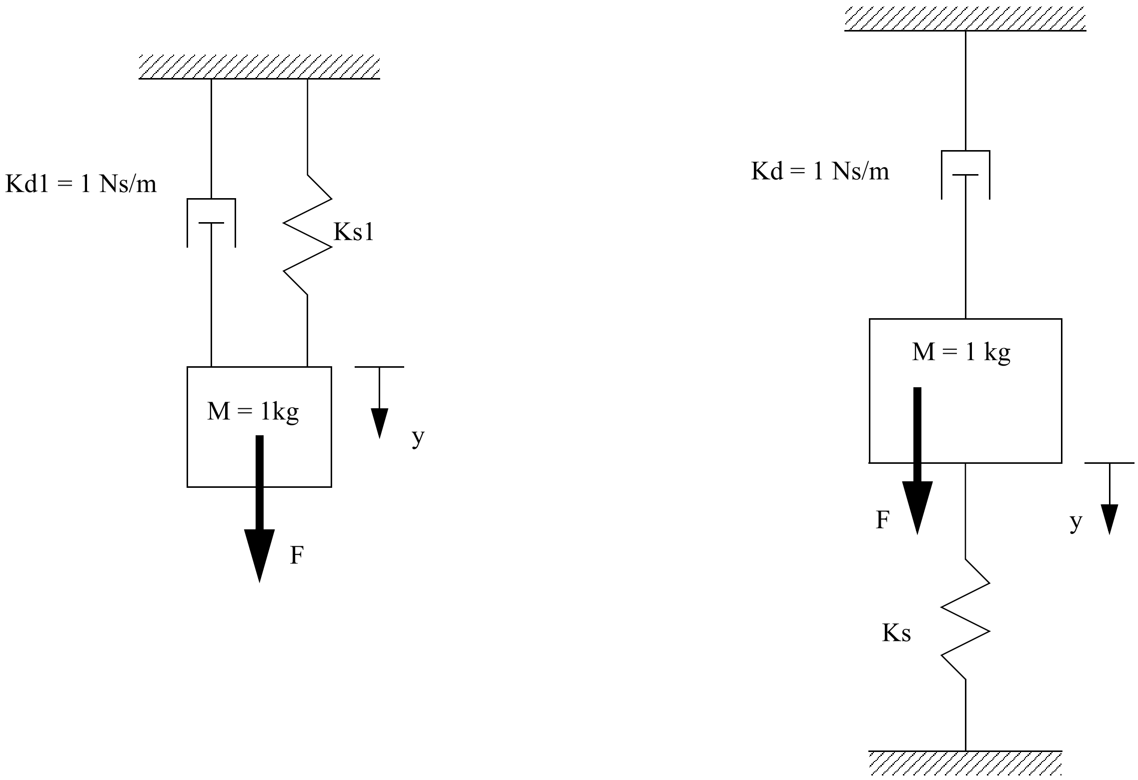


Problem 15.9 Draw a Root Locus plot for the transfer function.

$$G(D)H(D) = \frac{K(D+5)}{D(D^2 + 4D + 8)}$$

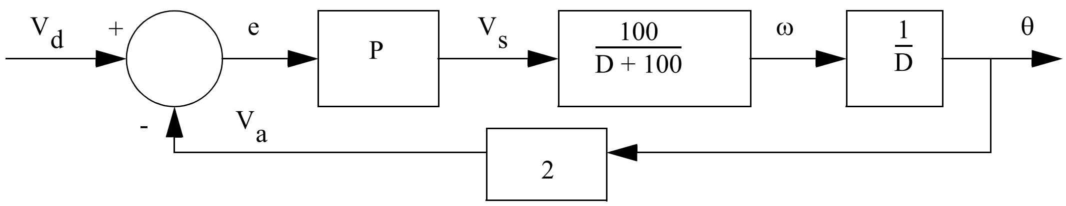
Problem 15.10 The systems below have a variable spring coefficient. For each of the systems below,

a) Write the differential equation and convert it to a transfer function.



b) If the input force is a step function of magnitude 1N, calculate the time response for ‘y’ by solving a differential equation for a Ks value of 10N/m.
c) Draw the poles for the transfer function on a real-complex plane.
d) Draw a Bode plot for Ks = 1N/m.

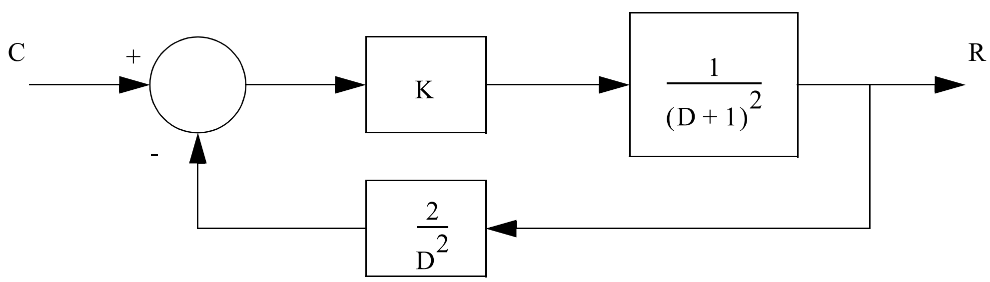
Problem 15.11 Draw a root locus diagram for the feedback system below given the variable parameter ‘P’.

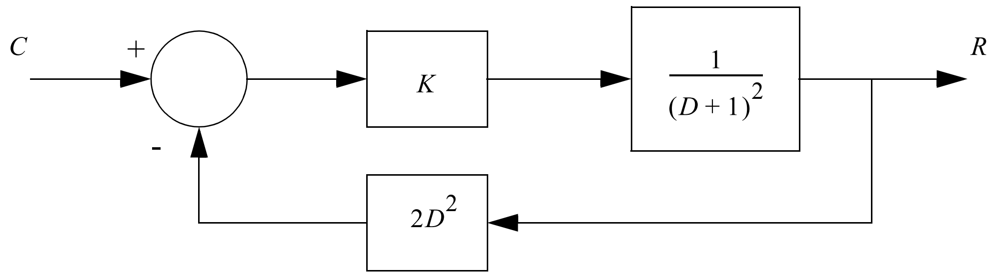


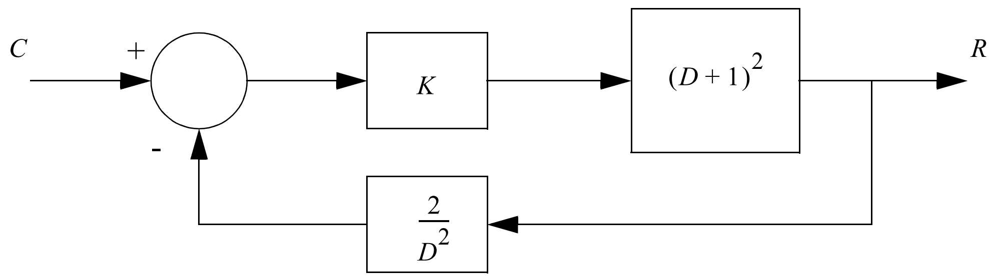
Problem 15.12 For the transfer functions below, draw the root locus plots assuming there is unity feedback, i.e., H(D) = 1. Draw an approximate time response for each for a step input.

$$G(s) = \qquad \frac{1}{D+1} \qquad \frac{1}{D^2+1} \qquad \frac{1}{(D+1)^2} \qquad \frac{1}{D^2+2D+2}$$

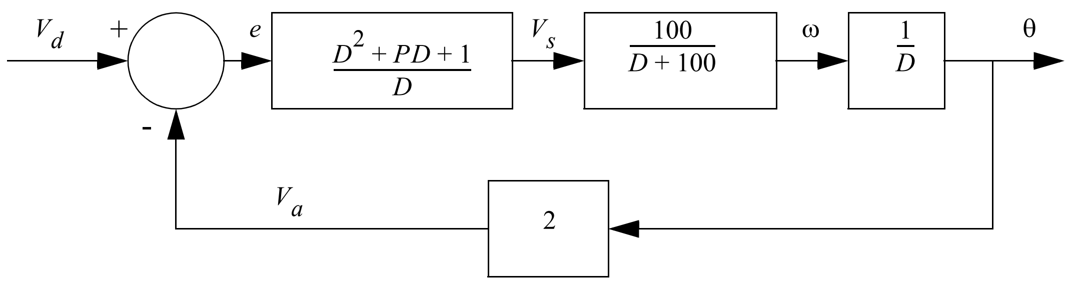
Problem 15.13 Draw a root-locus plot for the following feedback control systems.







Problem 15.14 The feedback loop below is for controlling a DC motor with a PID controller.



a) Find the transfer function for the system.
b) Draw a root locus diagram for the variable parameter 'P'.
c) Find the response of the system in to a unit step input using explicit integration.

Answer 15.14 a)

$$\frac{\theta}{V_d} = \frac{D^2(100) + D(100P) + (100)}{D^3 + D^2(300) + D(200P) + (200)}$$

b)
c)

16. Motion Control

Topic 16.1 *Motion controllers.*
Topic 16.2 *Motion profiles, trapezoidal and smooth.*
Topic 16.3 *Gain schedulers.*
Objective 16.1 *To understand single and multi axis motion control systems.*

16.1 Introduction

A system with a feedback controller will attempt to drive the system to a state described by the desired input, such as a velocity. In earlier chapters we simply chose step inputs, ramp inputs and other simple inputs to determine the system response. In practical applications this setpoint needs to be generated automatically. A simple motion control system is used to generate setpoints over time.

An example of a motion control system is shown in Figure 16.1. The motion controller will accept commands or other inputs to generate a motion profile using parameters such as distance to move, maximum acceleration and maximum velocity. The motion profile is then used to generate a set of setpoints, and times they should be output. The setpoint scheduler will then use a real-time clock to output these setpoints to the motor drive.

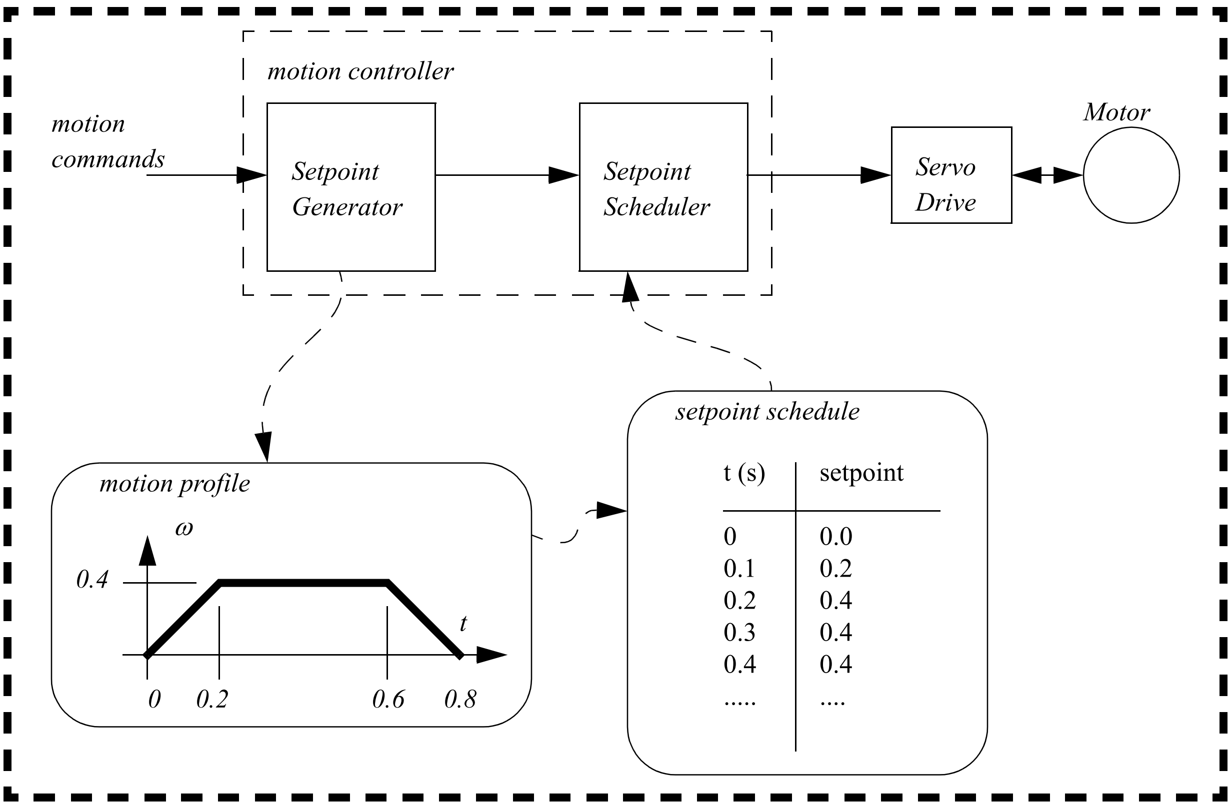


Figure 16.1 A motion controller

The combination of a motion controller, drive and actuator is called an axis. When there is more than one drive and actuator the system is said to have multiple axes. Complex motion control systems such as computer controlled milling machines (CNC) and robots have 3 to 6 axes which must be moved in coordination.

16.2 Motion Profiles

Velocity Profiles

A simple example of a velocity profile for a point-to-point motion is shown in Figure 16.2. In this example the motion starts at 20 deg and ends at 100 deg. (Note: in motion controllers it is more common to used encoder pulses, instead of degrees, for positions velocities, etc.) For position control we want a motion that has a velocity of zero at the start and end of the motion, and accelerates and decelerates smoothly.

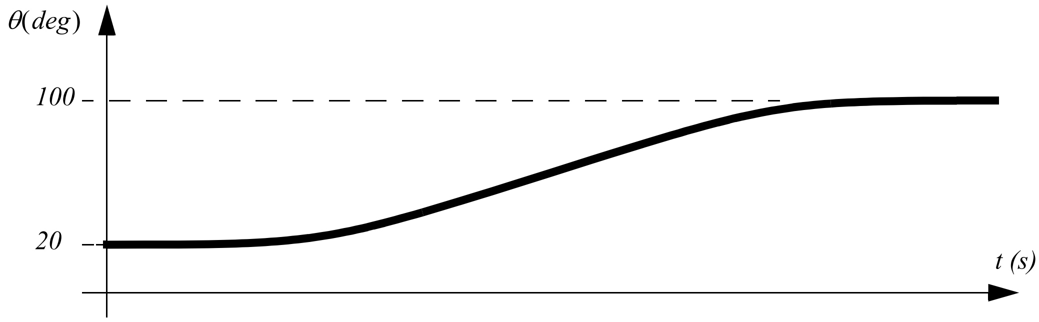
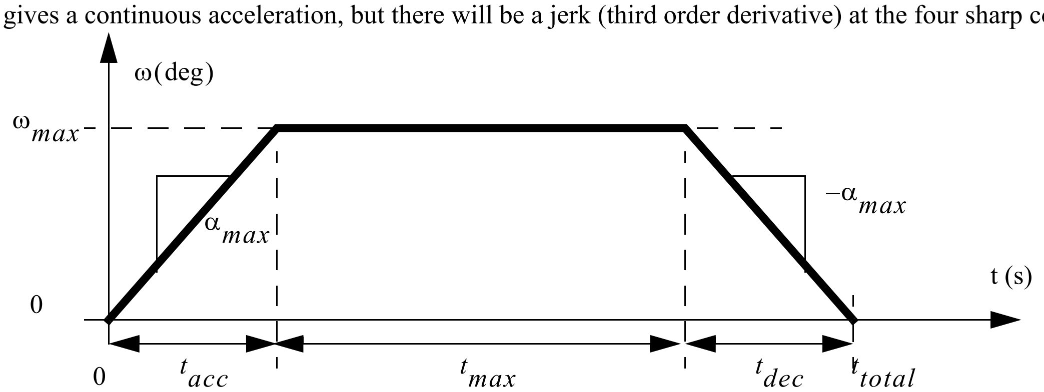


Figure 16.2 An example of a desired motion (position)

A trapezoidal velocity profile is shown in Figure 16.3. The area under the curve is the total distance moved. The slope of the initial and final ramp are the maximum acceleration and deceleration. The top level of the trapezoid is the maximum velocity. Some controllers allow the user to use the acceleration and deceleration times instead of the maximum acceleration and deceleration. This profile gives a continuous acceleration, but there will be a jerk (third order derivative) at the four sharp corners.



where,

ω_{max} = *the maximum velocity*

α_{max} = *the maximum acceleration*

t_{acc}, t_{dec} = *the acceleration and deceleration times*

t_{max} = *the times at the maximum velocity*

t_{total} = *the total motion time*

Figure 16.3 An example of a velocity profile

The basic relationships for these variables are shown in Figure 16.4. The equations can be used to find the acceleration and deceleration times. These equations can also be used to find the time at the maximum velocity. If this time is negative it indicates that the axis will not reach the maximum velocity, and the acceleration and deceleration times must be decreased. The resulting velocity profile will be a triangle.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}}$$ eqn 16.1

$$t_{total} = t_{acc} + t_{max} + t_{dec}$$ eqn 16.2

$$\theta = \frac{1}{2}t_{acc}\omega_{max} + t_{max}\omega_{max} + \frac{1}{2}t_{dec}\omega_{max} = \omega_{max}\left(\frac{t_{acc}}{2} + t_{max} + \frac{t_{dec}}{2}\right)$$ eqn 16.3

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2}$$ eqn 16.4

Note: if the time calculated in equation 4 is negative then the axis never reaches maximum velocity, and the velocity profile becomes a triangle.

Figure 16.4 Velocity profile basic relationships

For the example in Figure 16.5 the move starts at 100deg and ends at 20 deg. The acceleration and decelerations are completed in half a second. The system moves for 7.5 seconds at the maximum velocity.

Given, $\theta_{start} = 100deg$ $\theta_{end} = 20deg$

$\omega_{max} = 10\frac{deg}{s}$ $\alpha_{max} = 20\frac{deg}{s^2}$

The times can be calculated as,

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{10\frac{deg}{s}}{20\frac{deg}{s^2}} = 0.5s$$

$$\theta = \theta_{end} - \theta_{start} = 20deg - 100deg = -80deg$$

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2} = \frac{80deg}{10\frac{deg}{s}} - \frac{0.5s}{2} - \frac{0.5s}{2} = 7.5s$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = 0.5s + 7.5s + 0.5s = 8.5s$$

Figure 16.5 Velocity profile example

The motion example in Figure 16.6 is so short the axis never reaches the maximum velocity. This is made obvious by the negative time at maximum velocity. In place of this the acceleration and deceleration times can be calculated by using the basic acceleration position relationship. The result in this example is a motion that accelerates for 0.316s and then decelerates for the same time.

Given, $\theta_{start} = 20deg$ $\theta_{end} = 22deg$

$$\omega_{max} = 10\frac{deg}{s} \qquad \alpha_{max} = 20\frac{deg}{s^2}$$

The times can be calculated as,

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{10\frac{deg}{s}}{20\frac{deg}{s^2}} = 0.5s$$

$$\theta = \theta_{end} - \theta_{start} = 22deg - 20deg = 2deg$$

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2} = \frac{2deg}{10\frac{deg}{s}} - \frac{0.5s}{2} - \frac{0.5s}{2} = -0.3s$$

The time was negative so the acceleration and deceleration times become,

$$\frac{\theta}{2} = \frac{1}{2}\alpha_{max}t_{acc}^2$$

$$t_{acc} = \sqrt{\frac{\theta}{\alpha_{max}}} = \sqrt{\frac{2deg}{20\frac{deg}{s^2}}} = \sqrt{0.1s^2} = 0.316s$$

$$t_{max} = 0s$$

Figure 16.6 Velocity profile example without reaching maximum velocity

Given the parameters calculated for the motion, the setpoints for motion can be calculated with the equations in Figure 16.7.

Assuming the motion starts at 0s,

$0s \le t < t_{acc}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t^2 + \theta_{start}$$

$t_{acc} \le t < t_{acc} + t_{max}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t_{acc}^2 + \omega_{max}(t - t_{acc}) + \theta_{start}$$

$t_{acc} + t_{max} \le t < t_{acc} + t_{max} + t_{dec}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t_{acc}^2 + \omega_{max}t_{max} + \frac{1}{2}\alpha_{max}(t - t_{max} - t_{acc})^2 + \theta_{start}$$

$t_{acc} + t_{max} + t_{dec} \le t$

$$\theta(t) = \theta_{end}$$

Figure 16.7 Generating points given motion parameters

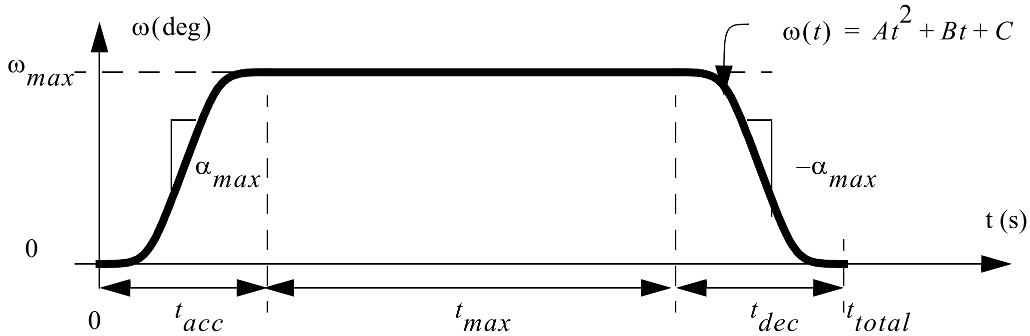
A subroutine that implements these is shown in Figure 16.8. In this subroutine the time is looped with fixed time steps.

The position setpoint values are put into the setpoint array, which will then be used elsewhere to guide the mechanism.

```
void generate_setpoint_table(
  double t_acc, double t_max, double t_step,
  double vel_max, double acc_max,
  double theta_start, double theta_end,
  double setpoint[], int *count){
        double t, t_1, t_2, t_total;
        t_1 = t_acc;
        t_2 = t_acc + t_max;
        t_total = t_acc + t_max + t_acc;
        *count = 0;
        for(t = 0.0; t <= t_total; t += t_step){
                if( t < t_1){
                    setpoint[*count] = 0.5*acc_max*t*t + theta_start;
                } else if ( (t >= t_1) && (t < t_2)){
                    setpoint[*count] = 0.5*acc_max*t_acc*t_acc
                        + vel_max*(t - t_1) + theta_start;
                } else if ( (t >= t_2) && (t < t_total)){
                        setpoint[*count] = 0.5*acc_max*t_acc*t_acc
                            + vel_max*(t_max)
                            + 0.5*acc_max*(t-t_2)*(t-t_2) + theta_start;
                } else {
                        setpoint[*count] = theta_end;
                }
                *count++;
        }
        setpoint[*count] = theta_end;
        *count++;
}
```

Figure 16.8 Subroutine for calculating motion setpoints

In some cases the jerk should be minimized. This can be achieved by replacing the acceleration ramps with a smooth polynomial, as shown in Figure 16.9. In this case two quadratic polynomials will be used for the acceleration, and another two for the deceleration.



where,

ω_{max} = *the maximum velocity*

α_{max} = *the maximum acceleration*

t_{acc}, t_{dec} = *the acceleration and deceleration times*

t_{max} = *the times at the maximum velocity*

t_{total} = *the total motion time*

Figure 16.9 A smooth velocity profile

An example of calculating the polynomial coefficients is given in Figure 16.10. The curve found is for the first half of the acceleration. It can then be used for the three other required curves.

Given, θ_{start} θ_{end} ω_{max} α_{max}

The constraints for the polynomial are,

$$\omega(0) = 0 \qquad \omega\left(\frac{t_{acc}}{2}\right) = \frac{\omega_{max}}{2}$$

$$\frac{d}{dt}\omega(0) = 0 \qquad \frac{d}{dt}\omega\left(\frac{t_{acc}}{2}\right) = \alpha_{max}$$

These can be used to calculate the polynomial coefficients,

$$0 = A0^2 + B0 + C \qquad \therefore C = 0$$

$$0 = 2A0 + B \qquad \therefore B = 0$$

$$\omega_{max} = At_{acc}^2 \qquad A = \frac{\omega_{max}}{t_{acc}^2}$$

$$\alpha_{max} = 2At_{acc} \qquad A = \frac{\alpha_{max}}{2t_{acc}}$$

$$A = \frac{\omega_{max}}{t_{acc}^2} = \frac{\alpha_{max}}{2t_{acc}} \qquad t_{acc} = \frac{2\omega_{max}}{\alpha_{max}}$$

$$A = \frac{\alpha_{max}}{2t_{acc}} = \frac{\alpha_{max}}{2\left(\frac{2\omega_{max}}{\alpha_{max}}\right)} = \frac{\alpha_{max}^2}{4\omega_{max}}$$

The equation for the first segment is,

$$\omega(t) = \frac{\alpha_{max}^2}{4\omega_{max}}t^2 \qquad 0 \le t < \frac{t_{acc}}{2}$$

The equation for the second segment can be found using the first segment,

$$\omega(t) = \omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}}(t_{acc} - t)^2$$

$$\omega(t) = \omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}}\left(t^2 - 2t_{acc}t + t_{acc}^2\right) \qquad \frac{t_{acc}}{2} \le t < t_{acc}$$

Figure 16.10 A smooth velocity profile example

The distance covered during acceleration, the area under the curves, is,

$$\theta_{acc} = \int_0^{\frac{t_{acc}}{2}} \frac{\alpha_{max}^2}{4\omega_{max}} t^2 dt + \int_{\frac{t_{acc}}{2}}^{t_{acc}} \left(\omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}} \left(t^2 - 2t_{acc} t + t_{acc}^2 \right) \right) dt$$

$$\theta_{acc} = \left. \frac{\alpha_{max}^2}{12\omega_{max}} t^3 \right|_0^{\frac{t_{acc}}{2}} + \left. \left(\omega_{max} t - \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t^3}{3} - t_{acc} t^2 + t_{acc}^2 t \right) \right) \right|_{\frac{t_{acc}}{2}}^{t_{acc}}$$

$$\theta_{acc} = \frac{\alpha_{max}^2}{12\omega_{max}} \frac{t_{acc}^3}{8} + \omega_{max} t_{acc} - \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t_{acc}^3}{3} - t_{acc}^3 + t_{acc}^3 \right)$$

$$- \omega_{max} \frac{t_{acc}}{2} + \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t_{acc}^3}{24} - \frac{t_{acc}^3}{4} + \frac{t_{acc}^3}{2} \right)$$

$$\theta_{acc} = \frac{\alpha_{max}^2}{96\omega_{max}} t_{acc}^3 + \frac{\omega_{max} t_{acc}}{2} - \frac{\alpha_{max}^2}{12\omega_{max}} t_{acc}^3 + \frac{7\alpha_{max}^2}{96\omega_{max}} t_{acc}^3$$

$$\theta_{acc} = \frac{\omega_{max} t_{acc}}{2}$$

so the time required at the maximum velocity is,

$$t_{max} = \frac{(\theta - 2\theta_{acc})}{\omega_{max}}$$

Figure 16.11 A smooth velocity profile example (cont'd)

Controllers use a variety of motion profiles. The trapezoidal and quadratic equations presented previously can be used in simpler systems. Other motion profiles are possible for some or all of the curves, as shown in Figure 16.12.

Type	Equation
Trapezoidal	$X(t) = (X_1 - X_0)\left(\frac{t - t_0}{t_1 - t_0}\right) + X_0$
Quadratic	$X(t) = A(t - t_0)^2 + B(t - t_0) + C$
Sinusoidal	$X(t) = (X_1 - X_0)\sin\left(\frac{t - t_0}{t_1 - t_0}\frac{\pi}{2}\right) + X_0$
Sigmoid	$X(t) = (X_1 - X_0)\left(\frac{1}{1 + e^{-\left(\frac{t - t_0}{t_1 - t_0}\right) - 3}}\right)$

Figure 16.12 Profile Types

Position Profiles

A motion can be described using position points along a path. These methods are normally used when a controller does not have any velocity or acceleration limits. The method shown in Figure 16.13 controls motion using a parametric function ‘p(u)’. The function value varies from 0 to 1 as the parameter ‘u’ varies from 0 to 1. However, the parameters of the function are selected so that the motion starts and stops with a velocity of zero. In this case the final polynomial equation, (3), is fairly simple. This equation can then be used in equation (1) to generate a smooth motion path between any arbitrary start and end point, with arbitrary start and end times.

$$\theta(t) = \theta_{start} + (\theta_{end} - \theta_{start})p\left(\frac{t - t_{start}}{t_{end} - t_{start}}\right) \qquad \text{eqn 16.5}$$

where,

$$\theta_{start}, \theta_{end} = \text{start and end positions of motion}$$

$$t_{start}, t_{end} = \text{start and end times for the motion}$$

$$p(u) = Au^3 + Bu^2 + Cu + D \qquad \text{eqn 16.6}$$

The constraints for the polynomial are,

$$p(0) = 0 \qquad p(1) = 1$$

$$\frac{d}{dt}p(0) = 0 \qquad \frac{d}{dt}p(1) = 0$$

These can be used to calculate the polynomial coefficients,

$$0 = A0^3 + B0^2 + C0 + D \qquad \backslash D = 0$$

$$0 = 3A0^2 + 2B0 + C \qquad \backslash C = 0$$

$$0 = 3A1^2 + 2B1 \qquad \backslash B = \left(-\frac{3}{2}\right)A$$

$$1 = A1^3 + B1^2 + (0)0 + 0 \qquad \backslash A = -2$$

$$\backslash B = 3$$

$$p(u) = -2u^3 + 3u^2 \qquad \text{eqn 16.7}$$

Figure 16.13 Generating smooth motion paths

The example in Figure 16.14 shows the use of a trigonometric function, instead of a polynomial. This function was used to generate the points in the following sample program in Figure 16.15.

where,

$$p(u) = A sin(Bt + C) + D$$

The coefficients can be calculated using the conditions used previously,

$$\frac{d}{dt}p(0) = AB cos(B(0) + C) = 0$$

$$\backslash cos(C) = 0 \qquad \backslash C = -\frac{\pi}{2}$$

$$\frac{d}{dt}p(1) = AB cos(B(1) + C) = 0$$

$$\backslash cos(B + C) = 0$$

$$\backslash B + C = \frac{\pi}{2} \qquad \backslash B = \pi$$

$$p(0) = A sin\left(B(0) - \frac{\pi}{2}\right) + D = 0 \qquad \backslash A = D$$

$$p(1) = A sin\left(\pi(1) - \frac{\pi}{2}\right) + D = 1$$

$$\backslash A(1) + A = 1 \qquad \backslash A = \frac{1}{2}$$

The final relationship is,

$$p(u) = \frac{1}{2} sin\left(\pi t - \frac{\pi}{2}\right) + \frac{1}{2}$$

Figure 16.14 Generating smooth motion paths

The program in Figure 16.15 generates a motion table that can then be used to update setpoints. The function ‘table_init()’ must be called once when the program starts to set up global time and table values. When a new target position has been specified the ‘table_generate()’ function is called to generate the setpoint table. The ‘table_update()’ function is called once every interrupt scan to check the setpoint table, and update the global setpoint variable, ‘point_current’ at scheduled times. This function also includes a simple clock to keep track of the system time.

```
#define        TABLE_SIZE     11
int     point_master[TABLE_SIZE] = {0, 24, 95, 206, 345, 500, 655, 794, 905, 976, 1000};
int     point_position[TABLE_SIZE];
int     point_time[TABLE_SIZE];
int     point_start_time;
int     point_index;

int     ticks; /* variables to keep a system clock count */
int     point_current; /* a global variable to track position */


int table_init(){ /* initialize the setpoint table */
        ticks = 0; /* set the clock to zero */
        point_current = 0; /* start the system at zero */
        point_index = TABLE_SIZE; /* mark the table as empty */
}


void table_generate(int start, int end, int duration_sec){
        unsigned i;

        point_time[0] = ticks + 10; /* delay the start slightly */
        point_position[0] = start;

        for(i = 1; i < TABLE_SIZE; i++){
                point_time[i] = point_time[0] +
                        (unsigned long)i * duration_sec * 250 / (TABLE_SIZE - 1);
                point_position[i] = start + (long int)(end - start) * point_master[i] / 1000;
        }
        point_index = 0;
}

int table_update(){/* interrupt driven encoder update */
        ticks++; /* update the clock */

        if(point_index < TABLE_SIZE){
                if(point_time[point_index] == ticks){
                        point_current = point_position[point_index++];
                        outint16(point_current);
                        putch("\n");
                }
        }
        return point_current;
}
```

Figure 16.15 Subroutines for motion profile generation and use

16.1 Multi-Axis Motion

In a machine with multiple axes the motions of individual axes must often be coordinated. A simple example would be a robot that needs to move two joints to reach a new position. We could extend the motion of the slower joints so that the motion of each joint would begin and end together.

Slew Motion

When the individual axis of a machine are not coordinated this is known as slew motion. Each of the axes will start moving at the same time, but finish at separate times. Consider the example in Figure 16.16. A three axis motion is required from the starting angles of (40, 80, -40)deg, and must end at (120, 0, 0)deg. The maximum absolute accelerations and decelerations are (50, 100, 150) degrees/sec/sec, and the maximum velocities are (20, 40, 50) degrees/sec.

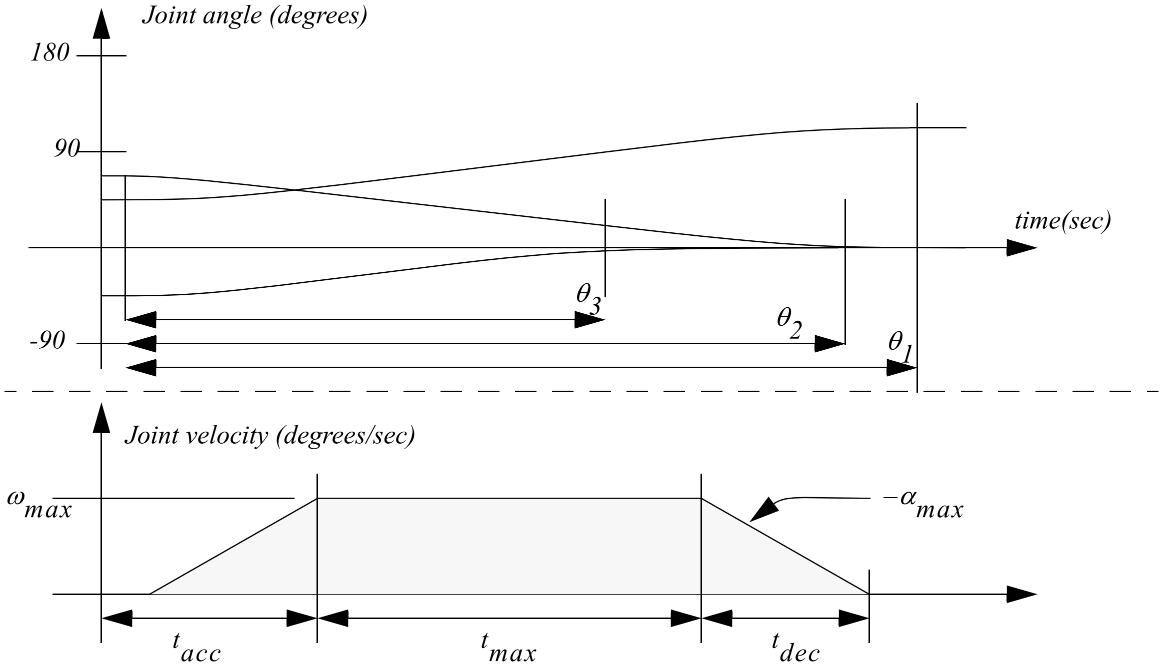


Figure 16.16 Multi-axis slew motion

The calculations for the motion parameters are shown in Figure 16.17. These are done in vector format for simplicity. All of the joints reach the maximum acceleration. The fastest motion is complete in 1.13s, while the longest motion takes 4.4s.

The area under the velocity curve is the distance (angle in this case) travelled. First we can determine the distance covered during acceleration, and deceleration and the time during acceleration, and deceleration.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \left(\frac{20}{50}, \frac{40}{100}, \frac{50}{150}\right) = (0.4, 0.4, 0.333)sec.$$

$$\theta_{acc.} = \theta_{dec.} = \frac{t_{acc}\omega_{max.vel.}}{2} = \left(\frac{0.4(20)}{2}, \frac{0.4(40)}{2}, \frac{0.333(50)}{2}\right) = (4, 8, 8.33)deg.$$

The next step is to examine the moves specified,

$$\theta_{move} = \theta_{end} - \theta_{start} = (120-40, 0-80, 0-(-40)) = (80, -80, 40)deg.$$

Remove the angles covered during accel./deccel., and find the travel time at maximum velocity.

$$t_{max} = \frac{|\theta_{move}| - 2\theta_{acc}}{\omega_{max}} = \left(\frac{80-2(4)}{20}, \frac{80-2(8)}{40}, \frac{40-2(8.333)}{50}\right)$$

$$t_{max} = (3.6, 1.6, 0.46668)sec.$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = (4.4, 2.4, 1.13)s$$

Figure 16.17 Calculated times for the slew motion

Interpolated Motion

In interpolated motion the faster joints are slowed so that they finish in coordination with the slowest. This is essential in devices such as CNC milling machines. If this did not occur a straight line cut in the x-y plane would actually be two straight lines. The slew motion example can be extended to be slew motion where all joints finish their motion at 4.4s. This can be done by accelerating at the maximum acceleration, but setting a new maximum velocity. This is shown in the example in Figure 16.18 using the results from the example in Figure 16.17.

The longest motion time is 4.4s for joint 1, and this can be used to prolong the other motions. The calculation begins by rewriting the velocity/position relationship using a new maximum velocity.

$$\Delta\theta = \frac{1}{2}\frac{\omega_{max}'}{\alpha_{max}}\omega_{max}' + \left(t_{total} - 2\frac{\omega_{max}'}{\alpha_{max}}\right)\omega_{max}' + \frac{1}{2}\frac{\omega_{max}'}{\alpha_{max}}\omega_{max}'$$

$$\therefore \Delta\theta = \omega_{max}'\left(t_{total} - \frac{\omega_{max}'}{\alpha_{max}}\right)$$

$$\therefore (\omega_{max}')^2 + \omega_{max}'(-t_{total}\alpha_{max}) + \Delta\theta\alpha_{max} = 0$$

$$\therefore \omega_{max}' = \frac{t_{total}\alpha_{max} \pm \sqrt{t_{total}^2\alpha_{max}^2 - 4\Delta\theta\alpha_{max}}}{2}$$

$$t_{acc}' = t_{acc}' = \frac{\omega_{max}'}{\alpha_{max}}$$

A new maximum velocity can be calculated for joint 2 using this equation.

$$\therefore \omega_{max}' = \frac{4.4(100) \pm \sqrt{(4.4)^2(100)^2 - 4(80)(100)}}{2}$$

$$\therefore \omega_{max}' = \frac{440 \pm 401.99502}{2} = 421, \boxed{19.0}$$

$$t_{acc}' = t_{acc}' = \frac{19.0}{100} = 0.19s$$

A new maximum velocity can be calculated for joint 3 using this equation.

$$\therefore \omega_{max}' = \frac{4.4(150) \pm \sqrt{(4.4)^2(150)^2 - 4(40)(150)}}{2}$$

$$\therefore \omega_{max}' = 651, \boxed{9.22}$$

$$t_{acc}' = t_{acc}' = \frac{9.22}{100} = 0.092s$$

Figure 16.18 Interpolated motion based upon Figure 16.17

Motion Scheduling

After the setpoint schedule has been developed, it is executed by the setpoint scheduler. The setpoint scheduler will use a clock to determine when an output setpoint should be updated. A diagram of a scheduler is shown in Figure 16.19. In this system the setpoint scheduler is an interrupt driven subroutine that compares the system clock to the total motion time. When enough time has elapsed the routine will move to the next value in the setpoint table. The frequency of the interrupt clock should be smaller than or equal to the time steps used to calculate the setpoints. The servo drive is implemented with an algorithm such a PID control.

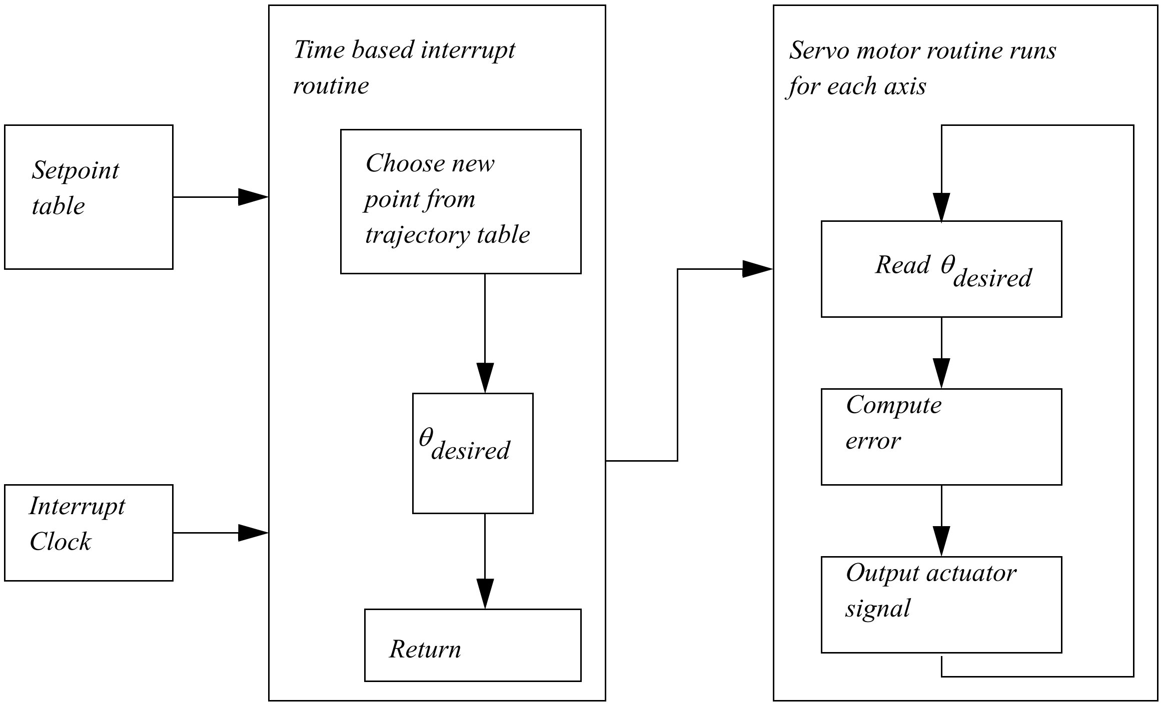


Figure 16.19 A setpoint scheduler

The output from the scheduler updates every time step. This then leads to a situation where the axis is always chasing the target value. This leads to small errors, as shown in Figure 16.20.

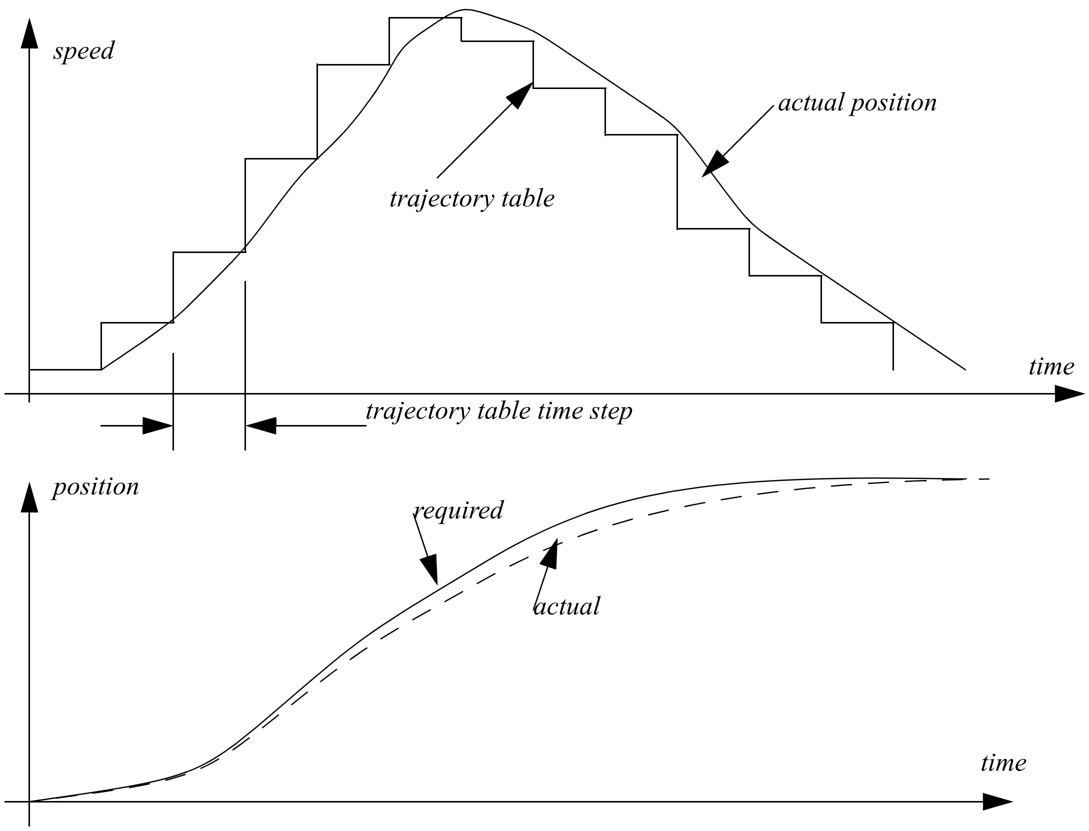


Figure 16.20 Errors in path following

16.2 Path Planning

Elevator systems are an example of path planning systems. For example an elevator could serve ten floors. While moving up or down it may have multiple target floors. The distances between the floors may vary. And, buttons pushed during motions may alter the path. The same system with multiple elevators could decrease the wait time by keeping idle elevators at the top and 7th floors.Similar path planning systems can improve the efficiency of delivery trucks, reduce hard disk read times, cutting more lumber from trees, or plan a path for a missile.

Consider the CNC machine path in Figure 16.21. The hole, or pocket, is cut using a cylindrical tool end that is moved through the work. The size of the tool determines the number of passes required, and how closely the tool can approach the pocket side. The path planning software will determine each of the segments for the cutting operation. For basic CNC control the path planning software generates sequential straight line segments for the motion. The motion controller will then perform a motion for each of the path segments until done. In other words each straight line is a self contained motion.

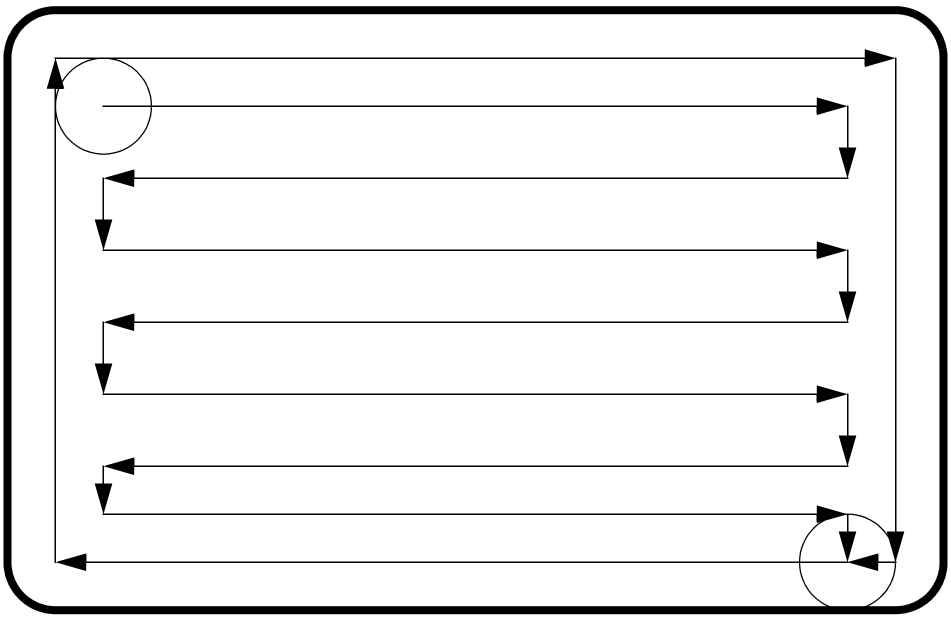


Figure 16.21 A Sample CNC Milling Machine Path for a Pocket

A simple approach to path planning for CNC machining is to break a surface into regularly spaced points. The surface patch in Figure 16.22 would normally be modelled by splines. Locations on the surface patch are specified with the 'u' and 'v' parameters. For example u=0.5 and v=0.5 should be near the geometric center of the surface. A set of equations are used to convert the 'u' and 'v' values to a '[x, y, z]' coordinates. The dashed line path on the surface can be calculated by keeping u=0.5, and then varying v from 0.0 to 1.0.

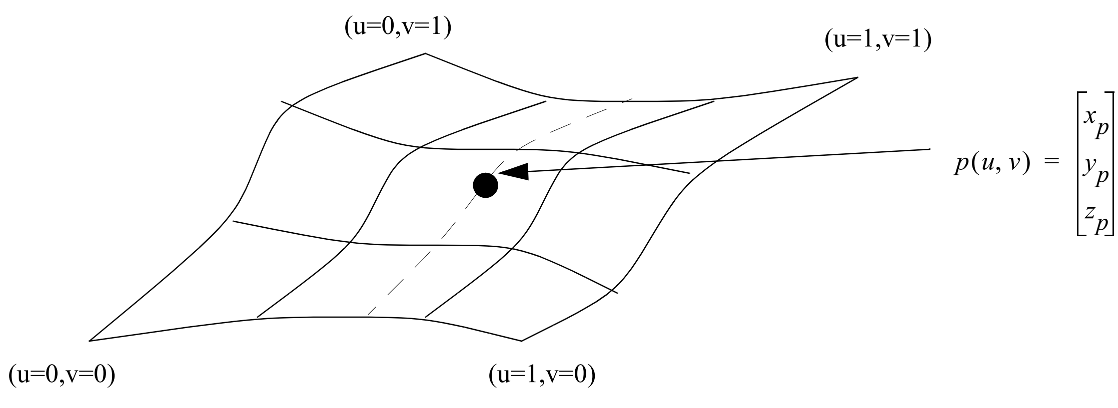


Figure 16.22 A parametric surface representation

The sample subroutine in Figure 16.23 can convert a four sided surface patch into a series of path segments. A function ‘p(u, v)’ is used to calculate surface points. This subroutine will break the surface into 10 passes with ten steps in each, for a total of 100 straight line cuts. The coordinates are printed with the format of CNC G-codes including ‘G00’ and ‘G01’.

```
dirn_flag = 1; a direction flag
n=10 ; number of passes to cut the surface
step=1.0/n ; step sizes for u and v directions
start=step/2 ; the start offset in the u and v directions
[xp,yp,zp] = p(start,start) ; calculate the start position
print(“G00 X”,xp,” Y”,yp,” Z”,zp+0.2) ; move the tool to above the start position
for i=0 to (n-1) ; will increment in the u direction
        for j=0 to (n-1) ; will increment in the v direction
        ; calculate next point
                if dirn_flag=-1 then [xp,yp,zp]=p(start+i*step,start+j*step)
                if dirn_flag=1 then [xp,yp,zp]=p(start+i*step,start+(n-j)*step)
                print(“G01 X”,xp,” Y”,yp,” Z”,zp) ; instruction to cut to next point
        next j ; make next step in v direction until done
        dirn_flag = -dirn_flag ; reverse direction to cut in opposite direction
next i ; move to next cut line in the u direction
print(“G00 Z”,zp+0.2) ; move the tool to above the end position
```

Figure 16.23 Linear interpolation of a surface patch

16.3 Kinematics

[[This section is still in point form. The core topics are present and correct, it will be expanded with text later.]]

A robot must be able to map between things that it can control, such as joint angles, to the position of the tool in space.

Describing the position of the robot in terms of joint positions/angles is Joint Space.

Real space is often described with a number of coordinate systems,

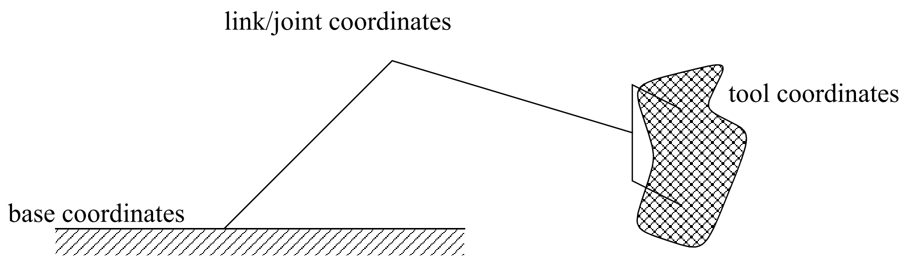
Cartesian

polar

spherical

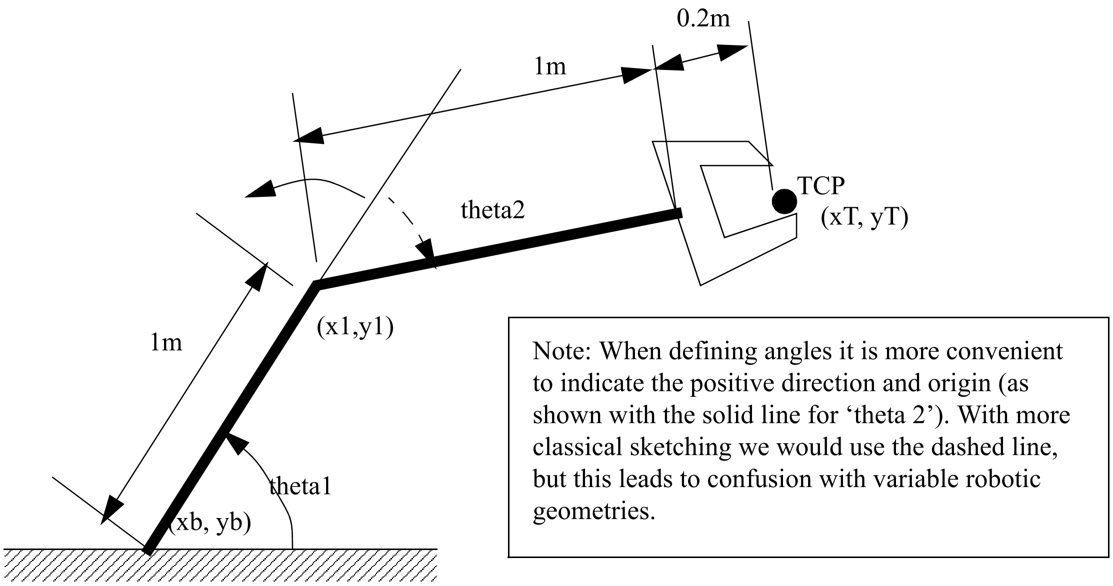
Positions can also be specified with respect to the robot base (Robot Coordinates), or globally (World Coordinates).

Robot Positions



- Robot base coordinates don’t move and are often used to specify robot tool position and orientation. (center of the robots world)
- Link/Joint Coordinates - specify where joints, endpoints or centers are located.
- Tool coordinates - determine where the tool is and what orientation it is in.
- World Coordinates - relates various robots to other robots and devices.

- Coordinate transformation - Can map from one set of coordinates to another. Most common method is matrix based. One special case of this is the Denavit-Hartenrberg transformation.



- Forward kinematics involves finding the endpoint of the robot (xT, yT) given the joint coordinates (theta1, theta2)
- There a number of simple methods for finding these transformations,
  - basic geometry
  - transformation matrices
  - Denavit-Hartenberg transformations

Geometry Methods for Forward Kinematics

- For simple manipulators (especially planar ones) this method is often very fast and efficient.
- The method uses basic trigonometry, and geometry relationships.
- To find the location of the robot above, we can see by inspection,

$$x_T = x_b + l_1 \cos\theta_1 + (l_2 + 0.2)\cos(\theta_1 + \theta_2)$$
$$y_T = y_b + l_1 \sin\theta_1 + (l_2 + 0.2)\sin(\theta_1 + \theta_2)$$

often set to zero

The general form of the operation is as below,

$(\theta_1, \theta_2, \ldots)$ Æ $\left(x_T, y_T, z_T, \theta_{T_x}, \theta_{T_y}, \theta_{T_z}\right)$

> Aside: later we will see that the opposite operation maps from tool coordinates, and is called the inverse kinematics.
>
> $(\theta_1, \theta_2, \ldots)$ ¨ $\left(x_T, y_T, z_T, \theta_{T_x}, \theta_{T_y}, \theta_{T_z}\right)$

Also note that the orientation of the tool is included, as well as position, therefore for the example,

$$\theta_{T_x} = 0$$

$$\theta_{T_y} = 0$$

$$\theta_{T_z} = \theta_1 + \theta_2$$

Figure 16.24 Kinematic equations for a two link robot arm

- The problem with geometrical methods are that they become difficult to manage when more complex robots are considered. This problem is overcome with systematic methods.

Geometry Methods for Inverse Kinematics

- To find the location of the robot above, we can see by inspection,

Inverse kinematics maps from the tool coordinates to the joint coordinates.

$$(\theta_1, \theta_2, \ldots) \leftarrow (x_T, y_T, z_T, \theta_{T_x}, \theta_{T_y}, \theta_{T_z})$$

Figure 16.25 Inverse kinematics for a two link robot

- Mathematically this calculation is difficult, and there are often multiple solutions.

Modeling the Robot

- If modeling only one link in motion, the model of the robot can treat all the links as a single moving rigid body,

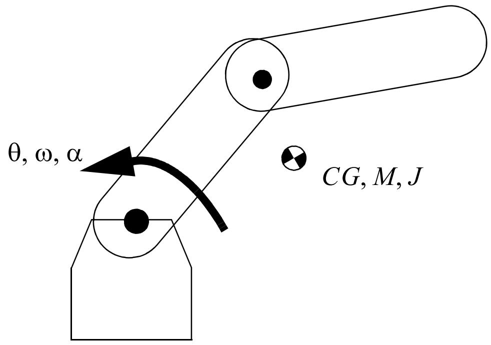


Figure 16.26 The dynamics of a two link robot

- If multiple joints move at the same time, the model becomes non-linear, in this case there are two approaches taken,
    1. Develop a full non-linear controller (can be very complicated).
    2. Develop linear approximations of the model/control system in the middle of the normal workspace.

16.4 Path Planning

- Basic - "While moving the robot arm from point A to B, or along a continuous path, the choices are infinite, with significant differences between methods used."

Straight-line motion

- In this method the tool of the robot travels in a straight line between the start and stop points. This can be difficult, and lead to rather erratic motions when the boundaries of the workspace are approached.
- NOTE: straight-line paths are the only paths that will try to move the tool straight through space, all others will move the tool in a curved path.

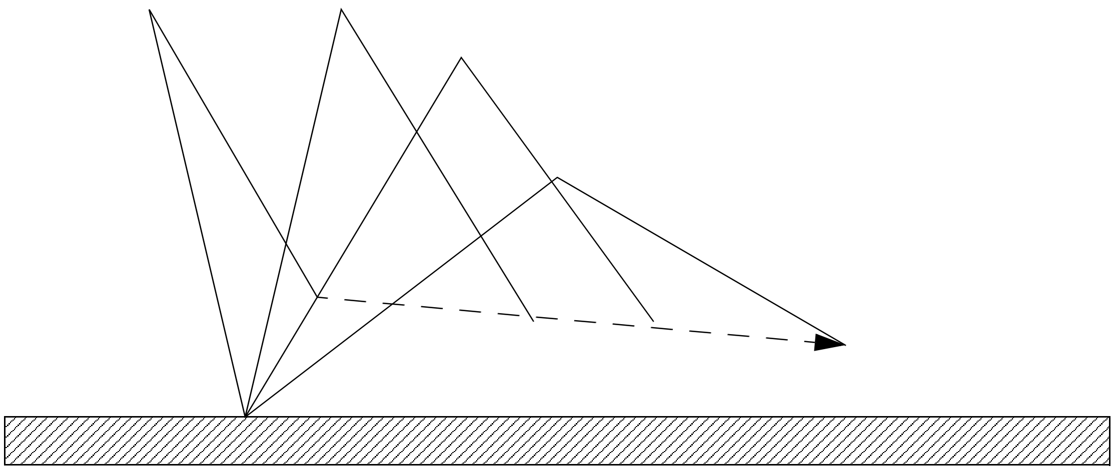


Figure 16.27 Straight line tool path

- The basic method is,

1. Develop a set of points from the start and stop points that minimize acceleration.
2. Do the inverse kinematics to find the joint angles of the robot at the specified points.

- Consider the example below,

Given,

$$P_0 = (5, 5, 5)\text{in.} \qquad P_1 = (-5, -5, 5)$$

$$\frac{d}{dt}P_0 = (0, 0, 0) \qquad \frac{d}{dt}P_1 = (0, 0, 0) \qquad t_0 = 0 \qquad t_1 = 2$$

Model the path with a function that allows acceleration/deceleration, in this case a third order polynomial will be used. The equation will be parameterized for simplicity (i.e., s = [0,1], where s=0 is the path start, and s=1 is the path end).

$$P(t) = P_0 + (P_1 - P_0)s(t)$$

$$s(t_0) = 0 \qquad s(t_1) = 1 \qquad \frac{d}{dt}s(t_0) = 0 \qquad \frac{d}{dt}s(t_1) = 0$$

$$s(t) = At^3 + Bt^2 + Ct + D \qquad \frac{d}{dt}s(t) = 3At^2 + 2Bt + C$$

Next, numerical values will be entered to find equation values

$$s(0) = A(0)^3 + B(0)^2 + C(0) + D = 0 \qquad \boxed{D = 0}$$

$$s(2) = A(2)^3 + B(2)^2 + C(2) + D = 1 \qquad \backslash 8A + 4B = 1$$

$$\frac{d}{dt}s(0) = 3A(0) + 2B(0) + C = 0 \qquad \boxed{C = 0}$$

$$\frac{d}{dt}s(2) = 3A(2) + 2B(2) + C = 0 \qquad \backslash\left(-\frac{3}{2}\right)A = B$$

$$8A + 4\left(-\frac{3}{2}\right)A = 1 \qquad \boxed{A = \frac{1}{2}} \qquad \boxed{B = -\frac{3}{4}}$$

This can now be put in the final form,

$$P(t) = P_0 + (P_1 - P_0)\left(\frac{t^3}{2} - \frac{3}{4}t^2\right)$$

Figure 16.28 Velocity controlled tool path

16.5 Case Studies

- the controller described in the block diagram below uses a model for a DC permanent magnet DC motor to estimate a voltage based upon a predicted velocity and position.
- the desired position and velocity are given in figure xxx based upon the motion position control derived in the

earlier section

$$\theta(t) = \theta_{start} + (\theta_{end} - \theta_{start})p\left(\frac{t - t_{start}}{t_{end} - t_{start}}\right)$$

$$p(u) = -2u^3 + 3u^2$$

$$\omega(t) = (\theta_{end} - \theta_{start})\dot{p}\left(\frac{t - t_{start}}{t_{end} - t_{start}}\right)$$

$$\dot{p}(u) = -6u^2 + 6u$$

Figure 16.29 Motion position control equation

- the controller that uses the desired position and velocity is shown in figure xxxx

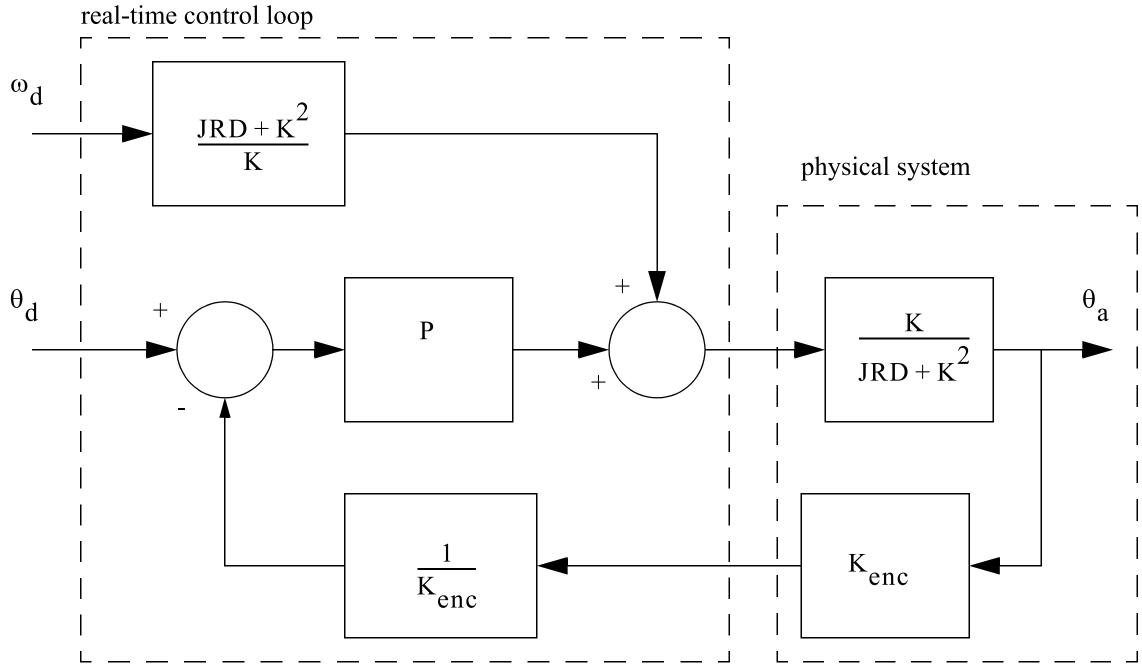


Figure 16.30 Feedforward trajectory controller

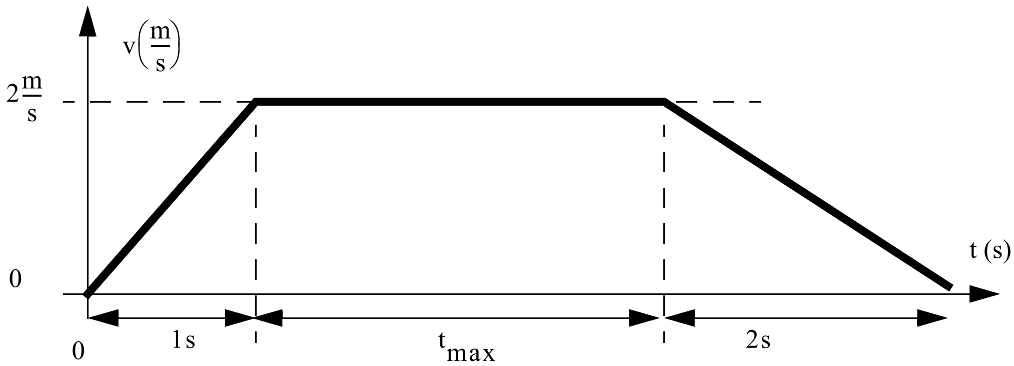
16.6 Summary

- Axis limits can be used to calculate motion profiles.
- Trapezoidal and smooth motion profiles were presented.
- Motion profiles can be used to generate setpoint tables.
- Values from the setpoints can then be output by a scheduler to drive an axis.

16.7 Problems With Solutions

Problem 16.1 The velocity profile shown is used for a system that must move 15m. a) Find the total time for the motion. b)

Indicate the positions at 1 second intervals over the duration of the motion.



Problem 16.2 a) Develop a motion profile for a joint that moves from -100 degrees to 100 degrees with a maximum velocity of 20 deg/s and a maximum acceleration of 100deg/s/s. b) Develop a setpoint table that has values for positions every 0.5 seconds for the entire motion.

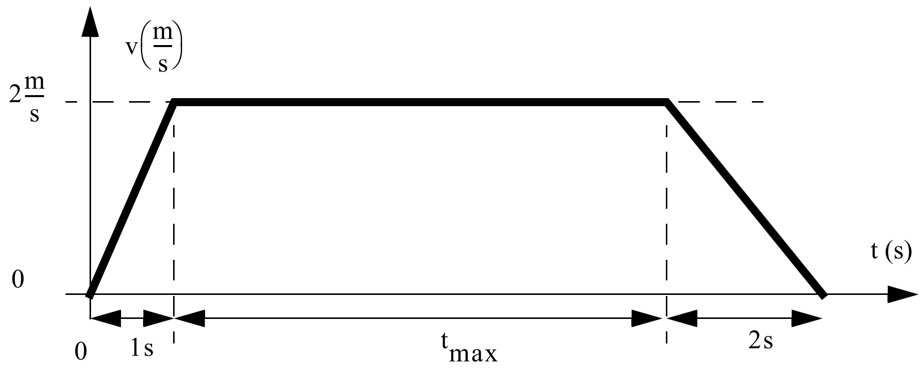
Problem 16.3 Consider a basic servo controller with encoder feedback. The motor will start at a position count of 100, and end the motion at a count of 3000. The motion is to have a maximum acceleration of 300 counts/s/s, and a maximum velocity of 100 counts/s. Find a motion profile that satisfies these constraints. Generate a table of setpoints for the desired position every 2 seconds.

Problem 16.4 Find a smooth path for a robot joint that will turn from $\theta = 75°$ to $\theta = -35°$ in 10 seconds. Do this by developing an equation then calculating points every 1.0 seconds along the path for a total motion time of 10 seconds.

Problem 16.5 We are designing motion algorithms for a 2 degree of freedom robot. To do this we are developing sample calculations to explore the basic process.
a) We want to move the tool in a straight line through space from (3”, 5”) to (8”, 7”). Develop equations that will give a motion that starts and stops smoothly. The motion should be complete in 1 second.
b) Find the velocity of the tool at t=0.5 seconds.
c) Plot out the tool position, joint positions and velocities as functions of time.

16.8 Problem Solutions

Answer 16.1



$$Dx = \frac{1s2\frac{m}{s}}{2} + 2\frac{m}{s}t_{max} + \frac{2s2\frac{m}{s}}{2} = 15m$$

$$t_{max} = \frac{15m - 1m - 2m}{2\frac{m}{s}} = 6s$$

t (s)	v (m/s)	x (m)
0.0	0.0	0.0
1.0	2.0	1.0
2.0	2.0	3.0
3.0	2.0	5.0
4.0	2.0	7.0
5.0	2.0	9.0
6.0	2.0	11.0
7.0	2.0	13.0
8.0	1.0	14.5
9.0	0.0	15.0

Answer 16.2

Given,

$$\omega_{max} = 20\frac{deg}{s} \qquad \alpha_{max} = 100\frac{deg}{s^2} \qquad \Delta\theta = 100 - (-100) = 200°$$

The motion times can be calculated.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{20\frac{deg}{s}}{100\frac{deg}{s^2}} = 0.2s$$

$$t_{max} = \frac{\Delta\theta - \omega_{max}t_{acc}}{\omega_{max}} = \frac{200deg - 20\frac{deg}{s}0.2s}{20\frac{deg}{s}} = 9.8s$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = 0.2s + 9.8s + 0.2s = 10.2s$$

$$\theta_{0.5s} = \frac{1}{2}100\frac{deg}{s^2}(0.2s)^2 + 20\frac{deg}{s}(0.5s - 0.2s) - 100deg$$

$$\theta_{0.5s} = -92deg$$

$$\theta_{1.0s} = \theta_{0.5s} + 20\frac{deg}{s}(0.5s) = -92deg + 10deg$$

t (s)	angle (deg)
0.0	-100
0.5	-92
1.0	-82
1.5	-72
2.0	-62
2.5	-52
3.0	-42
3.5	-32
4.0	-22
4.5	-12
5.0	-2
5.5	8
6.0	18
6.5	28
7.0	38
7.5	48
8.0	58
8.5	68
9.0	78
9.5	88
10.0	98
10.5	100
11.0	100

Answer 16.3

t (s)	counts
0	100
2	283
4	483
6	683
8	883
10	1083
12	1283
14	1483
16	1683
18	1883
20	2083
22	2283
24	2483
26	2683
28	2883
30	3000

Answer 16.4

$$\theta(t) = At^3 + Bt^2 + Ct + D \qquad \theta(0) = 75 \qquad \theta(10) = -35$$

$$\frac{d}{dt}\theta(t) = 3At^2 + 2Bt + C \qquad \frac{d}{dt}\theta(0) = 0 \qquad \frac{d}{dt}\theta(10) = 0$$

Solving

$$75 = A(0)^3 + B(0)^2 + C(0) + D$$

$$-35 = A(10)^3 + B(10)^2 + C(10) + D$$

$$0 = 3A(0)^2 + 2B(0) + C$$

$$0 = 3A(10)^2 + 2B(10) + C$$

For A, B, C, D we get

$$\theta(t) = (0.22)t^3 + (-3.3)t^2 + (75)$$

t (sec)	theta(t)
0	75
1	71.92
2	63.56
3	51.24
4	36.28
5	20
6	3.72
7	-11.24
8	-23.56
9	-31.92
10	-35

Answer 16.5 a)

$$P(t) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + (-2t^3 + 3t^2)\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

b)

$$\frac{d}{dt}P(t) = \begin{bmatrix} 7.5 \\ 3 \end{bmatrix}$$

16.9 Problems Without Solutions

Problem 16.6 We want to move a tool in a straight line through space from (3”, 5”) to (8”, 7”). Develop equations that will give a motion that starts and stops smoothly. The motion should be complete in 1 second

Problem 16.7 Find a smooth path for a robot joint that will turn from θ= 75° to θ = -35° in 10 seconds. Do this by developing an equation then calculating points every 1.0 seconds along the path for a total motion time of 10 seconds. Do not assume a maximum velocity.

Problem 16.8 Paths are to be planned for a three axis motion controller. Each of the joints has a maximum velocity of 20 deg/s, and a maximum acceleration of 30 deg/s/s. Assuming all of the joints start at an angle of 0 degrees. Joints 1, 2 and 3 move to 40 deg, 100deg and -50deg respectively. Develop the motion profiles assuming,
a) slew motion
b) interpolated motion

Problem 16.9 Develop a smooth velocity profile for moving a cutting tool that starts at 1000 inches and moves to -1000 inches. The maximum velocity is 100 in/s and the maximum acceleration is 50in/s/s.

Problem 16.10 An axis has a maximum velocity of 2.0 rad/s and a maximum acceleration/deceleration of 0.5 rad/s^2. Sketch a trapezoidal motion profile and find the motion time for a -4.0 rad motion.

Problem 16.11 An axis has a maximum velocity of 5m/s and an acceleration/deceleration time of 1s. Sketch a trapezoidal motion profile and find the motion time for a 25m motion.

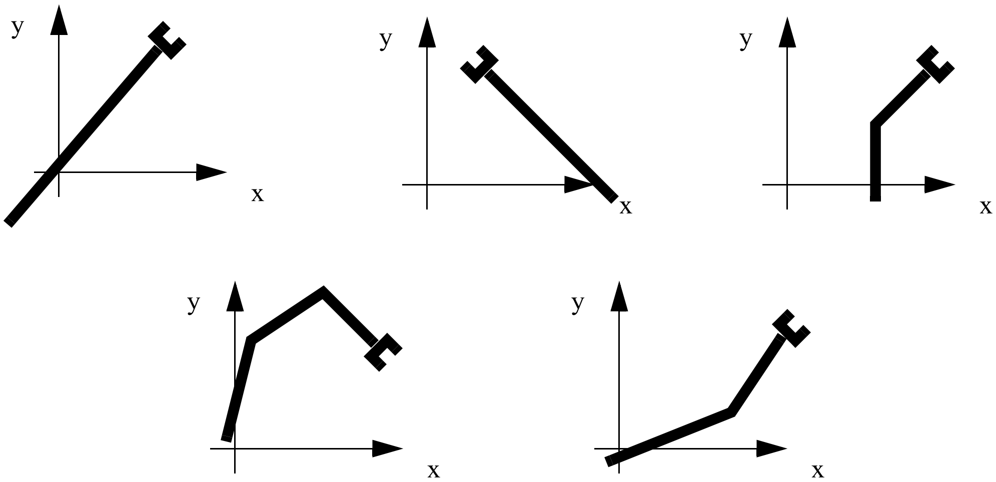
Problem 16.12 Develop a smooth velocity profile for moving a cutting tool that starts at 1000 inches and moves to -1000 inches. The maximum velocity is 100 in/s and the maximum acceleration is 50in/s/s.
a) A stepping motor is to be used to actuate one joint of a robot arm in a light duty pick and place application. The step angle of the motor is 10 degrees. For each pulse received from the pulse train source the motor rotates through a distance of one step angle.
i) What is the resolution of the stepper motor?
ii) Relate this value to the definitions of control resolution, spatial resolution, and accuracy, as discussed in class.
b) Solve part a) under the condition that the three joints move at different rotational velocities. The first joint moves at 10 degrees/sec., the second joint moves at 25 degrees/sec, and the third joint moves at 30°/sec.

Problem 16.13 A stepping motor is to be used to drive each of the three linear axes of a Cartesian coordinate robot. The motor output shaft will be connected to a screw thread with a screw pitch of 0.125”. It is desired that the control resolution of each of the axes be 0.025”
a) to achieve this control resolution how many step angles are required on the stepper motor?
b) What is the corresponding step angle?
c) Determine the pulse rate that will be required to drive a given joint at a velocity of 3.0”/sec.

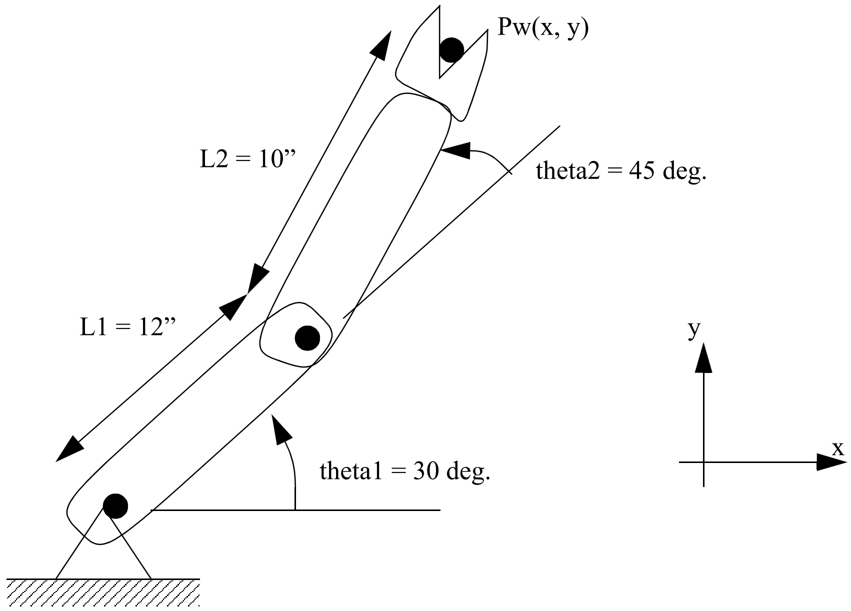
Problem 16.14 For the stepper motor of question 6, a pulse train is to be generated by the robot controller.
a) How many pulses are required to rotate the motor through three complete revolutions?
b) If it is desired to rotate the motor at a speed of 25 rev/min, what pulse rate must be generated by the robot controller?

Problem 16.15 A stepping motor is to be used to actuate one joint of a robot arm in a light duty pick and place application. The step angle of the motor is 10 degrees. For each pulse received from the pulse train source the motor rotates through a distance of one step angle.
a) What is the resolution of the stepper motor?
b) Relate this value to the definitions of control resolution, spatial resolution, and accuracy, as discussed in class.

Problem 16.16 Find the forward kinematics for the robots below using geometry methods.



Problem 16.17 Consider the forward kinematic transformation of the two link manipulator below.



a) Given the position of the joints, and the lengths of the links, determine the location of the tool centre point using basic geometry.
b) Determine the inverse kinematics for the robot. (i.e., given the position of the tool, determine the joint angles of the robot.) Keep in mind that in this case the solution will have two different cases.
c) Determine two different sets of joint angles required to position the TCP at x=5”, y=6”.
d) What mathematical conditions would indicate the robot position is unreachable? Are there any other cases that are indeterminate?

Problem 16.18 A jointed arm robot has three rotary joints, and is required to move all three axes so that the first joint is rotated through 50 degrees; the second joint is rotated through 90 degrees, and the third joint is rotated through 25 degrees. Maximum speed of any of these rotational joints is 10 degrees/sec. Ignore effects of acceleration and deceleration and,
a) determine the time required to move each joint if slew motion (joint motion is independent of all other joints) is used.
b) determine the time required to move the arm to a desired position and the rotational velocity of each joint, if joint interpolated motion (all joints start and stop simultaneously) is used.
c) Solve question 4 under the condition that the three joints move at different rotational velocities. The first joint moves at 10 degrees/sec., the second joint moves at 25 degrees/sec, and the third joint moves at 30°/sec.

Problem 16.19 Consider the following motion planning problem.
a) A jointed arm robot has three rotary joints, and is required to move all three axes so that the first joint is rotated through 50 degrees; the second joint is rotated through 90 degrees, and the third joint is rotated through 25 degrees. Maximum speed of any of these rotational joints is 10 degrees/sec. Ignore effects of acceleration and deceleration and,
b) determine the time required to move each joint if slew motion (joint motion is independent of all other joints) is used.
c) determine the time required to move the arm to a desired position and the rotational velocity of each joint, if joint interpolated motion (all joints start and stop simultaneously) is used.

Problem 16.20 Why do robots not follow exact mathematical paths?

Problem 16.21 We are designing motion algorithms for a 2 degree of freedom robot. To do this we are developing sample calculations to explore the basic process. We want to move the tool in a straight line through space from (8”, 7”) to (3”, 5”). Develop equations that will give a motion that starts and stops smoothly. The motion should be complete in 2 seconds. Show all derivations.

17. Laplace Transforms

Topic 17.1 *Laplace transforms.*
Topic 17.2 *Using tables to do Laplace transforms.*
Topic 17.3 *Using the s-domain to find outputs.*
Topic 17.4 *Solving Partial Fractions.*
Objective 17.1 *To be able to find time responses of linear systems using Laplace transforms.*

Laplace transforms provide a method for representing and analyzing linear systems using algebraic methods. In systems that begin undeflected and at rest the Laplace 's' can directly replace the d/dt operator in differential equations. It is a super set of the phasor representation in that it has both a complex part, for the steady state response, but also a real part, representing the transient part. As with the other representations the Laplace s is related to the rate of change in the system.

$$D = s \qquad \text{(if the initial conditions/derivatives are all zero at t=0s)}$$

$$s = \sigma + j\omega$$

Figure 17.1 The Laplace 's'

The basic definition of the Laplace transform is shown in Figure 17.2. The normal convention is to show the function of time with a lower case letter, while the same function in the s-domain is shown in upper case. Another useful observation is that the transform starts at t=0s. Examples of the application of the transform are shown in Figure 17.3 for a step function and in Figure 17.4 for a first order derivative.

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

where,

$$f(t) = \text{the function in terms of time t}$$

$$F(s) = \text{the function in terms of the Laplace s}$$

Note: To avoid confusion, functions of time are often shown in lower case, while functions of 's' are often shown as upper case. Sometimes '(t)' or '(s)' will be shown for clarity, and sometimes omitted for brevity.

Note: The Laplace transform always starts at 0 and goes to infinity. Please note that in the definition the limit of '0-' is used. It is also possible to use '0+', but results in another set of transforms that are somewhat less clear. All of the transforms used in this book assume '0+'.
Ref: Lundberg, K.H., Miller, H.R., Trumper, D.L., "Initial Conditions, Generalized Functions, and the Laplace Transform", IEEE Control Systems Magazine, February, 2007, Vol. 27, No. 1, pg. 22-35.

Figure 17.2 The Laplace transform

Aside: Proof of the step function transform.

For f(t) = 5,

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} 5e^{-st}dt = -\frac{5}{s}e^{-st}\Big|_{0}^{\infty} = \left[-\frac{5}{s}e^{-s\infty}\right] - \left[-\frac{5e^{-s0}}{s}\right] = \frac{5}{s}$$

Figure 17.3 Proof of the step function transform

Aside: Proof of the first order derivative transform

Given the derivative of a function g(t)=df(t)/dt,

$$G(s) = L[g(t)] = L\left[\frac{d}{dt}f(t)\right] = \int_0^{\infty}(d/dt)f(t)e^{-st}dt$$

we can use integration by parts to go backwards,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_0^{\infty}(d/dt)f(t)e^{-st}dt$$

therefore,

$$du = df(t) \qquad v = e^{-st}$$

$$u = f(t) \qquad dv = -se^{-st}dt$$

$$\therefore \int_0^{\infty} f(t)(-s)e^{-st}dt = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty}(d/dt)f(t)e^{-st}dt$$

$$\therefore \int_0^{\infty}(d/dt)f(t)e^{-st}dt = [f(t)e^{-\infty s} - f(t)e^{-0s}] + s\int_0^{\infty} f(t)e^{-st}dt$$

$$\therefore L\left[\frac{d}{dt}f(t)\right] = -f(0) + sL[f(t)]$$

Figure 17.4 Proof of the first order derivative transform

The previous proofs were presented to establish the theoretical basis for this method, however tables of values will be presented in a later section for the most popular transforms.

17.1 Applying Laplace Transforms

The process of applying Laplace transforms to analyze a linear system involves the basic steps listed below.

1. Convert the system transfer function, or differential equation, to the s-domain by replacing ‘D’ with ‘s’. (Note: If any of the initial conditions are non-zero refer to the transform tables for extra terms.)
2. Convert the input function(s) to the s-domain using the transform tables.
3. Algebraically combine the input and transfer function to find an output function.
4. Use partial fractions to reduce the output function to simpler components.
5. Convert the output equation back to the time-domain using the tables.

A Few Transform Tables

Laplace transform tables are shown in Figure 17.5, Figure 17.6 and Figure 17.7. These are commonly used when analyzing systems with Laplace transforms. The transforms shown in Figure 17.5 are general properties normally used for manipulating equations, and for converting them to/from the s-domain.

Note: The following tables all assume that the system is at rest before t=0, unless initial conditions are specified. Additionally, each of the time domain functions implicitly includes a unit step function, u(t), to turn the function on at t=0. For example if the input function is 5sin(10t), it should practically be interpreted as 5sin(10t)u(t). Please note that the s-domain (frequency domain) functions are written to address this assumption.

Time Domain	Frequency Domain (s-domain)
$f(t)$	$F(s)$
$Kf(t)$	$KL[f(t)]$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$F_1(s) + F_2(s) - F_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sL[f(t)] - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 L[f(t)] - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n L[f(t)] - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^{n-1} f(0^-)}{dt^{n-1}}$
$\int_{0-}^{t} f(t) dt$	$\frac{L[f(t)]}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as} L[f(t)]$
$e^{-at} f(t)$	$F(s+a)$
$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-dF(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(u) du$

Figure 17.5 Laplace transform tables

The Laplace transform tables shown in Figure 17.6 and Figure 17.7 are normally used for converting to/from the time/s-domain.

Time Domain		Frequency Domain (s-domain)
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

Figure 17.6 Laplace transform tables (continued)

Time Domain	Frequency Domain (s-domain)
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{\|A\|\angle\theta}{s+\alpha-\beta j}+\frac{\|A\|\angle-\theta}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{\|A\|\angle\theta}{(s+\alpha-\beta j)^2}+\frac{\|A\|\angle-\theta}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$
$Ce^{-\sigma t}\cos(\omega_d t+\theta)$	$\frac{As+B}{s^2+2\zeta\omega_n s+\omega_n^2}$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\sigma = \zeta\omega_n$, $\theta = \operatorname{atan}\left(\frac{A\sigma-B}{A\omega_d}\right)$, $C = \sqrt{A^2+\left(\frac{A\sigma-B}{\omega_d}\right)^2}$

Figure 17.7 Laplace transform tables (continued)

Proof:

$$\frac{As+B}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{F}{s+\sigma+(-\omega_d)j}+\frac{G}{s+\sigma+\omega_d j} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\sigma = \zeta\omega_n$$

$$= \frac{F(s+\sigma+\omega_d j)+G(s+\sigma+(-\omega_d)j)}{s^2+s(\sigma+(-\omega_d)j+\sigma+\omega_d j)+(\sigma+(-\omega_d)j)(\sigma+\omega_d j)}$$

$$= \frac{s(F+G)+(F(\sigma+\omega_d j)+G(\sigma+(-\omega_d)j))}{s^2+s(2\sigma)+(\sigma^2+\omega_d^2)}$$

$$A = F+G \qquad G = A-F$$

$$B = F(\sigma+\omega_d j)+G(\sigma+(-\omega_d)j)$$

$$G = \frac{B-F(\sigma+\omega_d j)}{\sigma+(-\omega_d)j} = A-F$$

$$B-F(\sigma+\omega_d j) = A(\sigma+(-\omega_d)j)-F(\sigma+(-\omega_d)j)$$

$$F = \frac{B-A(\sigma+(-\omega_d)j)}{2(\omega_d)j} = -j\left(\frac{B-A\sigma+jA\omega_d}{2\omega_d}\right) = \left(\frac{A}{2}\right)+j\left(\frac{A\sigma-B}{2\omega_d}\right)$$

$$A = \sqrt{\left(\frac{A}{2}\right)^2+\left(\frac{A\sigma-B}{2\omega_d}\right)^2}\angle\operatorname{atan}\left(\frac{\left(\frac{A\sigma-B}{2\omega_d}\right)}{\left(\frac{A}{2}\right)}\right) = \frac{1}{2}\sqrt{A^2+\left(\frac{A\sigma-B}{\omega_d}\right)^2}\angle\operatorname{atan}\left(\frac{A\sigma-B}{A\omega_d}\right)$$

$$= \sqrt{A^2+\left(\frac{A\sigma-B}{\omega_d}\right)^2}e^{-\sigma t}\cos\left(\omega_d t+\operatorname{atan}\left(\frac{A\sigma-B}{A\omega_d}\right)\right)$$

Example:

$$\frac{2s+14}{s^2+6s+25} \qquad \omega_n = \sqrt{25} = 5 \qquad 2\zeta\omega_n = 6 \qquad \zeta = 0.6$$

$$\sigma = 3 \qquad \omega_d = 5\sqrt{1-0.6^2} = 4$$

$$= \sqrt{A^2+\left(\frac{A\sigma-B}{\omega_d}\right)^2}e^{-\sigma t}\cos\left(\omega_d t+\operatorname{atan}\left(\frac{A\sigma-B}{A\omega_d}\right)\right)$$

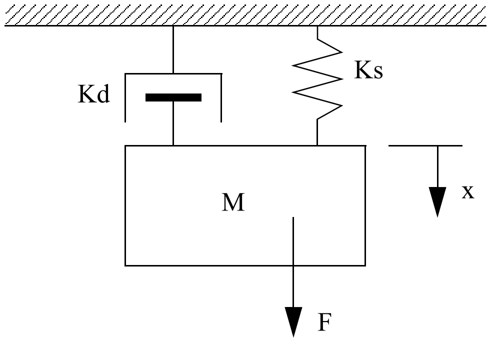
$$= \sqrt{2^2+\left(\frac{2(3)-14}{4}\right)^2}e^{-3t}\cos\left(4t+\operatorname{atan}\left(\frac{2(3)-14}{2(4)}\right)\right)$$

$$= 2.828e^{-3t}\cos(4t-0.7854)$$

Figure 17.8 Simplified Form for Solving Second Order Systems

17.2 Modeling Transfer Functions in the s-Domain

In previous chapters differential equations, and then transfer functions, were derived for mechanical and electrical systems. These can be converted to the s-domain, as shown in the mass-spring-damper example in Figure 17.9. In this case we assume the system starts undeflected and at rest, so the ‘D’ operator may be directly replaced with the Laplace ‘s’. If the system did not start at rest and undeflected, the ‘D’ operator would be replaced with a more complex expression that includes the initial conditions.



$$f = MD^2x + K_dDx + K_sx$$

$$\frac{f(t)}{x(t)} = MD^2 + K_dD + K_s$$

$$L\left[\frac{f(t)}{x(t)}\right] = \frac{F(s)}{X(s)} = Ms^2 + K_ds + K_s$$

ASIDE: Here 'D' is simply replaced with 's'. Although this is very convenient, it is only valid if the initial conditions are zero, otherwise the more complex form, shown below, must be used.

$$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nL[f(t)] - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \ldots - \frac{d^nf(0^-)}{dt^n}$$

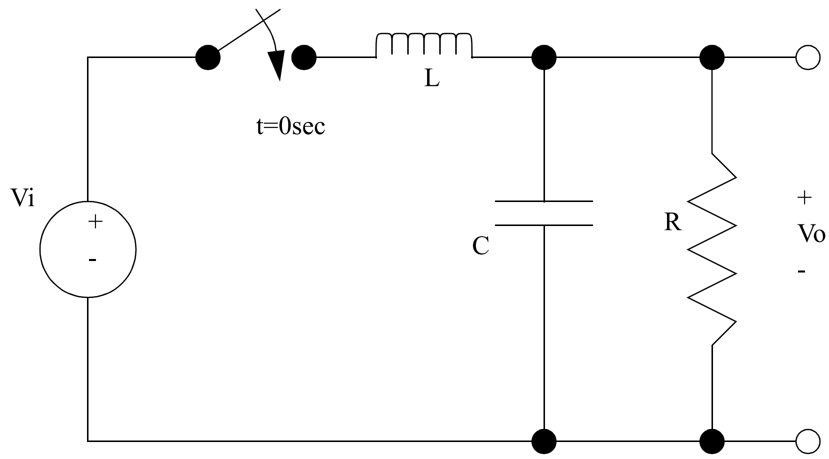
Figure 17.9 A mass-spring-damper example

Impedances in the s-domain are shown in Figure 17.10. As before these assume that the system starts undeflected and at rest.

Device	Time domain	s-domain	Impedance
Resistor	$v(t) = Ri(t)$	$V(s) = RI(s)$	$Z = R$
Capacitor	$v(t) = \frac{1}{C}\int i(t)dt$	$V(s) = \left(\frac{1}{C}\right)\frac{I(s)}{s}$	$Z = \frac{1}{sC}$
Inductor	$v(t) = L\frac{d}{dt}i(t)$	$V(s) = LsI(s)$	$Z = Ls$

Figure 17.10 Impedances of electrical components

Figure 17.11 shows an example of circuit analysis using Laplace transforms. The circuit is analyzed as a voltage divider, using the impedances of the devices. The switch that closes at t=0s ensures that the circuit starts at rest. The calculation result is a transfer function.



Treat the circuit as a voltage divider,

$$V_o = \frac{Vi\left(\frac{1}{sC+\frac{1}{R}}\right)}{sL+\left(\frac{1}{sC+\frac{1}{R}}\right)} = \frac{V_i\left(\frac{R}{1+sCR}\right)}{sL+\left(\frac{R}{1+sCR}\right)} = V_i\left(\frac{R}{s^2RLC+sL+R}\right)$$

$$\frac{V_o}{V_i} = \left(\frac{R}{s^2RLC+sL+R}\right)$$

Figure 17.11 A circuit example

At this point two transfer functions have been derived. To state the obvious, these relate an output and an input. To find an output response, an input is needed.

Examples.....

$$L[e^{-5(t-3)}u(t-3)] = e^{-3s}L[e^{-5t}] = \frac{e^{-3s}}{s+5}$$

$$L[10e^{-5t}\cos(8t+5)] \qquad A = \frac{10}{2}\angle 5 = 1.418 - 4.795j$$

$$= \frac{1.418-4.795j}{s+(-5)-(8)j} + \frac{1.418-4.795j}{s+(-5)+(8)j}$$

$$= \frac{-9.589s+25.253}{s^2-10s+89}$$

$$L^{-1}\left[\frac{-9.589s+25.253}{s^2-10s+89}\right]$$

$$= L^{-1}\left[\frac{1.418-4.795j}{s+(-5)-(8)j} + \frac{1.418-4.795j}{s+(-5)+(8)j}\right]$$

$$A = \frac{10}{2}\angle 5 = 1.418 - 4.795j$$

$$= 10e^{-5t}\cos(8t+5)$$

Figure 17.12 Transform table examples

17.3 Finding Output Equations

An input to a system is normally expressed as a function of time that can be converted to the s-domain. An example of this conversion for a step function is shown in Figure 17.13.

Apply a constant force of A, starting at time t=0 sec.
(*Note: a force applied instantly is impossible but assumed)

f(t) = 0 for t < 0
= A for t >= 0

Perform Laplace transform using tables

$$F(s) = L[f(t)] = \frac{A}{s}$$

Figure 17.13 An input function

In the previous section we converted differential equations, for systems, to transfer functions in the s-domain. These transfer functions are a ratio of output divided by input. If the transfer function is multiplied by the input function, both in the s-domain, the result is the system output in the s-domain.

Given,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$F(s) = \frac{A}{s}$$

Therefore,

$$X(s) = \left(\frac{X(s)}{F(s)}\right)F(s) = \left(\frac{1}{Ms^2 + K_d s + K_s}\right)\frac{A}{s}$$

Assume,

$$K_d = 3000\frac{Ns}{m}$$

$$K_s = 2000\frac{N}{m}$$

$$M = 1000kg$$

$$A = 1000N$$

$$\backslash X(s) = \frac{1}{(s^2 + 3s + 2)s}$$

Figure 17.14 A transfer function multiplied by the input function

Output functions normally have complex forms that are not found directly in transform tables. It is often necessary to simplify the output function before it can be converted back to the time domain. Partial fraction methods allow the functions to be broken into smaller, simpler components. The previous example in Figure 17.14 is continued in Figure 17.15 using a partial fraction expansion. In this example the roots of the third order denominator polynomial, are calculated. These provide three partial fraction terms. The residues (numerators) of the partial fraction terms must still be calculated. The example shows a method for finding resides by multiplying the output function by a root term, and then finding the limit as s approaches the root.

$$X(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \lim_{s \to 0}\left[s\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

$$B = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = -1$$

$$C = \lim_{s \to -2}\left[(s+2)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

Aside: the short cut above can reduce time for simple partial fraction expansions. A simple proof for finding 'B' above is given in this box.

$$\frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$(s+1)\left[\frac{1}{(s+1)(s+2)s}\right] = (s+1)\left[\frac{A}{s}\right] + (s+1)\left[\frac{B}{s+1}\right] + (s+1)\left[\frac{C}{s+2}\right]$$

$$\frac{1}{(s+2)s} = (s+1)\left[\frac{A}{s}\right] + B + (s+1)\left[\frac{C}{s+2}\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1}\left[(s+1)\left[\frac{A}{s}\right]\right] + \lim_{s \to -1} B + \lim_{s \to -1}\left[(s+1)\left[\frac{C}{s+2}\right]\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1} B = B$$

$$X(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}$$

Figure 17.15 Partial fractions to reduce an output function

After simplification with partial fraction expansion, the output function is easily converted back to a function of time as shown in Figure 17.16.

$$x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}\right]$$

$$x(t) = L^{-1}\left[\frac{0.5}{s}\right] + L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{0.5}{s+2}\right]$$

$$x(t) = [0.5] + [(-1)e^{-t}] + [(0.5)e^{-2t}]$$

$$x(t) = 0.5 - e^{-t} + 0.5e^{-2t}$$

Figure 17.16 Partial fractions to reduce an output function (continued)

17.4 Inverse Laplace Transforms and Partial Fractions

The flowchart in Figure 17.17 shows the general procedure for converting a function from the s-domain to a function of time. In some cases the function is simple enough to immediately use a transfer function table. Otherwise, partial fraction expansion is normally used to reduce the complexity of the function.

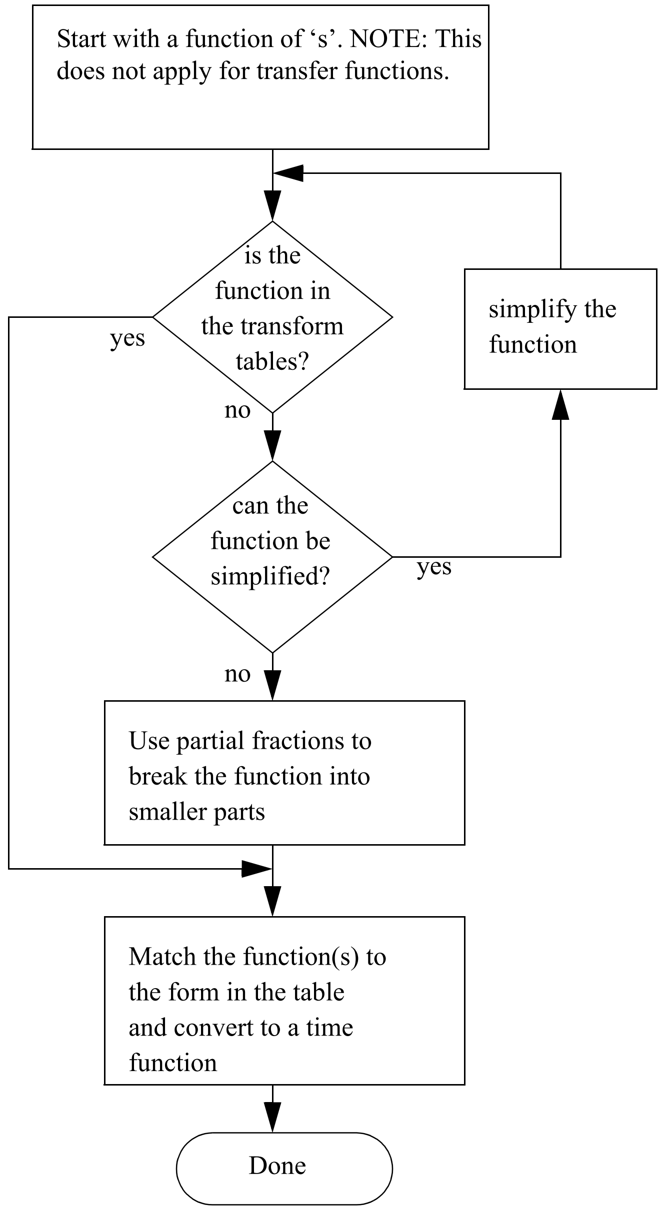


Figure 17.17 The methodology for doing an inverse transform of an output function

Figure 17.18 shows the basic procedure for partial fraction expansion. In cases where the numerator is greater than the denominator, the overall order of the expression can be reduced by long division. After this the denominator can be reduced from a polynomial to multiplied roots. Calculators or computers are normally used when the order of the polynomial is greater than second order. This results in a number of terms with unknown residues that can be found using a limit or algebra based technique.

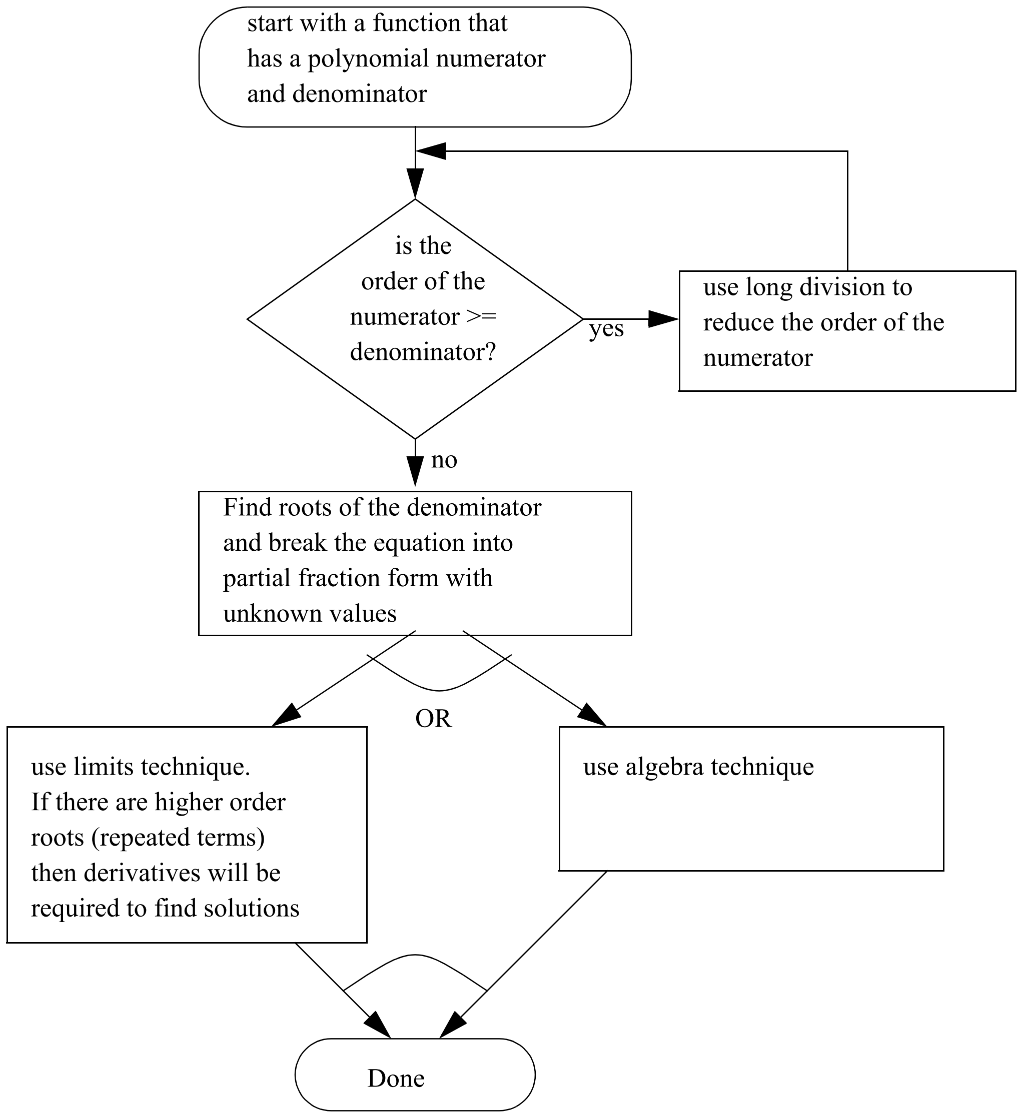


Figure 17.18 The methodology for solving partial fractions

Figure 17.19 shows an example where the order of the numerator is greater than the denominator. Long division of the numerator is used to reduce the order of the term until it is low enough to apply partial fraction techniques. This method is used infrequently because this type of output function normally occurs in systems with extremely fast response rates that are infeasible in practice.

$$X(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r l}
 & 5s + 3 \\
s^2 + 4 \,\big) & 5s^3 + 3s^2 + 8s + 6 \\
 & 5s^3 + 20s \\
\hline
 & 3s^2 - 12s + 6 \\
 & 3s^2 + 12 \\
\hline
 & -12s - 6
\end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$X(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

Figure 17.19 Partial fractions when the numerator is larger than the denominator

Partial fraction expansion of a third order polynomial is shown in Figure 17.20. The s-squared term requires special treatment. Here it produces two partial fraction terms divided by s and s-squared. This pattern is used whenever there is a root to an exponent.

$$X(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}\left[-(s+1)^{-2}\right] = -1$$

Figure 17.20 A partial fraction example

Figure 17.21 shows another example with a root to an exponent. In this case each of the repeated roots is given with the highest order exponent, down to the lowest order exponent. The reader will note that the order of the denominator is fifth order, so the resulting partial fraction expansion has five first order terms.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

Figure 17.21 Partial fractions with repeated roots

Algebra techniques are a reasonable alternative for finding partial fraction residues. The example in Figure 17.22 extends the example begun in Figure 17.21. The equivalent forms are simplified algebraically, until the point where an inverse matrix solution is used to find the residues.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 17.22 Solving partial fractions algebraically

For contrast, the example in Figure 17.22 is redone in Figure 17.23 using the limit techniques. In this case the use of repeated roots required the differentiation of the output function. In these cases the algebra techniques become more attractive, despite the need to solve simultaneous equations.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$A = \lim_{s \to 0}\left[\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{5}{(s+1)^3}\right] = 5$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{(s+1)^3}\right)\right] = \lim_{s \to 0}\left[\frac{5(-3)}{(s+1)^4}\right] = -15$$

$$C = \lim_{s \to -1}\left[\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{5}{s^2}\right] = 5$$

$$D = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{-2(5)}{s^3}\right] = 10$$

$$E = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{30}{s^4}\right] = 15$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 17.23 Solving partial fractions with limits

An inductive proof for the limit method of solving partial fractions is shown in Figure 17.24.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$\lim_{s \to -1}\left[\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right]$$

$$\lim_{s \to -1}\left[(s+1)^3\left(\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right)\right]$$

$$\lim_{s \to -1}\left[\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right]$$

For C, evaluate now,

$$\frac{5}{(-1)^2} = \frac{A(-1+1)^3}{(-1)^2} + \frac{B(-1+1)^3}{-1} + C + D(-1+1) + E(-1+1)^2$$

$$\frac{5}{(-1)^2} = \frac{A(0)^3}{(-1)^2} + \frac{B(0)^3}{-1} + C + D(0) + E(0)^2 \qquad \boxed{C = 5}$$

For D, differentiate once, then evaluate

$$\lim_{s \to -1}\left[\frac{d}{dt}\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\frac{-2(5)}{s^3} = A\left(-\frac{2(s+1)^3}{s^3} + \frac{3(s+1)^2}{s^2}\right) + B\left(-\frac{(s+1)^3}{s^2} + \frac{3(s+1)^2}{s}\right) + D + 2E(s+1)\right]$$

$$\frac{-2(5)}{(-1)^3} = \boxed{D = 10}$$

For E, differentiate twice, then evaluate (the terms for A and B will be ignored to save space, but these will drop out anyway).

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)^2\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)\left(\frac{-2(5)}{s^3} = A(\ldots) + B(\ldots) + D + 2E(s+1)\right)\right]$$

$$\lim_{s \to -1}\left[\frac{-3(-2(5))}{s^4} = A(\ldots) + B(\ldots) + 2E\right]$$

$$\frac{-3(-2(5))}{(-1)^4} = A(0) + B(0) + 2E \qquad \boxed{E = 15}$$

Figure 17.24 A proof of the need for differentiation for repeated roots

```
-->s=poly(0,'s');

-->res=pfss( (s + 4) / (s^5 + 5 * s^3 + 40 * s + 200) );

-->res(1)
 ans  =

     0.0950737 - 0.0244323s
    ------------------------
                         2
    9.293225 - 4.3103716s + s

-->res(2)
 ans  =

      0.0621686 + 0.0170142s
    --------------------------
                          2
    9.6469792 + 2.0795122s + s

-->res(3)
 ans  =

      0.0074181
    -------------
    2.2308594 + s

-->res(4)
       !--error 21
invalid index

-->
```

Figure 17.25 Finding Partial Fractions in Scilab

17.5 Examples

Mass-Spring-Damper Vibration

A mass-spring-damper system is shown in Figure 17.26 with a sinusoidal input.

Given,

$$\backslash\frac{X(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{K_d}{M}s + \frac{K_s}{M}}$$

Component values are,

$$M = 1kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m}$$

The sinusoidal input is converted to the s-domain,

$$f(t) = 5\cos(6t)N$$

$$\backslash F(s) = \frac{5s}{s^2 + 6^2}$$

This can be combined with the transfer function to obtain the output function,

$$X(s) = F(s)\left(\frac{x(s)}{F(s)}\right) = \left(\frac{5s}{s^2 + 6^2}\right)\left(\frac{\frac{1}{1}}{s^2 + 0.5s + 2}\right)$$

$$\backslash X(s) = \frac{5s}{(s^2 + 36)(s^2 + 0.5s + 2)}$$

$$X(s) = \frac{A}{s + 6j} + \frac{B}{s - 6j} + \frac{C}{s - 0.25 + 1.39j} + \frac{D}{s - 0.25 - 1.39j}$$

Figure 17.26 A mass-spring-damper example

The residues for the partial fraction in Figure 17.26 are calculated and converted to a function of time in Figure 17.27. In this case the roots of the denominator are complex, so the result has a sinusoidal component.

$$A = \lim_{s \to -6j}\left[\frac{(s + 6j)(5s)}{(s - 6j)(s + 6j)(s^2 + 0.5s + 2)}\right] = \frac{-30j}{(-12j)(36 - 3j + 2)}$$

$$A = 73.2\times10^{-3}–3.05$$

$$B = A^* = 73.2 ¥ 10^{-3}—3.05$$

Continue on to find C, D same way

$$X(s) = \frac{73.2\times10^{-3}–3.05}{s + 6j} + \frac{73.2 ¥ 10^{-3}—3.05}{s - 6j} + \ldots$$

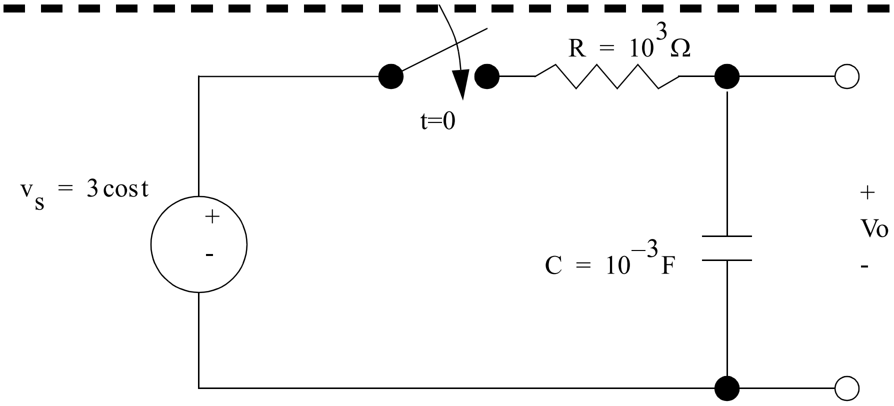
Do inverse Laplace transform

$$x(t) = 2(73.2\times10^{-3})e^{-0t}\cos(6t - 3.05) + \ldots$$

Figure 17.27 A mass-spring-damper example (continued)

Circuits

It is not necessary to develop a transfer functions for a system. The equation for the voltage divider is shown in Figure 17.28. Impedance values and the input voltage are converted to the s-domain and written in the equation. The resulting output function is manipulated into partial fraction form and the residues calculated. An inverse Laplace transform is used to convert the equation into a function of time using the tables.



As normal, relate the source voltage to the output voltage using component values in the s-domain.

$$V_o = V_s\left(\frac{Z_C}{Z_R + Z_C}\right) \qquad V_s(s) = \frac{3s}{s^2 + 1} \qquad Z_R = R \qquad Z_C = \frac{1}{sC}$$

Next, equations are combined. The numerator of resulting output function must be reduced by long division.

$$V_o = \frac{3s}{s^2+1}\left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right) = \frac{3s}{(s^2+1)(1+sRC)} = \frac{3s}{(s^2+1)(s10^3 10^{-3}+1)}$$

The output function can be converted to a partial fraction form and the residues calculated.

$$V_o = \frac{3s}{(s^2+1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} = \frac{As^2 + As + Bs + B + Cs^2 + C}{(s^2+1)(s+1)}$$

$$V_o = \frac{3s}{(s^2+1)(s+1)} = \frac{s^2(A+C) + s(A+B) + (B+C)}{(s^2+1)(s+1)}$$

$$B + C = 0 \qquad \therefore B = -C$$

$$A + C = 0 \qquad \therefore A = -C$$

$$A + B = 3 \qquad \therefore -C - C = 3 \qquad \therefore C = -1.5 \qquad \therefore A = 1.5 \qquad \therefore B = 1.5$$

$$V_o = \frac{1.5s + 1.5}{s^2 + 1} + \frac{-1.5}{s+1}$$

Figure 17.28 A circuit example

The output function can be converted to a function of time using the transform tables, as shown below.

$$V_o(t) = L^{-1}[V_o(s)] = L^{-1}\left[\frac{1.5s + 1.5}{s^2 + 1} + \frac{-1.5}{s + 1}\right] = L^{-1}\left[\frac{1.5s + 1.5}{s^2 + 1}\right] + L^{-1}\left[\frac{-1.5}{s + 1}\right]$$

$$\therefore V_o(t) = 1.5L^{-1}\left[\frac{s}{s^2 + 1}\right] + 1.5L^{-1}\left[\frac{1}{s^2 + 1}\right] - 1.5e^{-t}$$

$$\therefore v_o(t) = 1.5\cos t + 1.5\sin t - 1.5e^{-t}$$

$$\therefore v_o(t) = \sqrt{1.5^2 + 1.5^2}\sin\left(t + \operatorname{atan}\left(\frac{1.5}{1.5}\right)\right) - 1.5e^{-t}$$

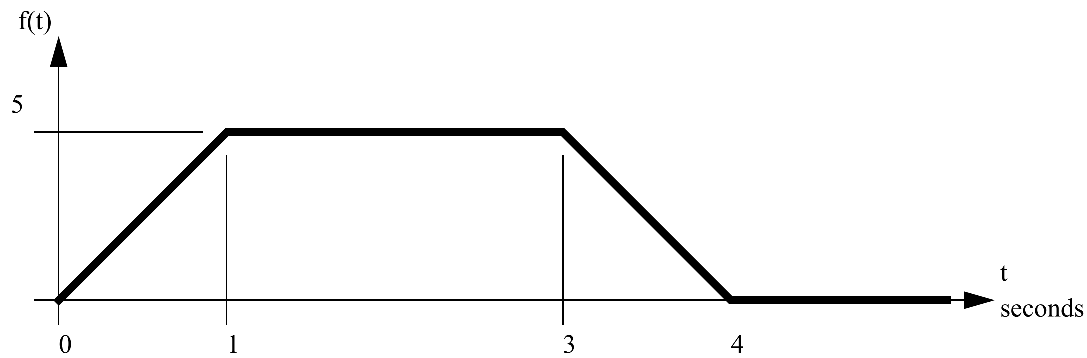
$$\therefore v_o(t) = 2.121\sin\left(t + \frac{\pi}{4}\right) - 1.5e^{-t}$$

Figure 17.29 A circuit example (continued)

17.6 Advanced Topics

Input Functions

In some cases a system input function is comprised of many different functions, as shown in Figure 17.30. The step function can be used to switch function on and off to create a piecewise function. This is easily converted to the s-domain using the e-to-the-s functions.



$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

$$F(s) = \frac{5}{s^2} - \frac{5e^{-s}}{s^2} - \frac{5e^{-3s}}{s^2} + \frac{5e^{-4s}}{s^2}$$

Figure 17.30 Switching on and off function parts

Initial and Final Value Theorems

The initial and final values an output function can be calculated using the theorems shown in Figure 17.31.

$$x(t \to \infty) = \lim_{s \to 0} [sx(s)]$$ Final value theorem

$$\therefore x(t \to \infty) = \lim_{s \to 0}\left[\frac{1s}{(s^2 + 3s + 2)s}\right] = \lim_{s \to 0}\left[\frac{1}{s^2 + 3s + 2}\right] = \frac{1}{(0)^2 + 3(0) + 2} = \frac{1}{2}$$

$$x(t \to 0) = \lim_{s \to \infty} [sx(s)]$$ Initial value theorem

$$\therefore x(t \to 0) = \lim_{s \to \infty}\left[\frac{1(s)}{(s^2 + 3s + 2)s}\right] = \frac{1}{((\infty)^2 + 3(\infty) + 2)} = \frac{1}{\infty} = 0$$

Figure 17.31 Final and initial values theorems

17.7 Impulse Functions

An impulse is a very brief event, such as a hammer strike. A theoretical impulse has a duration of zero and a height of infinity. But integrating the impulse function has a value of one at the time of the impulse, Figure 17.32. Although the duration of zero and infinite amplitude are physically impossible, impulse functions are very useful for describing sudden changes in a system.

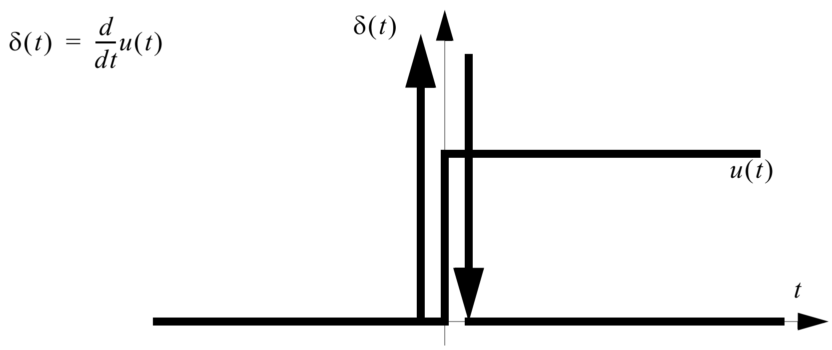


Figure 17.32 The impulse function

A unit impulse is the input in Figure 17.33. The input ‘f’ is a brief but nearly infinite impact. In terms of energy the force is the equivalent of 1Ns. In the s-domain the force becomes 1. The following analysis is similar to those done previously. The result on this impact is a motion resulting in two exponential decay functions.

$$\frac{Y}{X} = \frac{s+3}{s^2+3s+2} \qquad f = \delta(t)N \qquad \therefore F = 1$$

$$Y = \frac{s+3}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{s(A+B)+(2A+B)}{s^2+3s+2}$$

$$\begin{array}{rl} & 2A + B = 3 \\ - & A + B = 1 \\ \hline & A = 2 \end{array} \qquad B = -1$$

$$Y = \frac{2}{s+1} + \frac{-1}{s+2} \qquad y = 2e^{-t} - 1e^{-2t}$$

Figure 17.33 A sample system with a unit impulse input

When the result of a Laplace analysis is an impulse function this indicates a very brief event, such as an impact or release of energy.

17.8 A Map of Techniques for Laplace Analysis

The following map is to be used to generally identify the use of the various topics covered in the course.

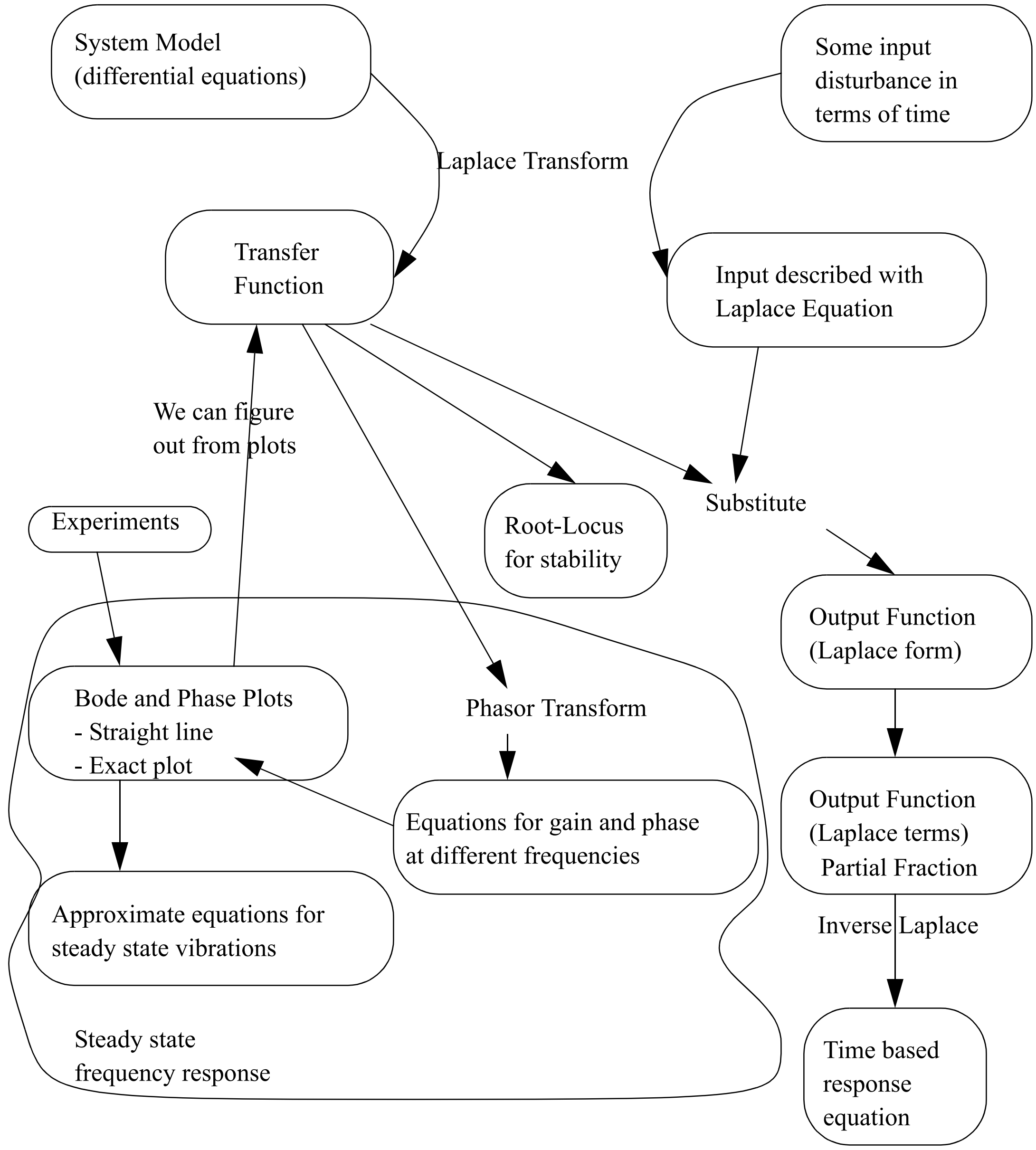


Figure 17.34   A map of Laplace analysis techniques

17.9 Summary

- Transfer and input functions can be converted to the s-domain
- Output functions can be calculated using input and transfer functions
- Output functions can be converted back to the time domain using partial fractions.

17.10 Problems With Solutions

Problem 17.1 Convert the following functions from time to Laplace functions using the tables.

a) $L[5]$

b) $L[e^{-3t}]$

c) $L[5e^{-3t}]$

d) $L[5te^{-3t}]$

e) $L[5t]$

f) $L[4t^2]$

g) $L[\cos(5t)]$

h) $L[3(t-1)+e^{-(t+1)}]$

i) $L[5e^{-3t}\cos(5t)]$

j) $L[5e^{-3t}\cos(5t+1)]$

k) $L[\sin(5t)]$

l) $L[\sinh(3t)]$

m) $L[t^2\sin(2t)]$

n) $L\left[\frac{d}{dt}t^2e^{-3t}\right]$

o) $L[\ddot{x}+5\dot{x}+3x], \dot{x}(0) = 8, x(0) = 7$

p) $L\left[\frac{d}{dt}\sin(6t)\right]$

q) $L\left[\left(\frac{d}{dt}\right)^3 t^2\right]$

r) $L\left[\int_0^t y\,dt\right]$

s) $L[3t^3(t-1)+e^{-5t}]$

t) $L[u(t-1)-u(t-2)]$

u) $L[e^{-2t}u(t-2)]$

v) $L[e^{-(t-3)}u(t-1)]$

w) $L[5e^{-3t}+u(t-1)-u(t-2)]$

x) $L[\cos(7t+2)+e^{t-3}]$

y) $L[\cos(5t+1)]$

z) $L[6e^{-2.7t}\cos(9.2t+3)]$

aa)

Problem 17.2 Convert the following functions below from the frequency to time domains using the tables.

a) $L^{-1}\left[\frac{1}{s+1}\right]$

b) $L^{-1}\left[\frac{5}{s+1}\right]$

c) $L^{-1}\left[\frac{6}{s^2}\right]$

d) $L^{-1}\left[\frac{6}{s^3}\right]$

e) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

f) $L^{-1}\left[\frac{6}{s^2+6}\right]$

g) $L^{-1}\left[\frac{5}{s}(1-e^{-4.5s})\right]$

h) $L^{-1}\left[\frac{4+3j}{s+1-2j}+\frac{4-3j}{s+1+2j}\right]$

i) $L^{-1}\left[\frac{6}{s^4}+\frac{6}{s^2+9}\right]$

j) $L^{-1}\left[\frac{6}{s^2+5s+6}\right]$

k) $L^{-1}\left[\frac{6}{4s^2+20s+24}\right]$

Problem 17.3 Convert the following differential equations to transfer functions.

a) $5\ddot{x} + 6\dot{x} + 2x = 5f$

b) $\dot{y} + 8y = 3x$

c) $\dot{y} - y + 5x = 0$

Problem 17.4 Do the following conversions as indicated.

a) $L[5e^{-4t}\cos(3t+2)] =$

b) $L[e^{-2t} + 5t(u(t-2) - u(t))] =$

c) $L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right)y + y\right] =$ where at t=0 $y_0 = 1$ $y_0' = 2$ $y_0'' = 3$ $y_0''' = 4$

d) $L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] =$

e) $L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2 + 4s + 40}\right] =$

Problem 17.5 Convert the output function to functions of time.

a) $\frac{s^3 + 4s^2 + 4s + 4}{s^3 + 4s}$

b) $\frac{s^2 + 4}{s^4 + 10s^3 + 35s^2 + 50s + 24}$

Problem 17.6 Convert the following functions below from the Laplace to time domains using partial fractions and the tables.

a) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

b) $L^{-1}\left[\frac{2s+2}{s^2 + s + 2}\right]$

c) $L^{-1}\left[\frac{s+1}{s^2 + s + 3}\right]$

d) $L^{-1}\left[\frac{s^2 + 2s}{s^3 + 4s^2 + 8s}\right]$

e) $L^{-1}\left[\frac{6}{s^2 + 5s}\right]$

f) $L^{-1}\left[\frac{9s^2 + 6s + 3}{s^3 + 5s^2 + 4s + 6}\right]$

g) $L^{-1}\left[\frac{s^3 + 9s^2 + 6s + 3}{s^3 + 5s^2 + 4s + 6}\right]$

h) $L^{-1}\left[\frac{9s+4}{(s+3)^3}\right]$

i) $L^{-1}\left[\frac{9s+4}{s^3(s+3)^3}\right]$

j) $L^{-1}\left[\frac{s^2 + 2s + 1}{s^2 + 3s + 2}\right]$

k) $L^{-1}\left[\frac{s^2 + 3s + 5}{6s^2 + 6}\right]$

l) $L^{-1}\left[\frac{s^2 + 2s + 3}{s^2 + 2s + 1}\right]$

Problem 17.7 Convert the output function below Y(s) to the time domain Y(t) using the tables.

$$Y(s) = \frac{5}{s} + \frac{12}{s^2 + 4} + \frac{3}{s + 2 - 3j} + \frac{3}{s + 2 + 3j}$$

Problem 17.8 Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G(s) = \frac{4}{s+2} = \frac{Y(s)}{X(s)} \qquad x(t) = 20 \quad \text{When } t >= 0$$

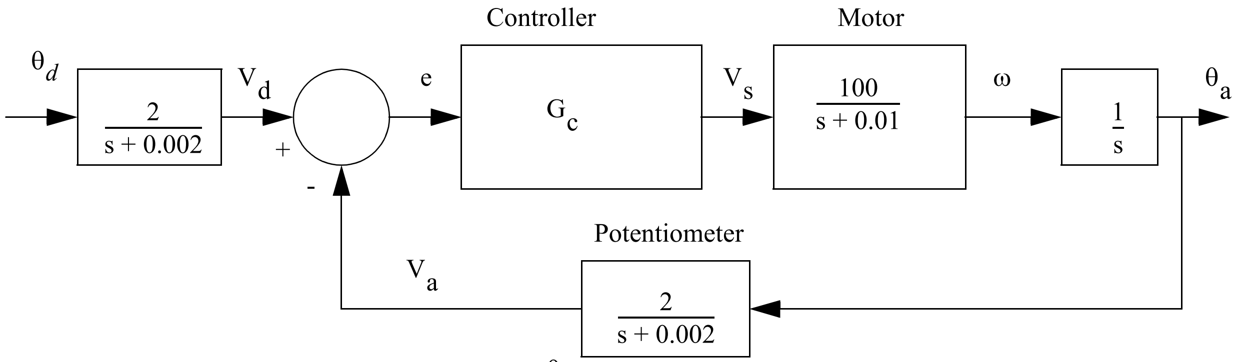
Problem 17.9 Given the following input functions and transfer functions, find the response in time.

	Transfer Function	Input
a)	$\frac{X(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{N}\right)$	$f(t) = 5N$
b)	$\frac{X(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{N}\right)$	$x(t) = 5m$

Problem 17.10 Solve the differential equation using Laplace transforms. Assume the system starts undeflected and at rest.

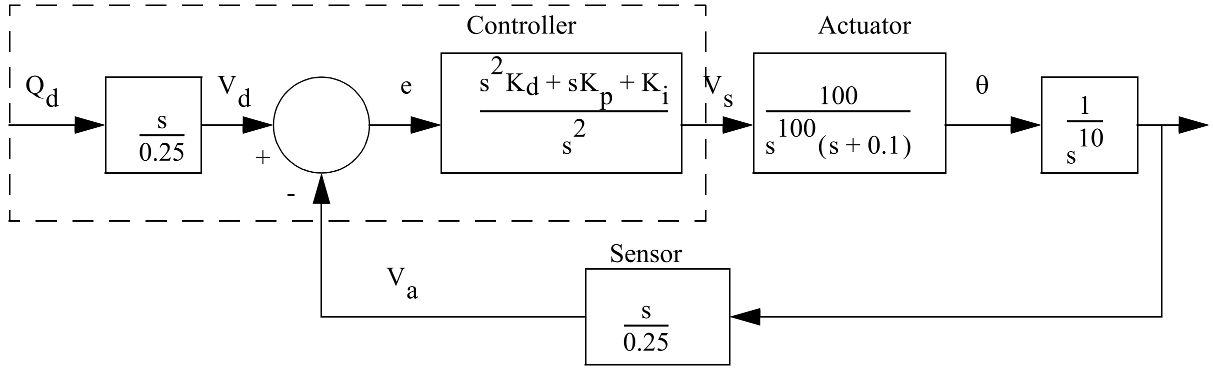
$$\dddot{\theta} + 40\ddot{\theta} + 20\dot{\theta} + 2\theta = 4$$

Problem 17.11 For the following control system select a controller transfer function, Gc, that will make the overall system performance match the desired transfer function.



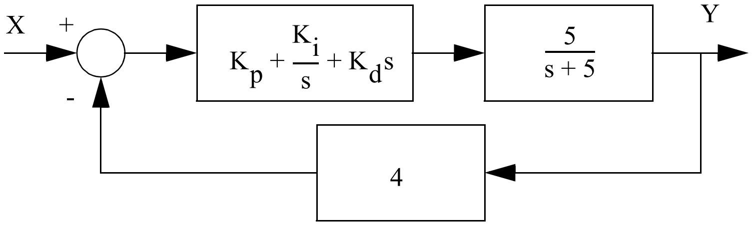
desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{s+0.1}$

Problem 17.12 Write a C program for an ATMega microcontroller to implement the control system in the dashed line below with an update time of 10ms.



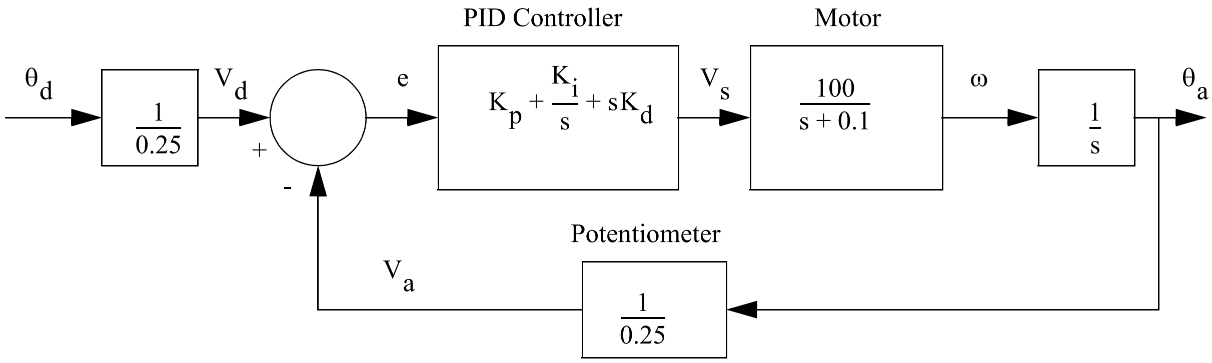
Problem 17.13 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer

function is given.



a) Develop the transfer function for the system.
b) Select controller values that will result in a response that includes a natural frequency of 2 rad/sec and damping factor of 0.5. Verify that the controller will be stable.
c) If the values of Kp = Ki = Kd = 1 find the response to an input of 5sin(10t) as a function of time using the Laplace Transforms.
d) Find the response in part c) using numerical methods. Show the results as a table and graph. The results should show the region(s) of greatest interest.
e) Find the system response to an input of X = 5sin(100t + 1) using phasor transforms.

Problem 17.14 a) The block diagram below is for an angular positioning system. The set point is a desired angle, which is converted to a desired voltage. This is compared to a feedback voltage from a potentiometer. A PID controller is used to generate an output voltage to drive a DC motor. Simplify the block diagram.



b) Given the transfer function below, select values for Kp, Ki and Kd that will result in a second order response that has a damping factor of 0.125 and a natural frequency of 10rad/s. (Hint: eliminate Ki).

$$\frac{\theta_a}{\theta_d} = \frac{s^2(400K_d) + s(400K_p) + (400K_i)}{s^3 + s^2(0.1 + 400K_d) + s(400K_p) + (400K_i)}$$

c) The function below has a step input of magnitude 1.0. Find the output as a function of time using numerical methods. Give the results in a table OR graph.

$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + 2.5s + 100}$$

d) The function below has a step input of magnitude 1. Find the output as a function of time by integrating the differential equation (i.e., using the homogeneous and particular solutions).

$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + 2.5s + 100}$$

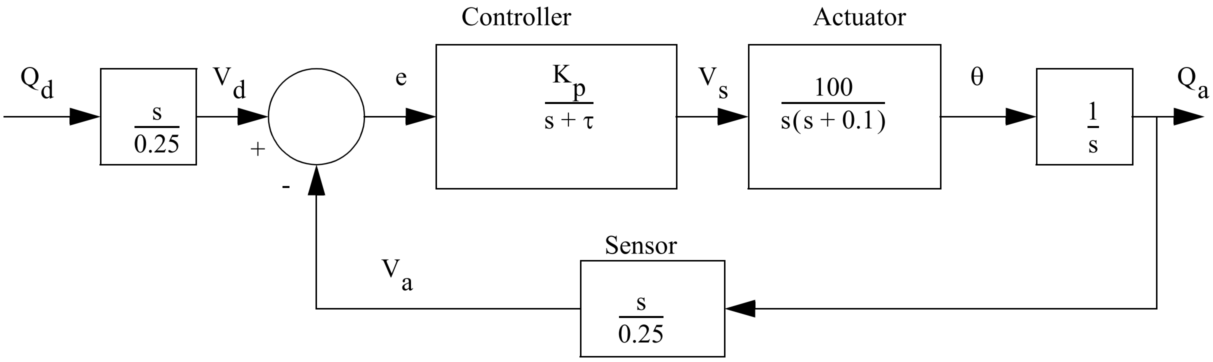
e) The function below has a step input of magnitude 1. Find the output as a function of time using Laplace transforms.

$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + 2.5s + 100}$$

f) Given the transfer function below; 1) apply a phasor transform and express the gain and phase angle as a function of frequency, 2) calculate a set of values and present them in a table, 3) use the values calculated in step 2) to develop a frequency response plot on semi-log paper, 4) draw a straight line approximation of the Bode plot on semi-log paper.

$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + s + 4}$$

Problem 17.15 a) Simplify the block diagram as far as possible.



b) Given the transfer function below, select values for Kp and tau that will include a second order response that has a damping factor of 0.125 and a natural frequency of 10rad/s.

$$\frac{Q_a}{Q_d} = \frac{400K_p}{s^3 + s^2(0.1 + \tau) + s(0.1\tau) + (400K_p)}$$

Hint: $$\frac{(\ )}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + A)}$$

c) The function below has a step input of magnitude 1.0. Find the output as a function of time using numerical methods. Give the results in a table OR graph from 0.0 to 0.010s.

$$\frac{Q_a}{Q_d} = \frac{-4156.8}{s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)}$$

d) The function below has a step input of magnitude 1. Find the output as a function of time by integrating the differential equation (i.e., using the homogeneous and particular solutions).

$$\frac{Q_a}{Q_d} = \frac{-4156.8}{s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)}$$

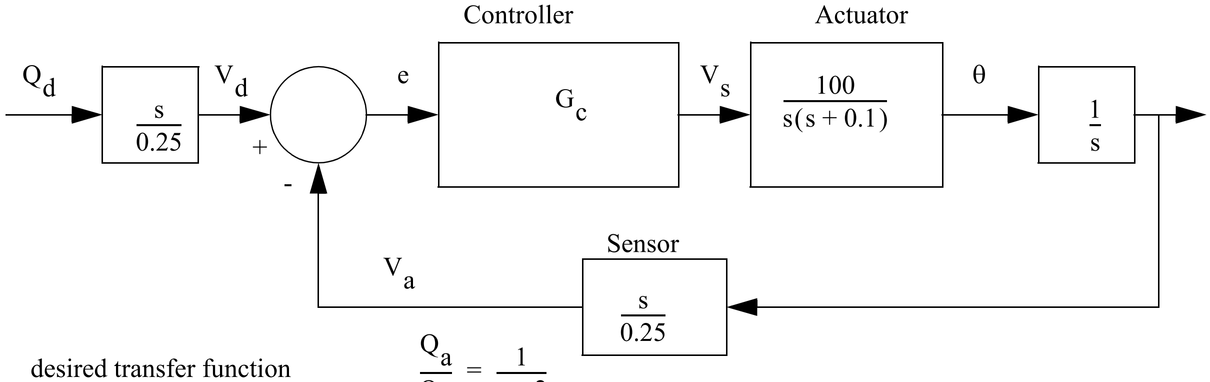
e) The function below has a step input of magnitude 1. Find the output as a function of time using Laplace transforms.

$$\frac{Q_a}{Q_d} = \frac{-4156.8}{s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)}$$

f) Given the transfer function below; 1) apply a phasor/Fourier transform and express the gain and phase angle as a function of frequency, 2) calculate a set of values and present them in a table, 3) use the values calculated in step 4) to develop a frequency response plot on semi-log paper, 5) draw a straight line approximation of the Bode plot on semi-log paper.

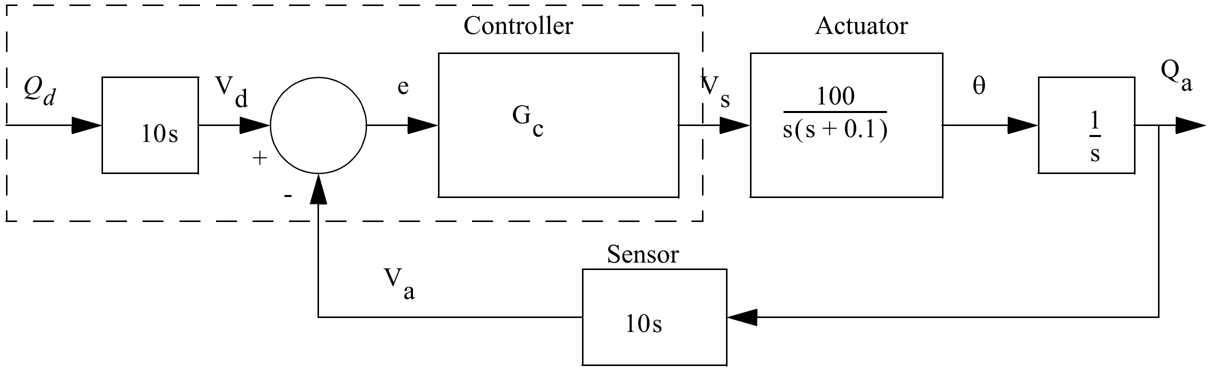
$$\frac{Q_a}{Q_d} = \frac{-4156.8}{s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)}$$

g) Select a controller transfer function, Gc, that will reduce the system to a first order system with a time constant of 0.5s, as shown below.



desired transfer function $$\frac{Q_a}{Q_d} = \frac{1}{s+2}$$

Problem 17.16 a) Find the simplified transfer function for the block diagram.



b) Given the transfer function below calculate a controller transfer function, Gc. The desired response should be first order with a time constant of 1s.

$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s^2 + (0.1)s + 1000G_c}$$

c) For the system given below, and the provided controller function, find the response to a unit step input using Laplace transforms.

$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s^2 + (0.1)s + 1000G_c} \qquad G_c = 0.01(s+1)$$

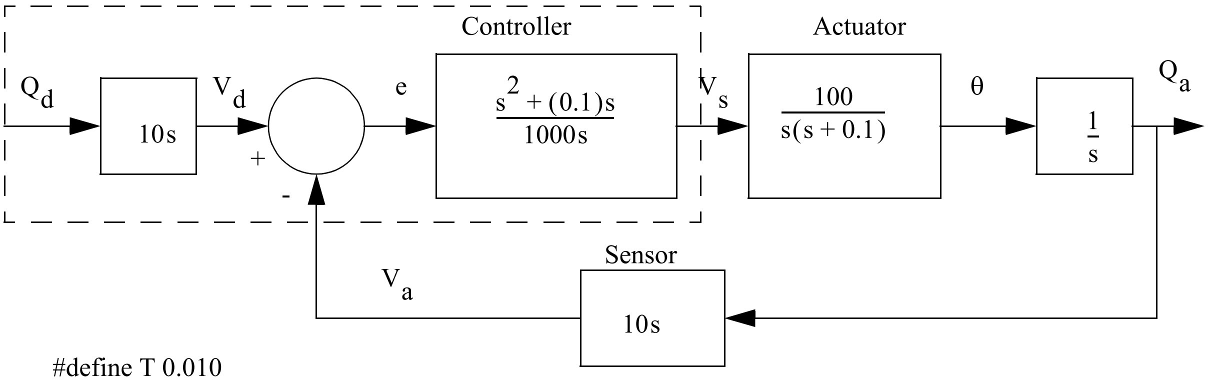
d) For the system given below, and the provided controller function, find the response to a unit step input by solving the differential equation.

$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s^2 + (0.1)s + 1000G_c} \qquad G_c = 0.01(s+1)$$

e) For the system given below, and the provided controller function, find the response to a unit step input using numerical methods.

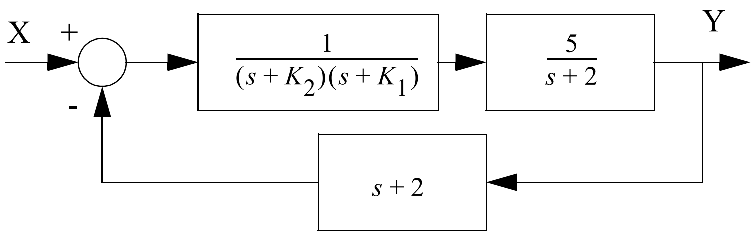
$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s^2 + (0.1)s + 1000G_c} \qquad G_c = 0.01(s+1)$$

f) Write a C program for an ATMega microcontroller to implement the control system in the dashed line below with an update time of 10ms.




```
#define T 0.010

double Vs(double Qd, double Va){
```

Problem 17.17 A feedback control system is shown below.



a) Develop the transfer function for the system.
b) Select controller values that will result in a response that includes a natural frequency of 10 rad/sec and damping factor of 2.0.

Problem 17.18 a) Using Laplace transforms find the system response for the given transfer function and input function.

$$\frac{\omega}{V_d} = \frac{200}{s^2 + 2s + 100} \qquad V_d(t) = 5$$

b) Convert the transfer function to a differential equation and then solve the differential equation with the given input.

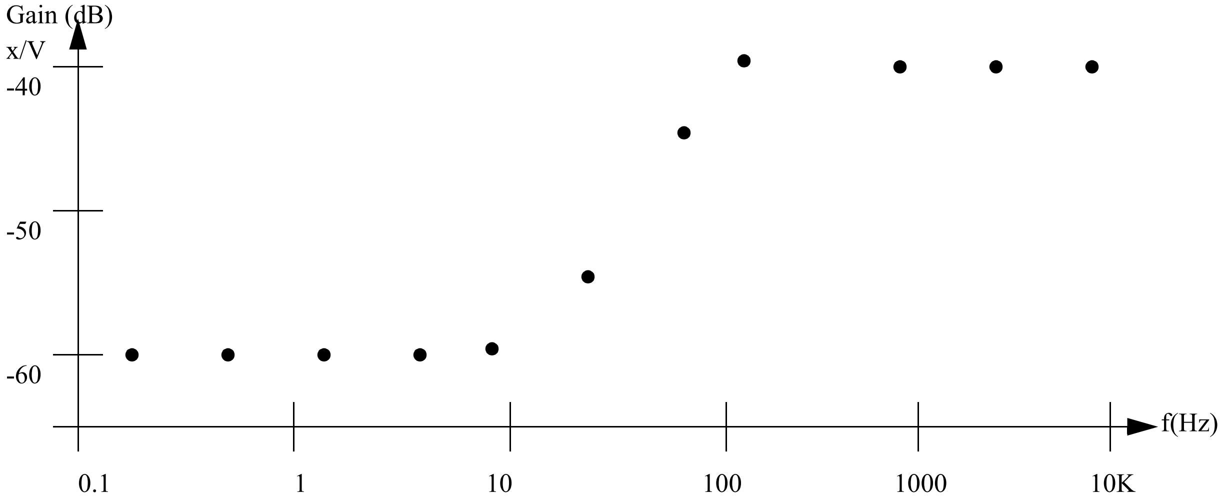
$$\frac{\omega}{V_d} = \frac{200}{s^2 + 2s + 100} \qquad V_d(t) = 5$$

c) Convert the given transfer function to state equations and then solve it numerically. Show the results for the first 2 seconds in 0.2s intervals.

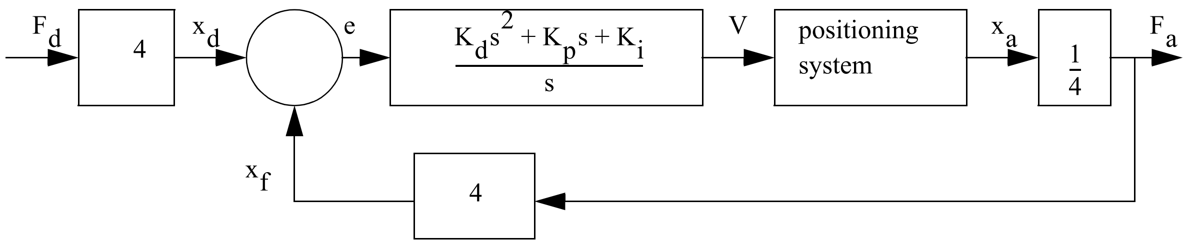
$$\frac{\omega}{V_d} = \frac{200}{s^2 + 2s + 100} \qquad V_d(t) = 5$$

d) Compare the solutions for parts a), b), and c). Show that they are equal.

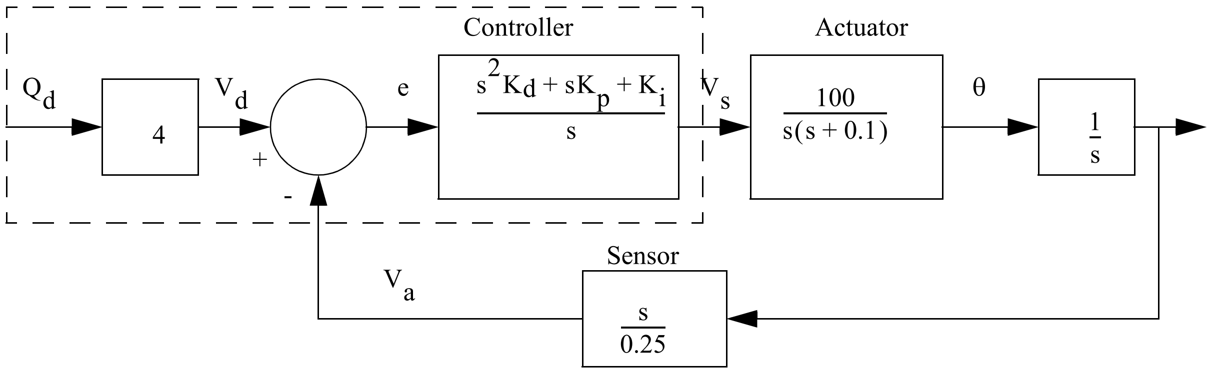
Problem 17.19 a) Given the experimental Bode (Frequency Response Function) plot below, find a transfer function to model a positioning system. The input is a voltage ‘V’ and the output is a displacement ‘x’.



b) The transfer function found in step a) will be used for the positioning system in the block diagram below. Find the overall transfer function for the system.



Problem 17.20 Write a C subroutine for an ATMega microcontroller to implement the control system in the dashed line below with an update time of 10ms. The subroutine should use integer math for all calculations



17.11 Problem Solutions

Answer 17.1

a) $\frac{5}{s}$

b) $\frac{1}{s+3}$

c) $\frac{5}{s+3}$

d) $\frac{5}{(s+3)^2}$

e) $\frac{5}{s^2}$

f) $\frac{8}{s^3}$

g) $\frac{s}{s^2+25}$

h) $\frac{3}{s^2}-\frac{3}{s}+\frac{e^{-1}}{s+1}$

i) $\frac{5(s+3)}{(s+3)^2+5^2}$

j) $\frac{2.5\text{–}1}{s+3-5j}+\frac{2.5\text{—}1}{s+3+5j}$

$=\frac{s(\ )+}{s^2+6s+34}$

k) $\frac{5}{s^2+25}$

l) $\frac{0.5}{s-3}-\frac{0.5}{s+3}$

m) $\frac{-4}{(s^2+4)^2}+\frac{16s^2}{(s^2+4)^3}=\frac{12s^2-16}{(s^2+4)^3}$

n) $\frac{2s}{(s+3)^3}$

o) $(s^2x-7s-8)+5(sx-7)+3x$

p) $\frac{6s}{s^2+36}$

q) 0

r) $\frac{y}{s}$

s) $\frac{72}{s^5}-\frac{18}{s^4}+\frac{1}{s+5}$

t) $\frac{e^{-s}}{s}-\frac{e^{-2s}}{s}$

u) $\frac{e^{-4-2s}}{s+2}$

v) $\frac{e^{2-s}}{s+1}$

w) $\frac{5}{s+3}+\frac{e^{-s}}{s}-\frac{e^{-2s}}{s}$

x) $\frac{\cos(2)s-\sin(2)7}{s^2+49}+\frac{e^{-3}}{s-1}=\frac{-0.416s-6.37}{s^2+49}+\frac{e^{-3}}{s-1}$

y) $\frac{s\cos 1-5\sin 1}{s^2+25}$

z) $\frac{3\text{–}3}{s+2.7-9.2j}+\frac{3\text{—}3}{s+2.7+9.2j}=\frac{s(\ )+}{s^2+5.4s+91.93}$

Answer 17.2

a) e^{-t}

b) $5e^{-t}$

c) $6t$

d) $3t^2$

e) $-e^{-3t} + 2e^{-4t}$

f) $\sqrt{6}\sin(\sqrt{6}t)$

g) $5 - 5u(t - 4.5)$

h) $2(5)e^{-(1)t}\cos\left(2t + \operatorname{atan}\left(\frac{3}{4}\right)\right)$

i) $t^3 + 2\sin(3t)$

j) $6e^{-2t} - 6e^{-3t}$

k) $1.5e^{-2t} - 1.5e^{-3t}$

Answer 17.3

a) $\frac{X}{F} = \frac{5}{5s^2 + 6s + 2}$

b) $\frac{Y}{X} = \frac{3}{s + 8}$

c) $\frac{Y}{X} = \frac{-5}{s - 1}$

Answer 17.4 a)

$$L[5e^{-4t}\cos(3t + 2)] = L[2|A|e^{-\alpha t}\cos(\beta t + \theta)]$$

$\alpha = 4$ $\beta = 3$

$|A| = 2.5$ $\theta = 2$

$$A = 2.5\cos 2 + 2.5j\sin 2 = -1.040 + 2.273j$$

$$\frac{A}{s + \alpha - \beta j} + \frac{A^{\text{complex conjugate}}}{s + \alpha + \beta j} = \frac{-1.040 + 2.273j}{s + 4 - 3j} + \frac{-1.040 - 2.273j}{s + 4 + 3j}$$

b)

$$L[e^{-2t} + 5t(u(t-2) - u(t))] = L[e^{-2t}] + L[5tu(t-2)] - L[5tu(t)]$$

$$= \frac{1}{s+2} + 5L[tu(t-2)] - \frac{5}{s^2} = \frac{1}{s+2} + 5L[(t-2)u(t-2) + 2u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5L[(t-2)u(t-2)] + 10L[u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5e^{-2s}L[t] + 10e^{-2s}L[1] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + \frac{5e^{-2s}}{s^2} + \frac{10e^{-2s}}{s} - \frac{5}{s^2}$$

c)

$$\left(\frac{d}{dt}\right)^3 y = s^3 y + 1s^2 + 2s^1 + 3s^0$$

$$\left(\frac{d}{dt}\right)y = s^1 y + s^0 1$$

$$L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right)y + y\right] = (s^3 y + 1s^2 + 2s + 3) + (sy + 1) + (y)$$

$$= y(s^3 + s + 1) + (s^2 + 2s + 4)$$

d)

$$L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] = L^{-1}\left[\frac{A}{s+\alpha-\beta j} + \frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}\right]$$

$$|A| = \sqrt{1^2+1^2} = 1.141 \qquad \theta = \operatorname{atan}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} \qquad \alpha = 3 \qquad \beta = 4$$

$$= 2|A|e^{-\alpha t}\cos(\beta t + \theta) = 2.282e^{-3t}\cos\left(4t - \frac{\pi}{4}\right)$$

e)

$$L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2+4s+40}\right] = L[s] + L\left[\frac{1}{s+2}\right] + L\left[\frac{3}{s^2+4s+40}\right]$$

$$= \frac{d}{dt}\delta(t) + e^{-2t} + L\left[\frac{3}{(s+2)^2+36}\right] = \frac{d}{dt}\delta(t) + e^{-2t} + 0.5L\left[\frac{6}{(s+2)^2+36}\right]$$

$$= \frac{d}{dt}\delta(t) + e^{-2t} + 0.5e^{-2t}\sin(6t)$$

Answer 17.5 a)

$$\frac{s^3+4s^2+4s+4}{s^3+4s}$$

$$\begin{array}{r|l} & 1 \\ \hline s^3+4s & s^3+4s^2+4s+4 \\ & -(s^3+4s) \\ \hline & 4s^2+4 \end{array}$$

$$= 1 + \frac{4s^2+4}{s^3+4s} = 1 + \frac{A}{s} + \frac{Bs+C}{s^2+4} = 1 + \frac{s^2(A+B)+s(C)+(4A)}{s^3+4s}$$

$$A = 1 \qquad C = 0 \qquad B = 3$$

$$= 1 + \frac{1}{s} + \frac{3s}{s^2+4}$$

$$= \delta(t) + 1 + 3\cos(2t)$$

b)

$$\frac{s^2+4}{s^4+10s^3+35s^2+50s+24} = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}+\frac{D}{s+4}$$

$$A = \lim_{s\to -1}\left(\frac{s^2+4}{(s+2)(s+3)(s+4)}\right) = \frac{5}{6}$$

$$B = \lim_{s\to -2}\left(\frac{s^2+4}{(s+1)(s+3)(s+4)}\right) = \frac{8}{-2}$$

$$C = \lim_{s\to -3}\left(\frac{s^2+4}{(s+1)(s+2)(s+4)}\right) = \frac{13}{2}$$

$$D = \lim_{s\to -4}\left(\frac{s^2+4}{(s+1)(s+2)(s+3)}\right) = \frac{20}{-6}$$

$$\frac{5}{6}e^{-t}-4e^{-2t}+\frac{13}{2}e^{-3t}-\frac{10}{3}e^{-4t}$$

Answer 17.6

a) $-e^{-3t}+2e^{-4t}$

b)

c)

d)

e) $1.2-1.2e^{-5t}$

f) $8.34e^{-4.4t}+2(0.99)e^{-0.3t}\cos(1.13t+1.23)$

g)

h)

i)

j) $\delta(t)-e^{-2t}$

k) $\frac{\delta(t)}{6}+0.834\cos(t-0.927)$

l) $\delta(t)+2te^{-t}$

Answer 17.7

$$y(t) = 5+6\sin(2t)+2(3)e^{-2t}\cos(3t-0)$$

Answer 17.8

$$y(t) = 40-40e^{-2t}$$

Answer 17.9

a) $\frac{5}{6}+\frac{5}{3}e^{-3t}-\frac{5}{2}e^{-4t}$

b) $(5\delta(t)+30-5e^{-2t})N$

Answer 17.10

$$\theta(t) = -66 \cdot 10^{-6} e^{-39.50t} - 3.216 e^{-0.1383t} + 1.216 e^{-0.3368t} + 2.00$$

Answer 17.11

$$G_c = \frac{s^3 + 0.012s^2 + 0.00002s}{200s - 180}$$

Answer 17.12

$$\frac{V_s}{e} = \frac{s^2 K_d + sK_p + K_i}{s^2}$$

$$\therefore \ddot{V}_s = K_d e + K_p \int dt + K_i \iint e dt$$

$$\frac{V_d}{Q_d} = \frac{s}{0.25}$$

$$\therefore V_d = 4\dot{Q}_d$$

Answer 17.13

a) $$\frac{Y}{X} = \frac{5\left(K_p s + K_i + K_d s^2\right)}{s^2(1 + 20K_d) + s(5 + 20K_p) + (20K_i)}$$

b) $K_p = -0.15$ $\qquad K_i = 1.0$ (← picked arbitrarily) $\qquad K_d = \frac{1}{5}$

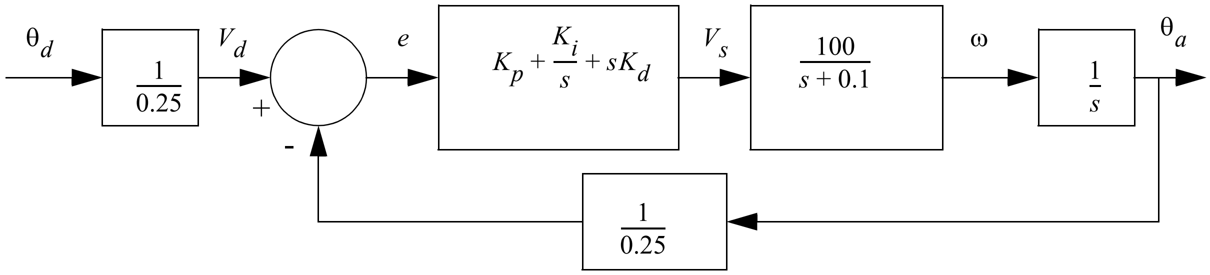
c) add

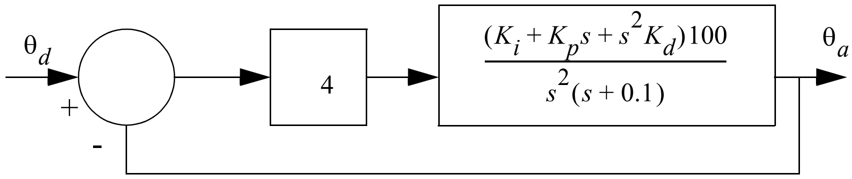
d) $\dot{y} = v$

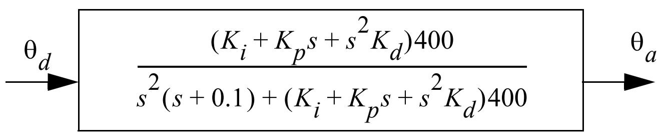
$$\dot{v} = \frac{-25}{21}v - \frac{20}{21}y + \frac{5}{21}\ddot{x} + \frac{5}{21}\dot{x} + \frac{5}{21}x$$

e) add

Answer 17.14 a)







$$\frac{\theta_a}{\theta_d} = \frac{s^2(400K_d) + s(400K_p) + (400K_i)}{s^3 + s^2(0.1 + 400K_d) + s(400K_p) + 400K_i}$$

b)

$$\frac{\theta_a}{\theta_d} = \frac{s^2(400K_d) + s(400K_p) + (400K_i)}{s^3 + s^2(0.1 + 400K_d) + s(400K_p) + (400K_i)} \qquad K_i = 0$$

$$\frac{\theta_a}{\theta_d} = \frac{s^2(400K_d) + s(400K_p)}{s^3 + s^2(0.1 + 400K_d) + s(400K_p)} = \frac{s(400K_d) + (400K_p)}{s^2 + s(0.1 + 400K_d) + (400K_p)}$$

$$400K_p = 10^2 \qquad K_p = 0.25$$

$$0.1 + 400K_d = 2(10)0.125 \qquad K_d = 0.0060$$

c)

$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + 2.5s + 100}$$

$$\ddot{\theta}_a + \dot{\theta}_a(2.5) + \theta_a(100) = \ddot{\theta}_d(0.9) + \theta_d(4)$$

$$\omega_a = \dot{\theta}_a$$

$$\dot{\omega}_a = \ddot{\theta}_d(0.9) + \theta_d(4) + \omega_a(-2.5) + \theta_a(-100)$$

$$\dot{\omega}_a = 4 + \omega_a(-2.5) + \theta_a(-100)$$

t	theta a
0.0	0.0
0.5	0.037
1.0	0.051
1.5	0.044
2.0	0.038
2.5	0.038
3.0	0.040
3.5	0.041
4.0	0.040

d)

$$\ddot{\theta}_a + \dot{\theta}_a(2.5) + \theta_a(100) = \ddot{\theta}_d(0.9) + \theta_d(4)$$

$$\frac{\theta_a}{\theta_d} = \frac{0.9s^2 + 4}{s^2 + 2.5s + 100} = \frac{A}{s + 1.25 - 9.9216} + \frac{B}{s + 1.25 + 9.9216}$$

$$A = -1.125 + 4.192j \qquad B = (-1.125) + (-4.192)j$$

first term:

$$\dot{\theta}_a + \theta_a(1.25 - 9.9216j) = \theta_d(-1.125 + 4.192j)$$

homog.:

$$\theta_a = C_1 e^{(-1.25 + 9.9216j)t}$$

part.:

$$\theta_a = -0.4300 - 0.05922j$$

init cond. of zero:

$$\theta_a = C_1 e^{(-1.25 + 9.9216j)t} - 0.4300 - 0.05922j$$

$$C_1 = 0.4300 + 0.05922j = 0.4341 e^{0.1369j}$$

$$\theta_a = 0.4341 e^{0.1369j} e^{(-1.25 + 9.9216j)t} - 0.4300 - 0.05922j$$

second term:

$$\dot{\theta}_a + \theta_a(1.25 + (9.9216)j) = \theta_d(-1.125 + (-4.192)j)$$

homog.:

$$\theta_a = C_1 e^{((-1.25) + (-9.9216)j)t}$$

part.:

$$\theta_a = -0.4300 + 0.05922j$$

init cond. of zero:

$$\theta_a = C_1 e^{(-1.25 + (-9.9216)j)t} - 0.4300 + 0.05922j$$

$$C_1 = 0.4300 + (-0.05922)j = 0.4341 e^{(-0.1369)j}$$

$$\theta_a = 0.4341 e^{(-0.1369)j} e^{(-1.25 + (-9.9216)j)t} - 0.4300 + 0.05922j$$

$$\theta_a = 0.4341 e^{0.1369j} e^{(-1.25 + 9.9216j)t} - 0.4300 - 0.05922j$$
$$+ 0.4341 e^{(-0.1369)j} e^{(-1.25 + (-9.9216)j)t} - 0.4300 + 0.05922j$$

$$\theta_a = 0.4341(e^{0.1369j} e^{(-1.25 + 9.9216j)t} + e^{(-0.1369)j} e^{(-1.25 + (-9.9216)j)t}) - 0.8600$$

$$\theta_a = 0.4341 e^{-1.25t}(e^{j(9.9216t + 0.1369)} + e^{-j(9.9216t + 0.1369)}) - 0.8600$$

$$\theta_a = 0.4341 e^{-1.25t}(2\cos(9.9216t + 0.1369)) - 0.8600$$

$$\theta_a = 0.8682 e^{-1.25t} \cos(9.9216t + 0.1369) - 0.8600$$

e)

$$\frac{\theta_a}{\theta_d} = \frac{(4)}{s^2 + 2.5s + 100} \qquad \theta_d = \frac{1}{s}$$

$$\theta_a = \frac{(4)}{s(s^2 + 2.5s + 100)} = \frac{A}{s + 1.25 + 9.922j} + \frac{B}{s + 1.25 + -(9.922)j} + \frac{C}{s}$$

$$A = \lim_{s \to -1.25 - 9.922j}\left(\frac{4}{(s + 1.25 - 9.922j)s}\right) = 0.0202\angle -3.016$$

$$B = \lim_{s \to -1.25 + 9.922j}\left(\frac{4}{(s + 1.25 + 9.922j)s}\right) = 0.0202\angle 3.016$$

$$C = \lim_{s \to 0}\left(\frac{4}{s^2 + 2.5s + 100}\right) = 0.04$$

$$\theta_a = \frac{0.0202\angle -3.016}{s + 1.25 + 9.922j} + \frac{0.0202\angle 3.016}{s + 1.25 + -(9.922)j} + \frac{0.04}{s}$$

$$\theta_a = 2(0.0202)e^{-1.25t}\cos(9.922t + 3.016) + 0.04$$

$$\theta_a = -0.0404e^{-1.25t}\sin\left(9.922t + 3.016 - \frac{3.14159}{2}\right) + 0.04$$

$$\theta_a = -0.0404e^{-1.25t}\sin(9.922t + 1.445) + 0.04$$

f)

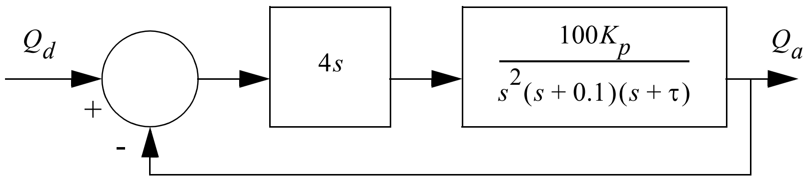
$$\frac{\theta_a}{\theta_d} = \frac{s(0.9) + (4)}{s^2 + s + 4} = \frac{j\omega(0.9) + (4)}{-\omega^2 + j\omega + 4}$$

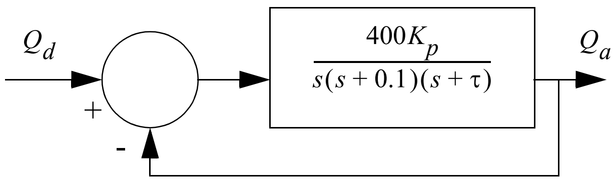
$$f = 1 \qquad \omega = 2\pi$$

$$\frac{\theta_a}{\theta_d} = \frac{j2\pi(0.9) + (4)}{-(2\pi)^2 + j2\pi + 4} = 0.1922\angle -2.011$$

$$gain = -14.32\,\text{dB} \qquad angle = -115\,\text{deg}$$

Answer 17.15 a)





$$\frac{Q_a}{Q_d} = \frac{400K_p}{s(s+0.1)(s+\tau)+400K_p} = \frac{400K_p}{s^3+s^2(0.1+\tau)+s(0.1\tau)+(400K_p)}$$

b)

$$\frac{Q_a}{Q_d} = \frac{400K_p}{s^3+s^2(0.1+\tau)+s(0.1\tau)+(400K_p)} = \frac{400K_p}{(s^2+2\zeta\omega_n s+\omega_n^2)(s+A)}$$

$$= \frac{400K_p}{s^3+2\zeta\omega_n s^2+s\omega_n^2+As^2+A2\zeta\omega_n s+A\omega_n^2}$$

$$= \frac{400K_p}{s^3+s^2(2\zeta\omega_n+A)+s(\omega_n^2+A2\zeta\omega_n)+A\omega_n^2}$$

$$400K_p = A\omega_n^2 = 100A$$

$$0.1\tau = \omega_n^2+A2\zeta\omega_n = 100+2.5A$$

$$0.1+\tau = 2\zeta\omega_n+A = 2.5+A$$

$$0.1\tau = 100+2.5(\tau-2.4) \qquad \tau = \frac{100+2.5(-2.4)}{-2.4} = -39.167$$

$$A = -2.5+0.1+(-39.167) = -41.567$$

$$K_p = \frac{100}{400}A = -10.392$$

c)

$$Q_a(s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)) = -4156.8Q_d$$

$$\dot{Q}_a = R$$

$$\dot{R} = S$$

$$\dot{S} + (-39.068)S + (-3.9168)R + (-4156.8)Q_a = -4156.8Q_d$$

$$\dot{S} = 4156.8Q_d + (39.068)S + (3.9168)R + (-4156.8)(1)$$

t	Qa
0.0	0.0
0.001	-1e-6
0.002	-6e-6
0.003	-19e-6
0.004	-46e-6
0.005	-91e-6
0.006	-159e-6
0.007	-255e-6
0.008	-384e-6
0.009	-553e-6
0.010	-766e-6

Note: This solution ‘blows up’ - it is not a practical system.

d)

$$\dddot{Q}_a + \ddot{Q}_a(-39.068) + \dot{Q}_a(-3.9168) + Q_a(-4156.8) = -4156.8$$

homog: $R = -41.57, 1.250 \pm 9.922j$

$$Q_h = C_1 e^{41.57t} + C_2 e^{-1.250t}\sin(9.922t + C_3)$$

part: $$Q_P = \frac{-4156.8}{-4156.8} = 1$$

$$Q_a = C_1 e^{41.57t} + C_2 e^{-1.250t}\sin(9.922t + C_3) + 1$$

$$0 = C_1 + C_2\sin(C_3) + 1 \qquad \therefore C_1 = -C_2\sin(C_3) - 1$$

$$\dot{Q}_a = 41.57C_1 e^{41.57t}$$

$$(-1.250C_2)e^{-1.250t}\sin(9.922t + C_3) + 9.922C_2 e^{-1.250t}\cos(9.922t + C_3)$$

$$0 = 41.57C_1 - 1.250C_2\sin(C_3) + 9.922C_2\cos(C_3)$$

$$C_1 = 0.03007C_2\sin C_3 + -0.2387C_2\cos C_3$$

$$\ddot{Q}_a = 41.57^2 C1 e^{41.57t} + (-1.250)^2 C_2 e^{-1.250t}\sin(9.922t + C_3)$$

$$+ (-1.250)(9.922)C_2 e^{-1.250t}\cos(9.922t + C_3)$$

$$+ (9.922)(-1.250)C_2 e^{-1.250t}\cos(9.922t + C_3)$$

$$-(9.922)^2 C2 e^{-1.250t}\sin(9.922t + C_3)$$

$$0 = 41.57^2 C1 + (-1.250)^2 C_2\sin(C_3) + (-1.250)(9.922)C_2\cos(C_3)$$

$$+ (9.922)(-1.250)C_2\cos(C_3) - (9.922)^2 C2\sin(C_3)$$

$$(-51.963)\sin C_3 + 412.49\cos C_3 = (1.5625)\sin(C_3) + (-12.4025)\cos(C_3) +$$

$$(-12.4025)\cos(C_3) - (98.446084)\sin(C_3)$$

$$(44.9205)\sin C_3 = (-437.295)\cos(C_3)$$

$$\frac{\sin C_3}{\cos(C_3)} = \frac{-437.295}{44.9205} \qquad C_3 = \operatorname{atan}\left(\frac{-437.295}{44.9205}\right) = -1.4684$$

$$C_1 = 0.03007C_2\sin C_3 + -0.2387C_2\cos C_3 = -C_2\sin(C_3) - 1$$

$$C_2 = \frac{-1}{0.03007\sin(-1.4684) + -0.2387\cos(-1.4684) + 1} = -1.057431$$

$$C_1 = -C_2\sin(C_3) - 1 = -2.0519$$

$$Q_a = (-2.0519)e^{41.57t} + (-1.0574)e^{-1.250t}\sin(9.922t - 1.4684) + 1$$

e)

$$Q_a = \left(\frac{-4156.8}{s^3 + s^2(-39.068) + s(-3.9168) + (-4156.8)}\right)\frac{1}{s}$$

$$Q_a = \frac{-4156.8}{(s-41.57)(s+1.250-9.922j)(s+1.250+9.922j)s}$$

$$Q_a = \frac{A}{(s-41.57)} + \frac{B}{(s+1.250-9.922j)} + \frac{C}{s+1.250+9.922j} + \frac{D}{s}$$

$$A = \lim_{s \to 41.57}\left[\frac{-4156.8}{(s+1.250-9.922j)(s+1.250+9.922j)s}\right] = -0.05176$$

$$B = \lim_{s \to -1.250+9.922j}\left[\frac{-4156.8}{(s-41.57)(s+1.250+9.922j)s}\right] = 0.4765\angle -3.039$$

$$D = \lim_{s \to 0}\left[\frac{-4156.8}{(s-41.57)(s+1.250-9.922j)(s+1.250+9.922j)}\right] = 1.000$$

$$Q_a = -0.05176e^{41.57t} + 0.2292e^{-1.250t}\cos(9.922t - 1.343) + 1.000$$

$$Q_a = -0.05176e^{41.57t} + 0.2292e^{-1.250t}\cos(9.922t - 1.343) + 1.000$$

Note: the results of d) and e) have different forms, but are equivalent.

f)

$$\frac{Q_a}{Q_d} = \frac{-4156.8}{(s-41.57)(s^2+2.5s+100)} = \frac{-4156.8}{(s-41.57)(s^2+2(10)0.125s+(10)^2)}$$

Poles:

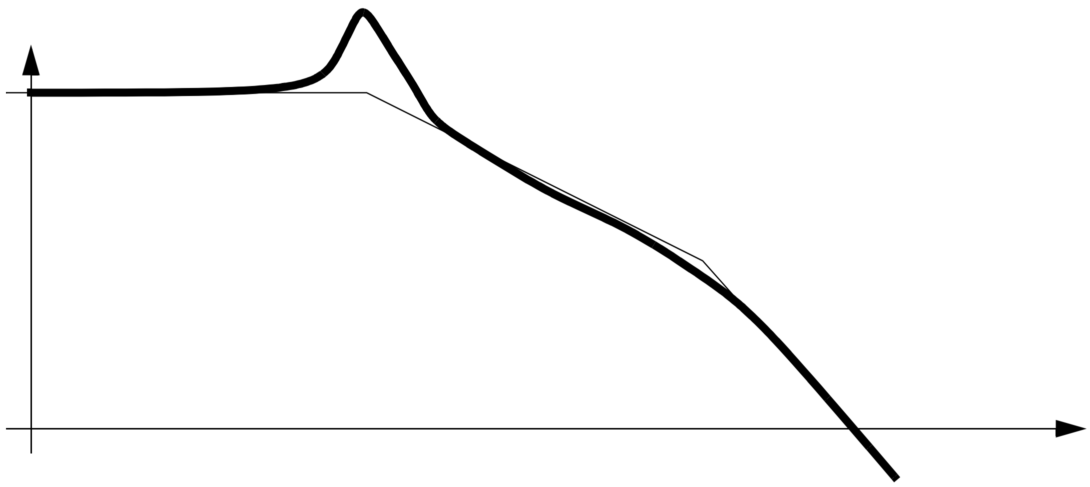
$\omega_1 = 41.57$ First order 20dB/dec down at 6.62 Hz

$$\omega_2 = 10\sqrt{1-(0.125)^2} = 9.922$$

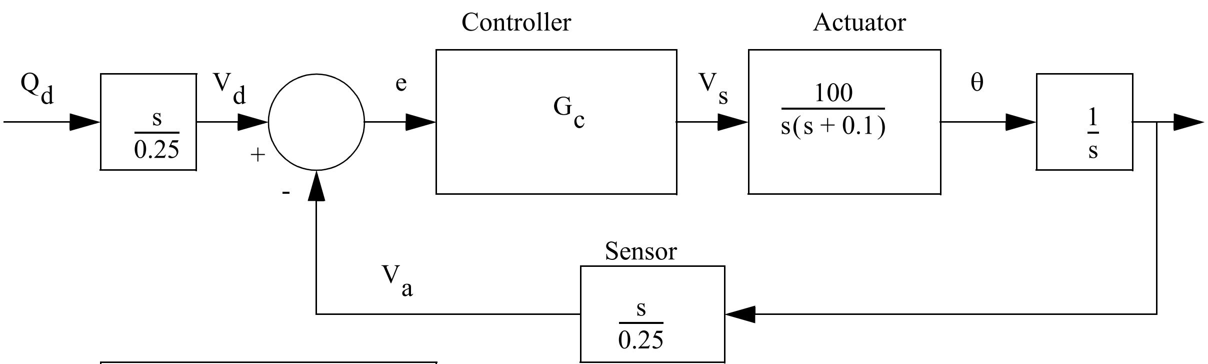
First order 40dB/dec down at 1.579 Hz
Underdamped resonant peak

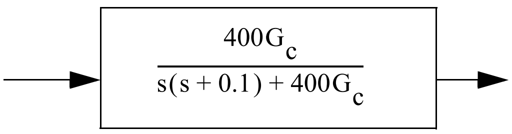
Initial Gain:

$$\frac{-4156.8}{(0-41.57)(0+2.5(0)+100)} = 0.9999 = 0\text{dB}$$



g)



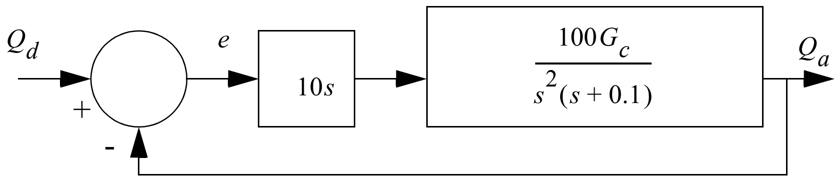


$$\frac{Q_a}{Q_d} = \frac{1}{s+2} = \frac{400G_c}{s(s+0.1)+400G_c}$$

$$s(s+0.1)+400G_c = (400)(s+2)G_c$$

$$G_c = \frac{s(s+0.1)}{400(s+1)}$$

Answer 17.16 a)



$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s(s+0.1)+1000G_c} = \frac{1000G_c}{s^2+(0.1)s+1000G_c}$$

b)

$$\frac{Q_a}{Q_d} = \frac{1000G_c}{s^2+(0.1)s+1000G_c} = \frac{1}{s+1}$$

$$1000G_c(s+1) = s^2+(0.1)s+1000G_c$$

$$G_c s(1000) = s^2+(0.1)s$$

$$G_c = \frac{s^2+(0.1)s}{1000s}$$

c)

$$Q_a = \left(\frac{1000(0.01)(s+1)}{s^2+(0.1)s+1000(0.01)(s+1)}\right)\frac{1}{s} = \frac{10s+10}{(s^2+(10.1)s+10)s}$$

$$= \frac{10s+10}{(s+1.1127)(s+8.9873)s} = \frac{A}{s+1.1127}+\frac{B}{s+8.9873}+\frac{C}{s}$$

$$A = \lim_{s\to-1.1127}\frac{10s+10}{(s+8.9873)s} = 0.12862$$

$$B = \lim_{s\to-8.9873}\frac{10s+10}{(s+1.1127)s} = -1.1286$$

$$C = \lim_{s\to 0}\frac{10s+10}{(s+1.1127)(s+8.9873)} = 1$$

$$Q_a = 0.12862e^{-8.9873t} - 1.1286e^{-1.1127t} + 1$$

d)

$$\frac{Q_a}{Q_d} = \frac{10(s+1)}{s^2 + (10.1)s + 10} = \frac{-0.1431}{s + 1.1127} + \frac{10.143}{s + 8.9873}$$

For the first term:

$$\dot{Q}_a + Q_a(1.1127) = Q_d(-0.1431) = -0.1431$$

homog:

$$Q_h = C_1 e^{-1.1127t}$$

part:

$$Q_p = \frac{-0.1431}{1.1127} = -0.1286$$

init cond zero:

$$Q_a = C_1 e^{-1.1127t} - 0.1286 \qquad C_1 = 0.1286$$

$$Q_a = 0.1286e^{-1.1127t} - 0.1286$$

For the second term:

$$\dot{Q}_a + Q_a(8.9873) = Q_d(10.143) = 10.143$$

homog:

$$Q_h = C_1 e^{-8.9873t}$$

part:

$$Q_p = \frac{10.143}{8.9873} = 1.1286$$

init cond zero:

$$Q_a = C_1 e^{-8.9873t} + 1.1286 \qquad C_1 = -1.1286$$

$$Q_a = (-1.1286)e^{-8.9873t} + 1.1286$$

Combined:

$$Q_a = 0.1286e^{-1.1127t} - 0.1286 + -1.1286e^{-8.9873t} + 1.1286$$

$$Q_a = 0.1286e^{-1.1127t} + (-1.1286)e^{-8.9873t} + 1$$

e)

$$\ddot{Q}_a(1) + \dot{Q}_a(10.1) + Q_a(10) = \dot{Q}_d(10) + Q_d(10)$$

$$\dot{Q}_a = R \qquad \text{eqn 17.1}$$

$$\ddot{Q}_a(1) + R(10.1) + Q_a(10) = 10$$

$$\dot{R}(1) = 10 - R(10.1) - Q_a(10) \qquad \text{eqn 17.2}$$

t	Qa
0.0	0.000
0.1	0.036
0.2	0.110
0.3	0.192
0.4	0.273
0.5	0.347
0.6	0.415
0.7	0.476
0.8	0.531
0.9	0.581
1.0	0.625

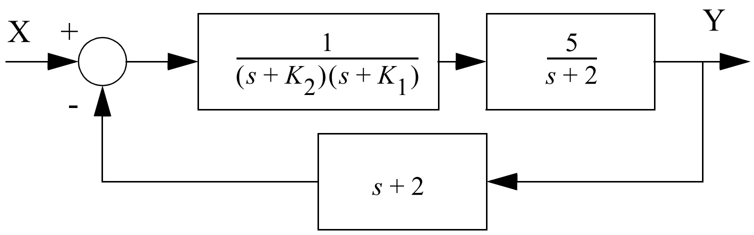
f)

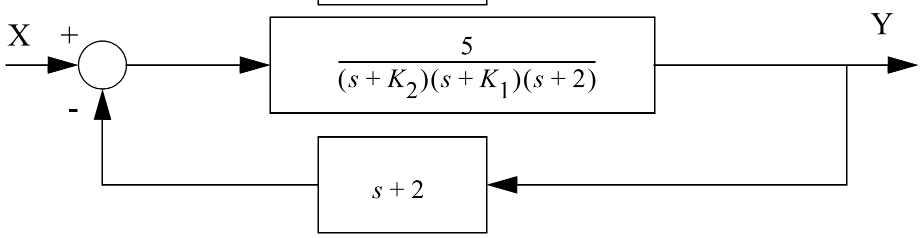
$$\frac{V_s}{e} = \frac{s^2 + (0.1)s}{1000s}$$

$$V_s = \frac{es}{1000} + \frac{e}{10000}$$

```
#define T 0.010

double Vs(double Qd, double Va){
        double e; // Error term
        double e_last = 0; // the last e value for the derivative
        double Vd; // The desired voltage
        double Qd_last = 0; // store the last value to calculate the derivative

        Vd = 10 * (Qd - Qd_last) / T;
        Qd_last = Qd;

        e = Vd - Va;

        Vs =  ((e - e_last) / T) / 1000 + e / 10000;
        e_last = e;

        return Vs;
}
```

Answer 17.17





$$\frac{Y}{X} = \frac{\frac{5}{(s+K_2)(s+K_1)(s+2)}}{1 + (s+2)\left(\frac{5}{(s+K_2)(s+K_1)(s+2)}\right)} = \frac{5}{(s+K_2)(s+K_1)(s+2) + (s+2)(5)}$$

$$= \frac{5}{(s+2)((s+K_2)(s+K_1)+5)} = \frac{5}{(s+2)(s^2 + s(K_1+K_2) + (5 + K_1K_2))}$$

$$2(10)2 = K_1 + K_2$$

$$10 = \sqrt{5 + K_1K_2} \qquad K_1K_2 = 95 = (40 - K_1)K_1$$

$$K_1^2 - 40K_1 + 95 = 0$$

$$K_1, K_2 = \frac{-(-40) \pm \sqrt{40^2 - 4(1)(95)}}{2} = 20 \pm 17.464249 = 37.46, 2.536$$

Answer 17.18 a)

$$\omega = \left(\frac{200}{s^2 + 2s + 100}\right)\left(\frac{5}{s}\right) = \frac{A}{s + 1 - 9.95j} + \frac{B}{s + 1 + 9.95j} + \frac{C}{s}$$

$$A = \lim_{s \to -1 + 9.95j}\left(\frac{1000}{s(s + 1 + 9.95j)}\right) = 5.025\angle 3.041$$

$$B = 5.025\angle -3.041$$

$$C = \lim_{s \to 0}\left(\frac{1000}{s^2 + 2s + 100}\right) = 10$$

$$\omega = \frac{5.025\angle 3.041}{s + 1 - 9.95j} + \frac{5.025\angle -3.041}{s + 1 + 9.95j} + \frac{10}{s}$$

$$\omega = 2(5.025)e^{-t}\cos(9.95t + 3.041) + 10$$

b)

$$\ddot{\omega} + \dot{\omega}(2) + \omega(100) = 200V_d = 1000$$

homog:

$$R = -1 \pm 9.95j$$

$$\omega_h = C_1 e^{-t}\cos(9.95t + C_2)$$

part:

$$\omega_p = 10$$

$$\omega = C_1 e^{-t}\cos(9.95t + C_2) + 10$$

$$0 = C_1\cos(C_2) + 10$$

$$\dot{\omega} = -C_1 e^{-t}\cos(9.95t + C_2) - 9.95C_1 e^{-t}\sin(9.95t + C_2)$$

$$0 = -C_1\cos(C_2) - 9.95C_1\sin(C_2)$$

$$\frac{-1}{9.95} = \frac{\sin(C_2)}{\cos(C_2)} \qquad C_2 = \operatorname{atan}\left(\frac{-1}{9.95}\right) = -0.100 + \pi = 3.04159$$

$$0 = C_1\cos(3.04159) + 10 \qquad C_1 = \frac{-10}{\cos(3.04159)} = 10.05$$

$$\omega = 10.05e^{-t}\cos(9.95t + 3.042) + 10$$

c)

$$\ddot{\omega} + \dot{\omega}(2) + \omega(100) = 1000$$

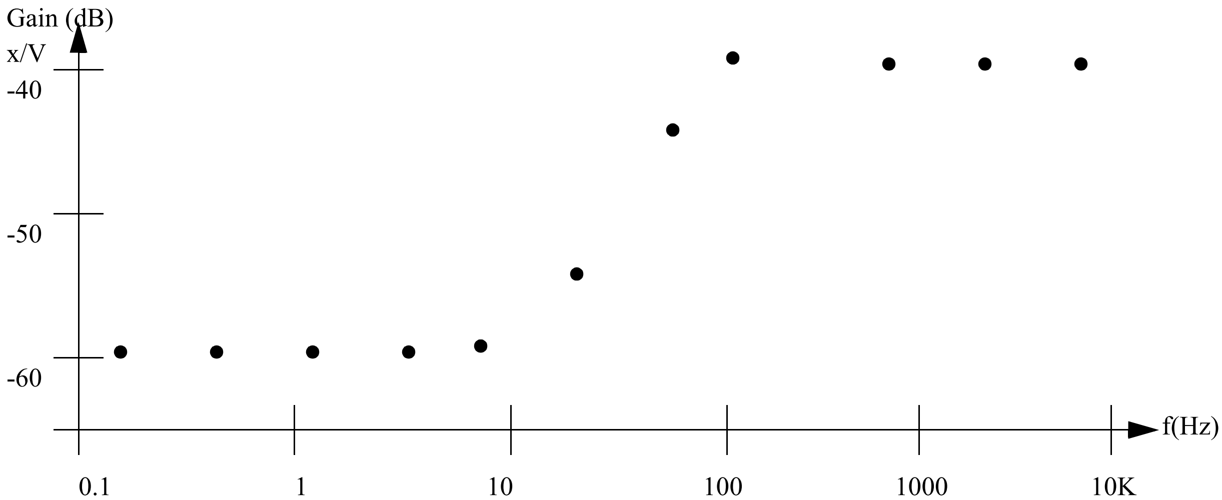
$$\dot{\omega} = \alpha$$

$$\dot{\alpha} = \alpha(-2) + \omega(-100) + 1000$$

t (s)	w	a
0	0	0.0
0.2	12.7	76.5
0.4	15.1	-52.5
0.6	4.6	-17.4
0.8	10.1	48.5
1.0	13.7	-21.1
1.2	7.40	-19.4
1.4	9.21	28.0
1.6	12.4	-5.68
1.8	8.94	-15.6
2.0	9.09	14.7

d) There are multiple ways to verify. The easiest is to calculate points at the time intervals from step c). They should be very close if the answers are correct.

Answer 17.19 a)

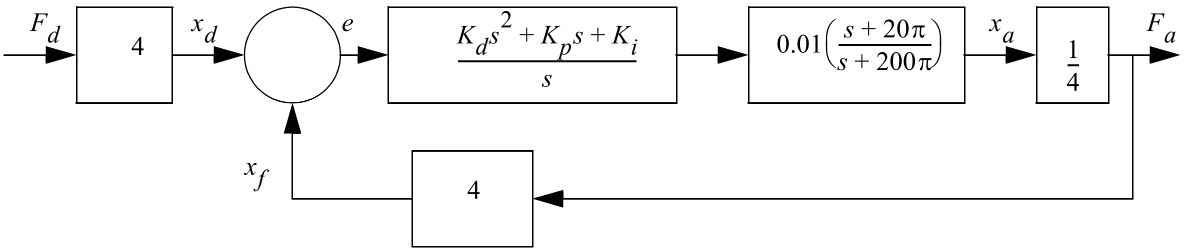


$$\frac{x}{V} = A\left(\frac{s + 20\pi}{s + 200\pi}\right)$$

$$10^{-\frac{60}{20}} = A\left(\frac{0 + 20\pi}{0 + 200\pi}\right) \qquad A = 10\left(10^{-\frac{60}{20}}\right) = 0.01$$

$$\frac{x}{V} = 0.01\left(\frac{s + 20\pi}{s + 200\pi}\right)$$

b)



$$\frac{F_a}{F_d} = \frac{\left(\frac{K_d s^2 + K_p s + K_i}{s}\right) 0.01\left(\frac{s+20\pi}{s+200\pi}\right)}{1 + \left(\frac{K_d s^2 + K_p s + K_i}{s}\right) 0.01\left(\frac{s+20\pi}{s+200\pi}\right)}$$

$$\frac{F_a}{F_d} = \frac{(K_d s^2 + K_p s + K_i)(s + 20\pi)}{100s(s + 200\pi) + (K_d s^2 + K_p s + K_i)(s + 20\pi)}$$

$$\frac{F_a}{F_d} = \frac{(K_d s^2 + K_p s + K_i)(s + 20\pi)}{100s(s + 200\pi) + (K_d s^2 + K_p s + K_i)(s + 20\pi)}$$

Answer 17.20

$$V_s = esK_d + eK_p + \frac{K_i}{s}e$$

```
#define T 10 // 10 ms loop
#define Kd 5
#define Ki 6
#define Kp 7

int Vs(int Qd, int Va){
        int Vd, e, e_diff;
        int e_last = 0;
        int e_sum = 0;

        Vd = Qd * 4;
        e = Vd - Va;
        e_sum += e * T;
        e_diff = (1000 * (e - e_last)) / T;
        e_last = e;

        return (Kd * e_diff + Kp * e + Ki * e_sum / 1000);
}
```

17.1 Problems Without Solutions

Problem 17.21 Convert the differential equation to the (Laplace) s-domain.

$$L[\ddot{x} + 7\dot{x} + 8x = 9] = $$

where,

$$\ddot{x}(0) = 1$$

$$\dot{x}(0) = 2$$

$$x(0) = 3$$

Problem 17.22 Convert the function to the s-domain.

$$f(t) = 5\sin(5t + 8)$$
$$F(s) = L[f(t)] =$$

Problem 17.23 Convert the equation from the s-domain.

$$F(s) = \frac{5}{s} + \frac{6}{s+7}$$

$$f(t) = L^{-1}[F(s)] =$$

Problem 17.24 Prove the following relationships.

a) $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

b) $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

c) $L[e^{-at}f(t)] = F(s+a)$

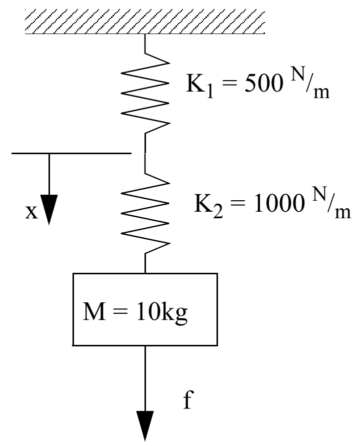
d) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

e) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

f) $L[tf(t)] = -\frac{d}{dt}F(s)$

Problem 17.25 The applied force ‘f’ is the input to the system, and the output is the displacement ‘x’.

a) find the transfer function.



b) What is the steady state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if ‘x’ is the input.
d) Find x(t), given F(t) = 10N for t >= 0 seconds using Laplace methods.

Problem 17.26 The following differential equation is supplied, with initial conditions.

$$\ddot{y} + \dot{y} + 7y = f \qquad y(0) = 1 \qquad \dot{y}(0) = 0$$

$$f(t) = 10 \qquad t > 0$$

a) Solve the differential equation using calculus techniques.
b) Write the equation in state variable form and solve it numerically.
c) Find the frequency response (gain and phase) for the transfer function using the phasor transform. Roughly

sketch the bode plots.
d) Convert the differential equation to the Laplace domain, including initial conditions. Solve to find the time response.

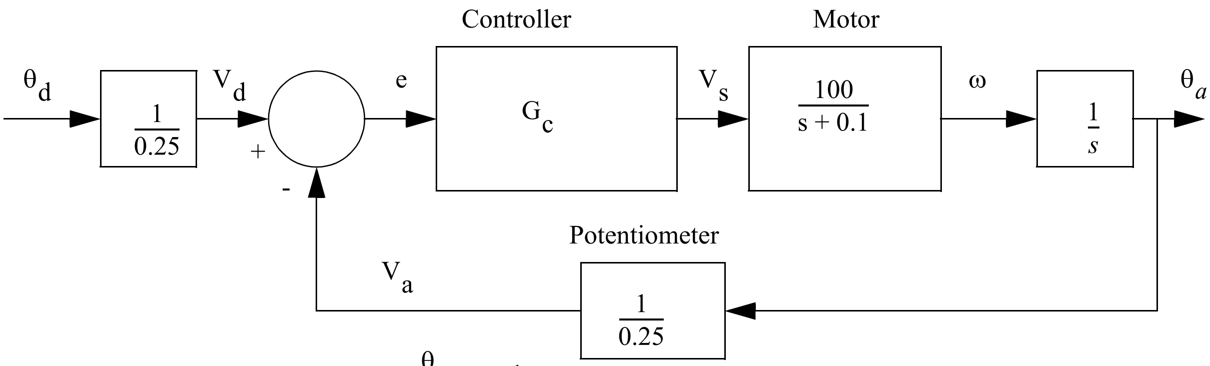
Problem 17.27 Given the transfer functions and input functions, F, use Laplace transforms to find the output of the system as a function of time. Indicate the transient and steady state parts of the solution.

$$\frac{x}{f} = \frac{D^2}{(D + 200\pi)^2} \qquad f = 5\sin(62.82t)$$

$$\frac{x}{f} = \frac{D(D + 2\pi)}{(D + 200\pi)^2} \qquad f = 5\sin(62.82t)$$

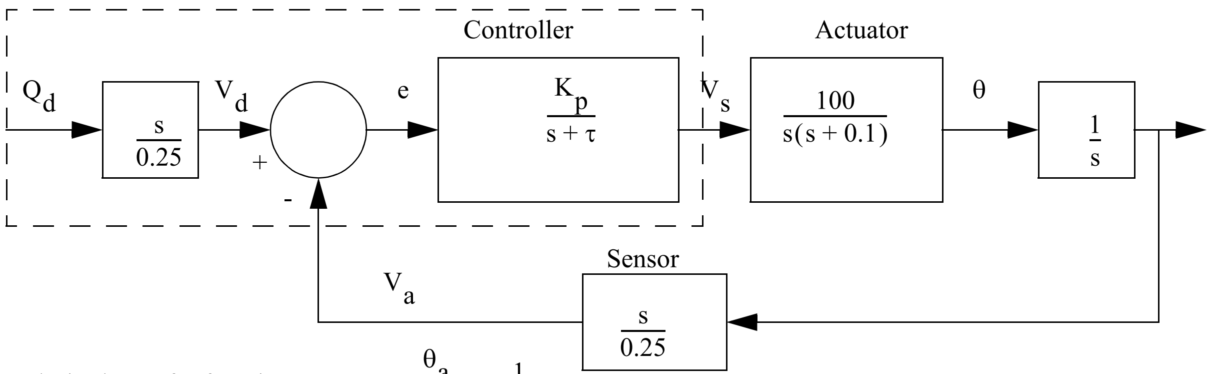
$$\frac{x}{f} = \frac{D^2(D + 2\pi)}{(D + 200\pi)^2} \qquad f = 5\sin(62.82t)$$

Problem 17.28 Select a controller transfer function, Gc, that will reduce the system to a first order system with a time constant of 0.5s, as shown below.



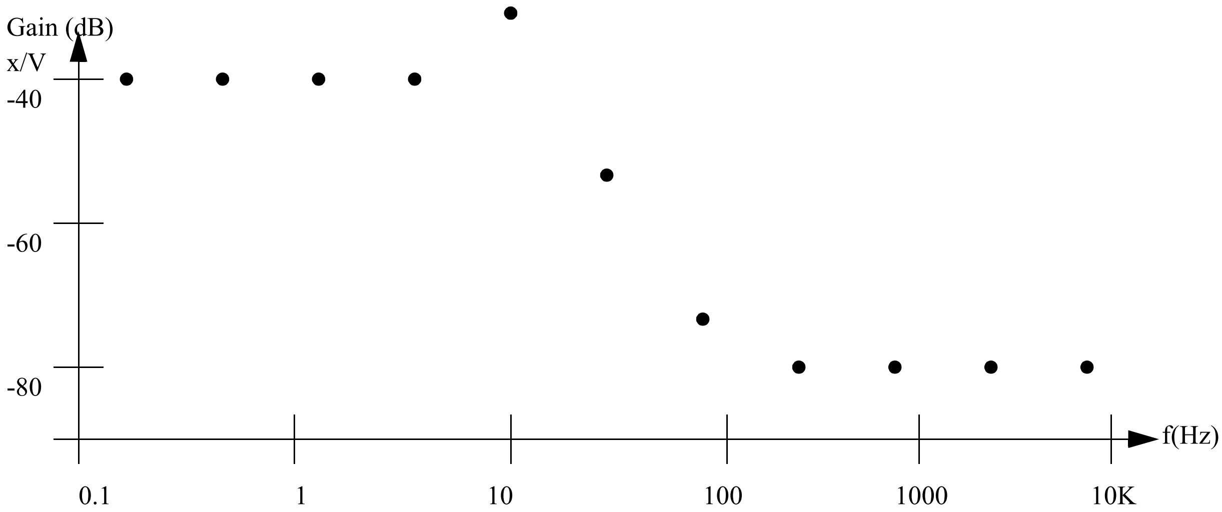
desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{s+2}$

Problem 17.29 Write a C program for an ATMega 32 microcontroller to implement the control system in the dashed line below with an update time of 10ms.

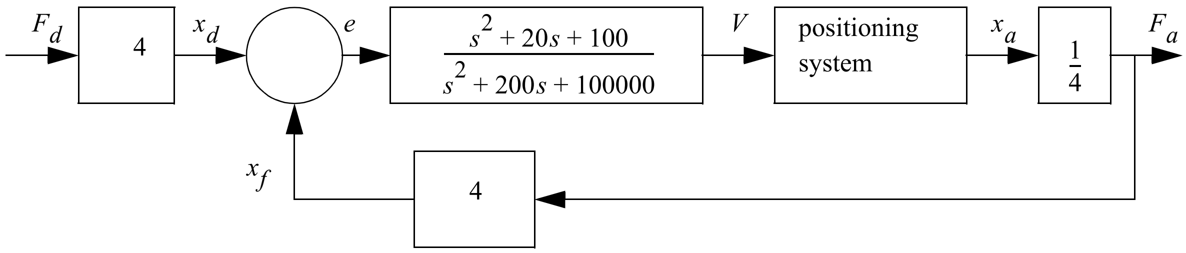


desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{s+2}$

Problem 17.30 a) Given the experimental Bode (Frequency Response Function) plot below, find a transfer function to model a positioning system. The input is a voltage 'V' and the output is a displacement 'x'. (Hint: after calculating the function develop a Bode plot to verify the system performance.)



b) The transfer function found in step a) will be used for the positioning system in the block diagram below. Find the overall transfer function for the system.



c) Find a new controller transfer function (V/e) that would give the overall response below.

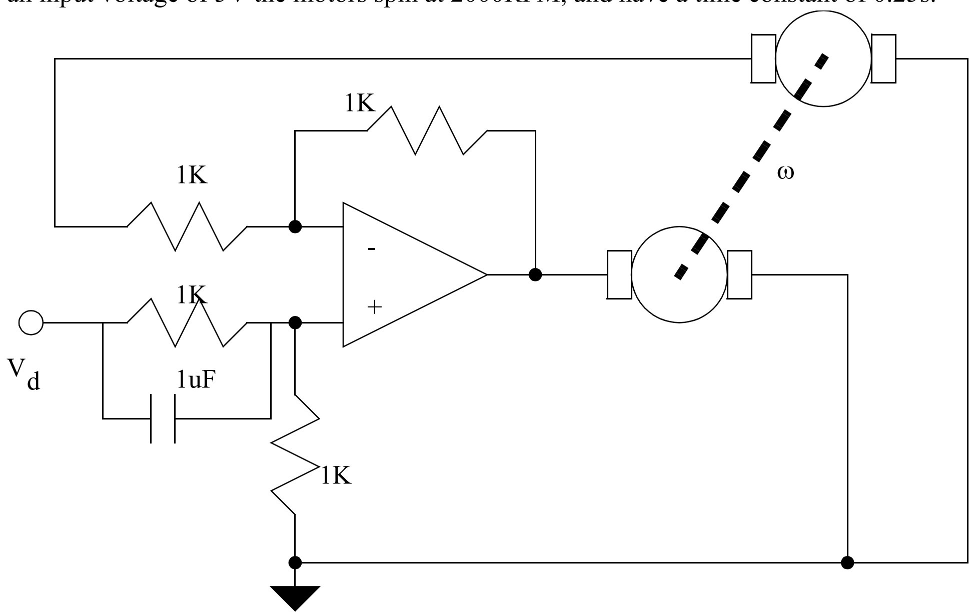
$$\frac{F_a}{F_d} = \frac{s + 10}{s + 100}$$

d) Write a subroutine that implements the control function (V/e) found in step c). (Hint: Convert it to state equations first.)

e) The controller function is replaced with the function below. Draw a Bode plot using the straight line approximation. Compare it to the Bode plot in part a).

$$\frac{V}{e} = 10$$

Problem 17.31 a) Develop differential and state equations describing the following system. The input is Vd and the output is the motor shaft speed. Assume all components are ideal. The motors are identical with a resistance of 20 ohms. With an input voltage of 5V the motors spin at 2000RPM, and have a time constant of 0.25s.



b) Find the system response to a unit step input explicitly (i.e. homogeneous and particular).
c) Find the system response to a unit step input using Laplace transforms.
d) Solve the problem numerically and report the results using a table of values.
e) Use phasors to find the steady state output response to an input of,

$$v_d = 5\sin(100t)$$

Problem 17.32 a) Using Laplace transforms find the system response for the transfer function and input function (given as a function of time).

$$\frac{\omega}{V_d} = \frac{5000s + 10^7}{s^2 + 2001s + 20000} \qquad V_d(t) = 5t$$

b) Use numerical methods to find the response of the system described with a transfer function. Report the results using a table of values.

$$\frac{\omega}{V_d} = \frac{5000s + 10^7}{s^2 + 2001s + 20000} \qquad V_d(t) = 5t$$

c) Use phasors to find the steady state output response to the input given below.

$$\frac{\omega}{V_d} = \frac{5000s + 10^7}{s^2 + 2001s + 20000} \qquad V_d = 5\sin(100t)$$

Problem 17.33 Convert the following differential equations to physical systems. Show the method.

a)

$$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

b)

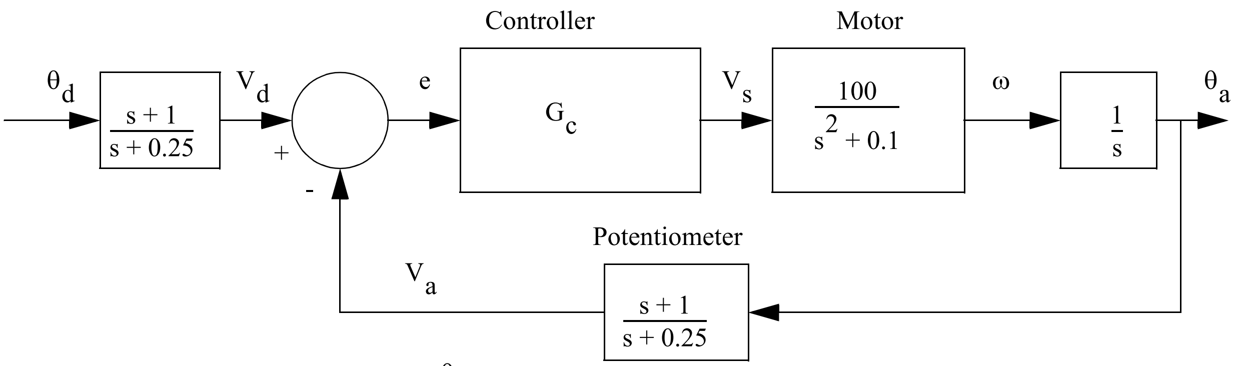
$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + g$$

$$\dot{x}_2 = v_2$$

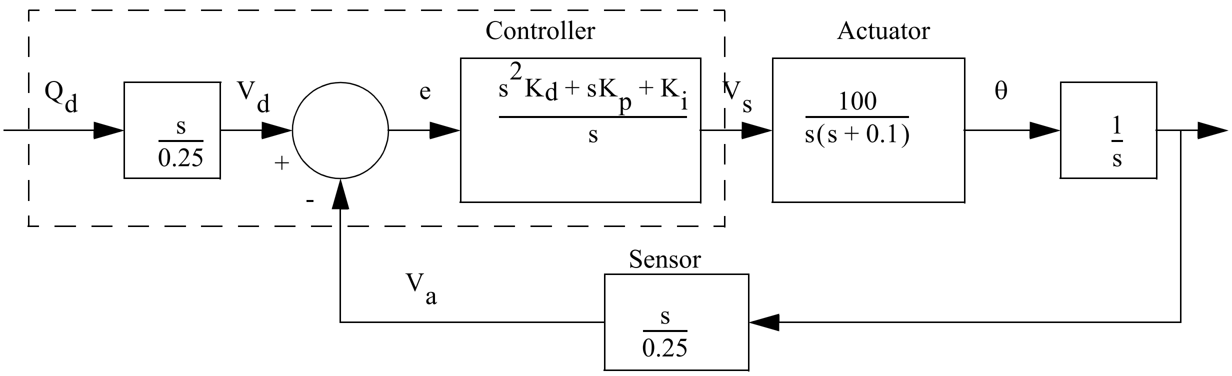
$$\dot{v}_2 = v_1\left(\frac{-R_2^2K_{d1}}{J_1}\right) + x_1\left(\frac{-R_2^2K_{s1}}{J_1}\right) + x_2\left(\frac{R_2^2K_{s1}}{J_1}\right) + \frac{-FR_1R_2}{J_1}$$

Problem 17.34 For the following control system select a controller transfer function, Gc, that will make the overall system performance match the desired transfer function.



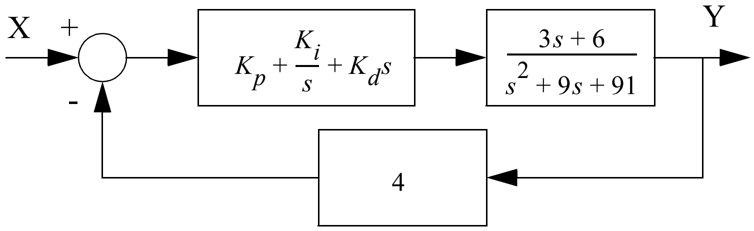
desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{s^2 + 2(2)(10)s + 10^2}$

Problem 17.35 Write a C program for an ATMega microcontroller to implement the control system in the dashed line below with an update time of 10ms.



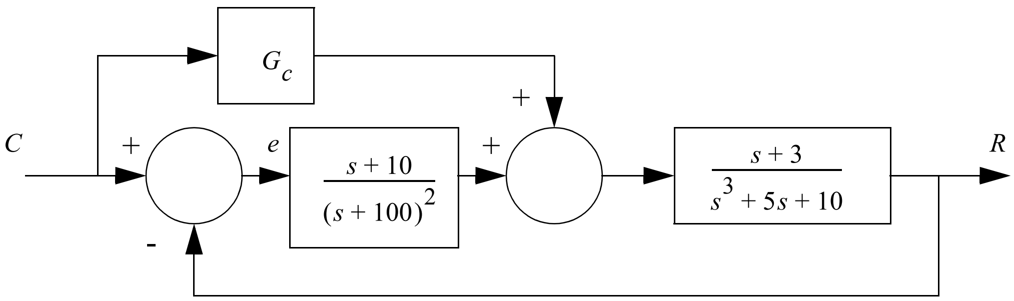
Problem 17.36 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer

function is given.



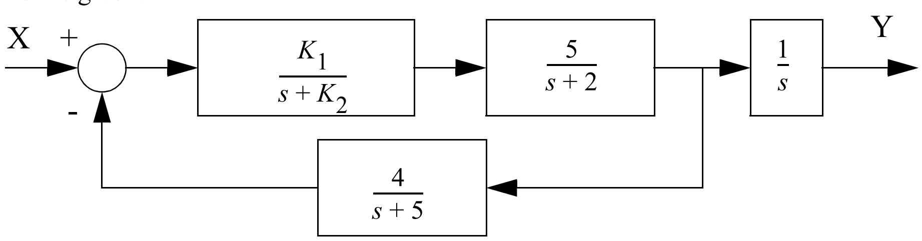
a) Develop the transfer function for the system.
b) Select controller values that will result in a response that includes a natural frequency of 2 rad/sec and damping factor of 0.5. Verify that the controller will be stable.
c) If the values of Kp = Ki = Kd = 1 find the response to a unit ramp input as a function of time using LaPlace Transforms.
d) Find the response in part c) using numerical methods.
e) Find the system response to an input of X = 5sin(100t + 1)
f) If the input X is a trapezoidal motion profile with an acceleration time of 2 seconds, and a maximum velocity of 5, what would the response Y look like.

Problem 17.37 For the following control system select a controller transfer function, Gc, that will make the overall system performance match the desired transfer function.



desired transfer function $\frac{R}{C} = \frac{1}{s+50}$

Problem 17.38 A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer function is given.



a) Develop the transfer function for the system.
b) Select controller values that will result in a response that includes a natural frequency of 2 rad/sec and damping factor of 0.5. Verify that the controller will be stable.
c) If the values of K1 = K2= 1 find the response to a unit ramp input as a function of time using LaPlace Transforms.
d) Find the response in part c) using numerical methods.
e) Find the system response to an input of X = 5sin(10t + 1) using phasor transforms.
f) If the input X is a trapezoidal motion profile with an acceleration time of 2 seconds, and a maximum velocity of 5, calculate the response Y.

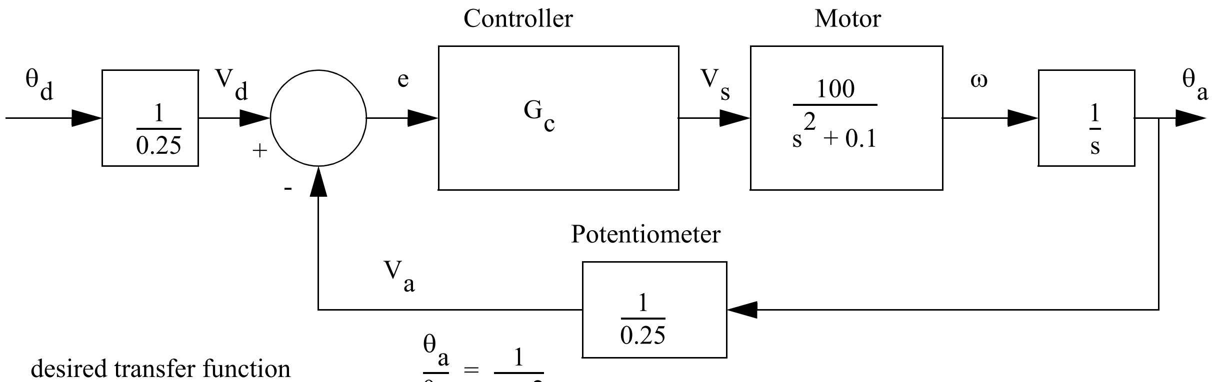
Problem 17.39 The following differential equation is supplied, with initial conditions.

$$\ddot{y} + \dot{y} + 7y = f \qquad y(0) = 1 \qquad \dot{y}(0) = 0$$

$$f(t) = 10 \qquad t > 0$$

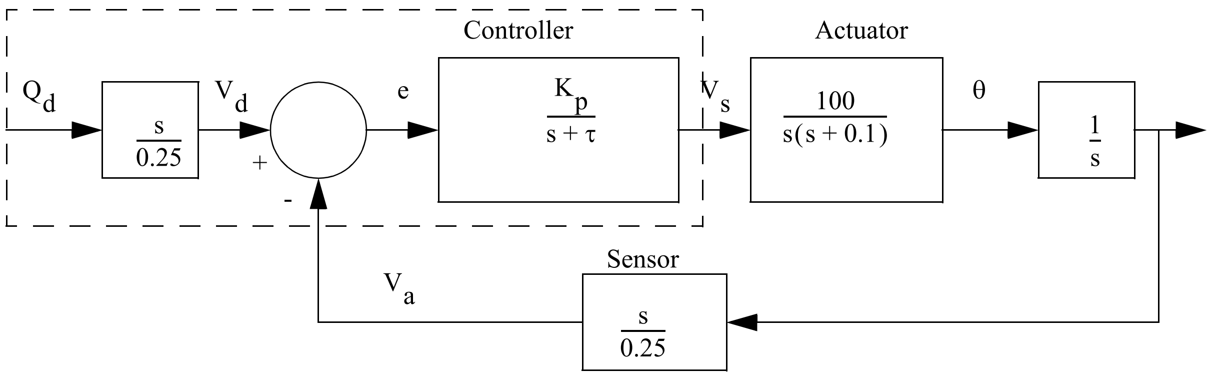
a) Solve the differential equation using calculus techniques.
b) Write the equation in state variable form and solve it numerically.
c) Find the frequency response (gain and phase) for the transfer function using the phasor transform. Sketch the bode plots.
d) Convert the differential equation to the Laplace domain. Solve to find the time response.

Problem 17.40 Select a controller transfer function, Gc, that will reduce the system to a first order system with a time constant of 0.5s, as shown below.

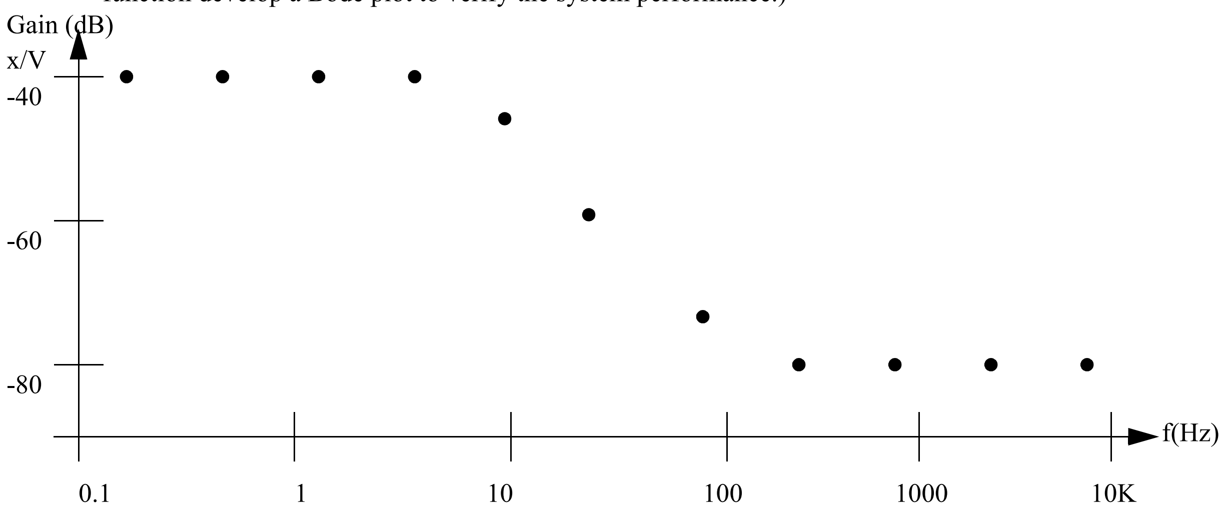


desired transfer function $\frac{\theta_a}{\theta_d} = \frac{1}{s+2}$

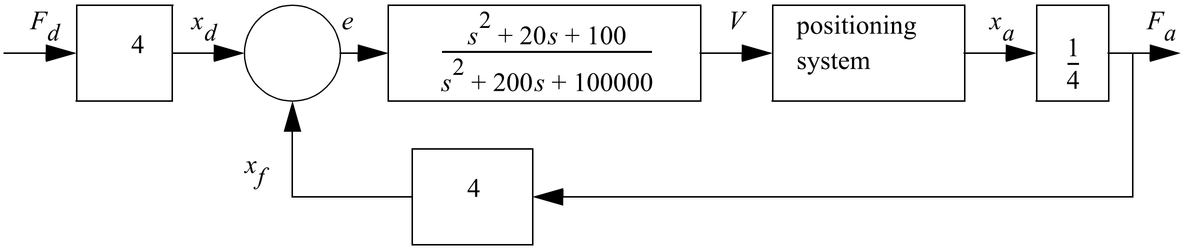
Problem 17.41 Write a C program for an ATMega 32 microcontroller to implement the control system in the dashed line below with an update time of 10ms.



Problem 17.42 a) Given the experimental Bode (Frequency Response Function) plot below, find a transfer function to model a positioning system. The input is a voltage 'V' and the output is a displacement 'x'. (Hint: after calculating the function develop a Bode plot to verify the system performance.)



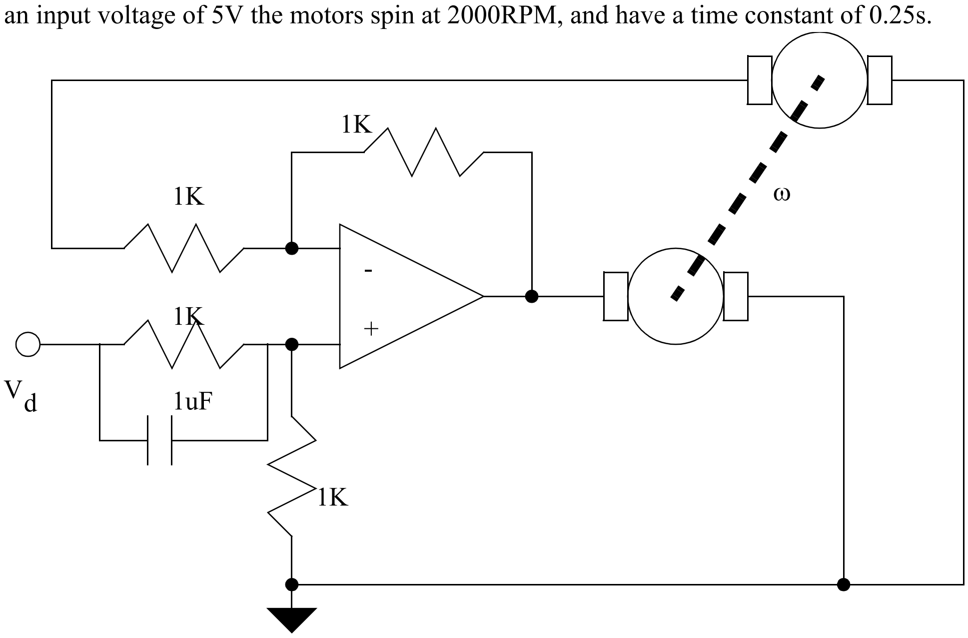
b) The transfer function found in step a) will be used for the positioning system in the block diagram below. Find the overall transfer function for the system.



c) Find a new controller transfer function (V/e) that would give the overall response below.

$$\frac{F_a}{F_d} = \frac{s + 10}{s + 100}$$

Problem 17.43 Develop differential and state equations describing the following system. The input is Vd and the output is the motor shaft speed. Assume all components are ideal. The motors are identical with a resistance of 20 ohms. With an input voltage of 5V the motors spin at 2000RPM, and have a time constant of 0.25s.



17.2 References

17.1 Close, C.M. and Frederick, D.K., “Modeling and Analysis of Dynamic Systems, second edition, John Wiley and Sons, Inc., 1995.